FINAL REPORT

of

MAJOR RESEARCH PROJECT

sanctioned by

UNIVERSITY GRANTS COMMISSION, NEW DELHI

[F. No. 43-414/2014 (SR), 17-08-2015

W.E.F. July 01, 2015]

for

MATHEMATICS SUBJECT ENTITLED

Zero divisor graphs of rings"

SUBMITTED BY:

Prof. Shariefuddin Pirzada

M.Sc Ph.D.

PRINCIPAL INVESTIGATOR, PROFESSOR

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF KASHMIR, SRINAGAR-190006

JAMMU AND KASHMIR, INDIA

DECEMBER 2018

Acknowledgement

Prof. Shariefuddin Pirzada, Principal investigator, UGC Major Research Project is thankful to the University Grants Commission, New Delhi, for the award of Major Research Project and financial assistance to pursue the research work in the Department of Mathematics, University of Kashmir, Srinagar. Prof. Shariefuddin Pirzada express his most sincere gratitude to the authorities of the University of Kashmir, Srinagar, for providing a lot of basic facilities in the Department of Mathematics, University of Kashmir, Srinagar throughout the tenure of the project. Prof. Shariefuddin Pirzada, is also the Finance Officer, University of Kashmir, Srinagar for extending timely accounts facilities. Further, Prof. Shariefuddin Pirzada is indebted to the services of project fellow, Mr. Mohmad Imran Bhat, for his sincere support in carrying out the project work in the Department of Mathematics, University of Kashmir, Srinagar. We are also thankful to Dr. Ramiz Raja for being a part of this project . Last, but not least w e are highly grateful to Prof. Shane Redmond, East Kentucky University, USA with who whom we made collaborations, for the cooperation and providing useful suggestions.

Encl. 1

Report of the Work Done

I. **Objectives of the project:** Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. The study of algebraic structures, using the properties of graphs, has become an exciting research topic and has been studied extensively, leading to many fascinating results and questions. The zero divisor graph helps us to study the algebraic properties of rings using graph theoretic tools. We translate some algebraic properties of a ring to graph theory and then the geometric properties of graphs helps us to explore some interesting results in the algebraic structure of rings and at advancing the application of one to the other. The main thrust of our project work intended to be carried out was to investigate locating sets and locating numbers of zero divisor graphs of rings and the extent to which this aspect of zero divisor graph of rings can be extended to the zero divisor graph determined by equivalence classes of zero divisors of a ring. Our chief motivation for choosing the setting of this later zero divisor graph is to explore more interesting properties, then compare and contrast with the existing zero divisor graph of rings. We endeavor to characterize various attributes of zero divisor graph of rings.

In view of this, the following problems were considered:

a. Locating number of some graphs and graphs associated to commutative rings and extend it to the ideal based zero divisor graph of rings.
b. Obtain bounds for the locating number of zero divisor graph associated with commutative rings and Cartesian product of zero divisor graphs. Relating locating sets to the cut vertices of zero divisor graphs and give a special code to the cut vertex of zero divisor graph.
c. Relation between the chromatic number, clique number, domination number and the locating number of zero divisor graph of rings.
d. Characterizations for the locating number of zero divisor graphs associated with finite product of rings, like Boolean rings. Moreover, finding a combinatorial formula for computing the locating number of zero divisor graphs.
e. We study the graphs associated with modules over commutative rings and study the concept of zero divisor graphs to modules over rings.
f. The notion of locating number is then to apply in zero divisor graphs determined by equivalence classes of zero divisors of a ring.
g. Relationship between diameter, girth and locating number of zero divisor graph of rings and the same in ideal based zero divisor graph of rings and zero divisor graph determined by equivalence classes of zero divisors of a ring.

II. *Work done so far and results achieved and publications, if any, resulting from the work:*

We have worked according to the objectives of the project. The results obtained during the tenure of the project were summarized in terms of research papers. Following are the research papers published in the reputed international journals:

[1] **S. Pirzada,** Rameez Raja and S. P. Redmond, Locating sets and numbers of graphs associated to commutative rings, *J. Algebra Appl*., **13, 7** (2014) 1450047 18 pp. DOI: 10.1142/S0219498814500479.

[2] **Rameez Raja** and S. Pirzada**,** , On locating numbers and codes of zero divisor graphs associated with commutative rings, *J. Algebra Appl*. , **15, 1** (2016) 1650014 22 pp. DOI: 10.1142/S0219498816500146.

[3] **S. Pirzada** and Rameez Raja, On the metric dimension of a zero divisor graph, *Communications in Algebra*, 45, 4 (2017) 1399-1408, DOI: 10.1080/00927872.2016.1175602.

[4] **S. Pirzada** and M. Imran Bhat, Computing metric dimension of compressed zero divisor graphs associated to rings, *Acta Univ. Sapientiae, Mathematica*, **10, 2** (2018)

[5] **Shariefuddin Pirzada** and Rameez Raja, On graphs associated with modules over commutative rings, J. Korean Math. Soc. **53**, **5** (2016) 1167-1182. http://dx.doi.org/10.4134/JKMS.j150457
pISSN: 0304-9914 / eISSN: 2234-3008,

III. *Has the progress been according to original plan of work and towards achieving the objective? If not, state reasons:* **The progress of the project was according to the original plan and there were no deviations to the original title and ideas.**

IV. Please indicate the difficulties, if any, experienced in implementing the project: **No difficulties have been experienced in the entire period of the project.**

V**.** *If project has not been completed, please indicate the approximate time by which it is likely to be completed. A summary of the work done for the period (Annual basis) may please be sent to the commission on a separate sheet.* **Project has been completed successfully.**

VI. *If the project has been completed, please mention the summary of the findings of the study.*

In the period of three years, we focused mainly to study the locating sets of zero divisor graphs of rings. In [1], we introduce and investigate the locating sets and numbers of graphs associated with commutative rings. We discussed various properties of locating number $loc(G)$ which includes the characterization of all finite commutative rings, relationship between girth, diameter and locating number. We then extended this concept to the ideal based zero divisor graphs of rings. In [2], we show that there is finite connected graph G with $loc(\Gamma(G))\ /\ |V(G)| = m\ /\ n$, where $m\ <\ n$ are positive integers. We examine two equivalence relations on the vertices of $\Gamma(R)$ and the relationship between locating sets n and the cut vertices of $\Gamma(R)$. Further, we obtain bounds for the locating number in zero divisor graphs of a commutative ring and discuss the relation between locating number, domination number, clique number and chromatic number of $\Gamma(R)$. We also investigate the locating number in $\Gamma(R)$. when R is a finite product of rings such that each ring in the decomposition of R is not

isomorphic to a finite Boolean ring. The special emphasis has been given to the zero divisor graph $\mathbf{\Gamma}(\prod_{i=1}^{n} Z_2)$ of Boolean rings, where it happens that locating number is less than n for $n = 1,2,3,4$ and for $n = 5$, $loc(G) = 5$, but the problem is still open when $n > 5$. In [3], we have shown that for a given rational $q \epsilon (0,1)$ there exists a finite graph G such that the ration $\frac{loc(G)}{V(H)} = q$, where H is any induced subgraph of G. We discussed the locating number of Cartesian product of zero divisor graphs and showed that there exists a zero divisor graph $\Gamma(R_1) \times \Gamma(R_2)$ such that there loc($\Gamma(R_1) \times \Gamma(R_2)$) lies between the numbers loc($\Gamma(R_1)$) and loc($\Gamma(R_1)$) $+1$ where R_1 and R_2 are any two finite commutative rings (not domains) with each having unity 1. In [4], we examined the concept of locating numbers in zero divisor graph $\Gamma_E(R)$ determined by equivalence classes of zero divisors associated with rings. We studied the relationship of locating number between the zero divisor graph and the later one. We obtained the locating number of local rings and obtain bounds for the locating number of this zero divisor graph. Further, we provide a combinatorial formula for computing the number of zero divisors and locating number of the family of graphs given by of the zero divisor graph $\Gamma(R \times F)$ and the zero divisor graph of rings determined by equivalence classes of its zero divisors $\Gamma_E(R \times F)$ **.** We then extended the concept of zero divisor graph to modules over rings. The three simple graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ associated to called as full annihilating , semi-annihilating and star-annihilating graph. a module M. Locating number is then investigated in these three graphs and various characterizations have been proven. Moreover, the equality of locating number is also discussed.

VII. Any other information which would help in evaluation of work done on the project:

a) *Manpower trained* : **The project fellow Mr. Mohmad Imran Bhat has worked in project for a period of 2 years till the end of the project.**
b) *Ph,D awarded/enrolled* : **The project fellow Mr. Mohmad Imran Bhat has been enrolled for Ph.D programme on 9−05-−2016 under the guidance of Prof. Shariefuddin Pirzada, Department of Mathematics, University of Kashmir, Srinagar-190006, J&K India.**
c) *Publication of results* : **As mentioned above.**
d) *Other Impact*: In this project, we have introduced locating sets and locating numbers of zero divisor graphs of rings.

VIII. **Achievements:**

In the current project, we have worked as per the objectives of the project and we were successful in obtaining important results in the form of research papers ([1] − [4]) which are published in the reputed international

ISI/SCI journals. We have made efforts to work and investigate the locating set of zero divisor graphs of rings. The significance of the study can be encountered in various areas as discussed in IX.

Encl. 1

IX . **Contribution to the Society:**

The outcomes of this project on which so for work has been done, especially on locating sets of a zero divisor graph which is a parameter that has appeared in various applications of graph theory and are mainly pertaining to the following:

Pharmaceutical Chemistry: A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. From this we can determine whether any two compounds in the collection share the same functional group at a particular position. This comparative statement plays a critical role in drug discovery whenever it is to be determined whether the features of a compound are responsible for its pharmacological activity.

Navigation can be studied in a graph-structured framework in which the navigation agent (which we shall assume to be a point robot) moves from node to node of a ‘‘graph space’’. The robot can locate itself by the presence of distinctly labeled ‘‘landmark’’ nodes in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive landmark provides information about the direction to the landmark, and allows the robot to determine its position by Triangulation.

Strategies for master mind game and coin weighing, the graphs that arise in fact Cartesian products. Further, dominating sets are being applied to hazard or full detection in networks and facilities. These zero divisor graph structures may be used to model computer networks, where servers, hosts or hubs in a network can be represented as vertices in a graph and edges could represent connections between them.

Coding theory, network designs, network discovery and verification, combinatorial optimization and sonar and coast guard Loran are some other important applications of locating sets. .

List of Research Publications

S.No	Title of the Research Paper	Journal Published	Year	Vol. No.	Pages
1	Locating sets and numbers of graphs associated to commutative rings	Journal and Algebra and its applications	2014	13 (7)	18
2	On locating numbers and codes of zero divisor graphs associated with commutative rings	Journal and Algebra and its applications	2016	15 (1)	22
3.	On graphs associated with modules over commutative rings	J. Korean Math. Soc.	2016	53(5)	1167-1182
4.	On the metric dimension of a zero divisor graph	Communications in Algebra	2017	45 (4)	1399-1408
5.	Computing metric dimension of compressed zero divisor graph associated to rings	Acta Universitatis Sapientiae, Mathematica	2018	10 (2)	298-318

Complete Statistics of Work Presented in the Final Report

Published - 05

Total - 05

Annexure -VIII

UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.

Final Report of the work done on the Major Research Project.

1. Project report No. **Final**

2. UGC Reference No. **F.No. 43-414/2014(SR) dated 17-08-2015**

3. Period of report: from **01-07-2015** to **30-06-2018**

4. Title of research project: **"Zero divisor graphs of rings"**

5. (a) Name of the Principal Investigator: **Prof. Shariefuddin Pirzada**

   (b) Deptt. **Mathematics**

   (c) University/College where work has progressed: **University of Kashmir, Srinagar-190006**

6. Effective date of starting of the project: **01-07-2015**

7. Grant approved and expenditure incurred during the period of the report:

   a. Total amount sanctioned: **Rs.13,80000/-**

   b. Total amount release : **Rs.11,13,248/-**

   c. Total expenditure: **Rs. 11,18662 /-**

d. Report of the work done: **A separate sheet has been enclosed herewith (Encl. 1)**

PRINCIPAL INVESTIGATOR

DEAN RESEARCH

Dean Research
University of Kashmir

Annexure – IX

UNIVERSITY GRANTS COMMISSION

BAHADUR SHAH ZAFAR MARG

NEW DELHI – 110 002

PROFORMA FOR SUBMISSION OF INFORMATION AT THE TIME SENDING THE FINAL REPORT OF THE WORK DONE ON THE PROJECT

1. Title of the Project: **"Zero divisor graphs of rings"**

2. NAME AND ADDRESS OF THE PRINCIPAL INVESTIGATON: **Dr. Shariefuddin Pirzada, Professor, Department of Mathematics, University of Kashmir, Srinagar-190006, Jammu and Kashmir, India.**

3. NAME AND ADDRESS OF THE INSTITUTION: **University of Kashmir, Srinagar-190006, J&K, India.**

4. UGC APPROVAL LETTER NO. AND DATE: **F.No. 43-414/2014(SR) dated 17-08-2015**

5. DATE OF IMPLEMENTATION: **01-07-2015**

6. TENURE OF THE PROJECT: **From 01-07-2015 to 30-06-2018 (36 months)**

7. TOTAL GRANT ALLOCATED: **Rs. 11,94720/-**

8. TOTAL GRANT RECEIVED : **Rs. 11,13248/-**

9. FINAL EXPENDITURE: **Rs. 11,18662/-**

10. TITLE OF THE PROJECT: **"Zero divisor graphs of rings"**

11. OBJECTIVES OF THE PROJECT: **A separate sheet has been enclosed herewith (Encl. 1).**

12. WHETHER OBJECTIVES WERE ACHIEVED: **Yes, the objectives are achieved and details are mentioned in the enclosed separate sheet (Encl. 1).**

13. ACHIEVEMENTS FROM THE PROJECT: **A separate sheet has been enclosed herewith (Encl. 1).**

14. SUMMARY OF THE FINDINGS: **A separate sheet has been enclosed herewith (Encl. 1).**

15. CONTRIBUTION TO THE SOCIETY: **A separate sheet has been enclosed herewith (Encl. 1).**

16. WHETHER ANY PH.D. ENROLLED/PRODUCED OUT OF THE PROJECT: Yes, the project fellow Mr. Mohmad Imran Bhat has enrolled to Ph.D on 09-05-2016 under the guidance of **Prof. Shariefuddin Pirzada, Department of Mathematics, University of Kashmir, Srinagar-190006, J&K, India.**

17. NO. OF PUBLICATIONS OUT OF THE PROJECT: **In this project, a total of 04(Four) Research Papers have been published and the details are given in the enclosed separate sheet (Encl. 1).**

PRINCIPAL INVESTIGATOR **DEAN RESEARCH**

Dean Research
University of Kashmir

UNIVERSITY GRANTS COMMISION
BAHADUR SHAH ZAFAR MARG
NEW DELHI-110002

MAJOR RESEARCH PROJECT COPY OF THE SPECIMEN OF HOUSE RENT FOR POST-DOCTROL FELLOW/PROJECT ASSOCIATE/PROJECT FELLOW

Certified that ~~Shri/Dr.~~ Mohmad Imran Bhat is paying House Rent of Rs. 2800 p/m and is eligible to draw House Rent Allowances @ 20% of 14000 as per University Rules.

P.I	Head of the Department	Registrar/Dean Research
Signature with seal	Signature with seal	Signature with seal
Prof. S. Pirzada, PI Major Research Project, (UGC)	Professor & Head, Postgraduate Department of Mathematics, University of Kashmir, Srinagar	Prof. Sheikh Javid Ahmad, Dean Research, University of Kashmir

Project title : Zero divisor graphs of rings

Annexure-XI

Final Report Assessment /Evaluation Certificate

(Two Members Expert Committee Not Belonging to the Institute of Principal Investigator)

(to be submitted with the final report)

It is certified that the final report of Major Research Project entitled **"Zero Divisor Graphs of Rings"** by **Prof. S. Pirzada,** Department of **Mathematics** has been assessed by the committee consisting the following members for final submission of the report to the UGC, New Delhi under the scheme of Major Research Project.

Comments / Suggestions of the Expert Committee:

The output of the Project is excellent.

Name & Signature of Experts with Date:

Name of Expert	University/College Name	Signature with Date
1. Prof. Mohammad Ashraf	Aligarh Muslim University , Aligarh	DEPARTMENT OF MATHEMATICS A.M.U., ALIGARH
2. Prof. K.S. Charak	University of Jammu	G. Deptt of Mathematics University of Jammu Jammu

It is also certified that the final report has been uploaded on UGC-MRP portal on .....

It is also certified that the final report, Executive summary of the report , Research documents, monograph academic papers provided under Major Research Project have been posted on the website of the University/College.

(Registrar /Principal)

Seal

Dean Research
University of Kashmir

Journal of Algebra and Its Applications
Vol. 13, No. 7 (2014) 1450047 (18 pages)

DOI: 10.1142/S0219498814500479

Locating sets and numbers of graphs associated to commutative rings

S. Pirzada* and Rameez Raja†

Department of Mathematics
University of Kashmir Srinagar
Kashmir 190006, India
**pirzadasd@kashmiruniversity.ac.in*
†rameeznaqash@gmail.com

Shane Redmond

Department of Mathematics and Statistics
Eastern Kentucky University
Richmond, KY 40475, USA
shane.redmond@eku.edu

Received 14 November 2013
Accepted 7 February 2014
Published 7 April 2014

Communicated by S. R. Lopez-Permouth

For a graph $G(V, E)$ with order $n \geq 2$, the locating code of a vertex v is a finite vector representing distances of v with respect to vertices of some ordered subset W of $V(G)$. The set W is a locating set of $G(V, E)$ if distinct vertices have distinct codes. A locating set containing a minimum number of vertices is a minimum locating set for $G(V, E)$. The locating number denoted by $\mathrm{loc}(G)$ is the number of vertices in the minimum locating set. Let R be a commutative ring with identity $1 \neq 0$, the zero-divisor graph denoted by $\Gamma(R)$, is the (undirected) graph whose vertices are the nonzero zero-divisors of R with two distinct vertices joined by an edge when the product of vertices is zero. We introduce and investigate locating numbers in zero-divisor graphs of a commutative ring R. We then extend our definition to study and characterize the locating numbers of an ideal based zero-divisor graph of a commutative ring R.

Keywords: Locating set; locating number; ring; zero-divisor; zero-divisor graph.

Mathematics Subject Classification: 05C69, 13A99

1. Introduction

Using the properties of graphs to study algebraic structures is an exciting research topic and has attracted considerable attention. The research in this subject aims at exposing the relationship between algebra and graph theory. The concept of the zero-divisor graph of a commutative ring was first introduced by Beck in [7]. In his

work, all the elements of the ring were vertices of the graph. Anderson and Naseer used this concept in [4]. A different method of associating a zero-divisor graph to commutative ring was proposed by Anderson and Livingston in [3]. They believed that this better illustrated the zero-divisor structure of the ring. The zero-divisor graph of a commutative ring has also been studied in [1, 2, 8, 11, 12, 14, 15], and extended to noncommutative rings in [13]. For basic definitions from graph theory we refer to [9, 17], and for commutative ring theory we refer to [6, 10].

A simple graph $G(V, E)$ is a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of $G(V, E)$ called edges. A graph $G(V, E)$ is connected if there is a path between every two distinct vertices of $G(V, E)$. The distance from a vertex v_1 to v_2, denoted by $d(v_1, v_2)$, is the length of the shortest path from v_1 to v_2 ($d(v, v) = 0$ and $d(v_1, v_2) = \infty$, if there is no such path). The diameter of $G = G(V, E)$ is $\mathrm{diam}(G) = \sup\{d(v_1, v_2) \mid v_1, v_2 \in V(G)\}$. The girth of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycle). A graph $G(V, E)$ in which each pair of distinct vertices is joined by an edge is called a complete graph. A complete graph with n vertices is denoted by K_n. A graph $G(V, E)$ is said to be bipartite if its vertex set can be partitioned into two sets $V_1(G)$ and $V_2(G)$ such that every edge of G has one end in $V_1(G)$ and another in $V_2(G)$. A complete bipartite graph is one whose each vertex of one partite set is joined to every vertex of the other partite set. We denote the complete bipartite graph with partite sets of size m and n by $K_{m,n}$. More generally a complete r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset and each vertex of a partite set is joined to every vertex of the another partite sets. A complete bipartite graph of the form $K_{1,n}$ is called a star graph. A graph $G(V, E)$ is Hamiltonian if it has a cycle which contains every vertex of the graph. Moreover, $N(v)$ denotes the set all vertices of $G(V, E)$ adjacent to the vertex v and $N[v] = N(v) \cup \{v\}$.

Throughout, R is a commutative ring (with 1), unless otherwise stated, $\mathrm{nil}(R)$ its set of nilpotent elements and $U(R)$ its set of units. We say R is reduced if $\mathrm{nil}(R) = 0$. Let $Z(R)$ be the set of zero-divisors of R. We use $Z^*(R)$ to denote the set of nonzero zero-divisors of R. We will denote the ring of integers by $\mathbb{Z}$, the ring of integers modulo n by $\mathbb{Z}_n$ and a finite field on q elements by $\mathbb{F}_q$, respectively. By the zero-divisor graph of R, denoted by $\Gamma(R) = (V(\Gamma(R)), E(\Gamma(R)))$, where $V(\Gamma(R))$ is the vertex set and $E(\Gamma(R))$ is an edge set of $\Gamma(R)$, we mean the graph whose vertices are the nonzero zero-divisors of R and for distinct x and y in $Z^*(R)$, there is an edge connecting x and y if and only if $xy = 0$. We adopt the approach used by Anderson and Livingston in [3] and consider only the nonzero zero-divisors as vertices of the graph.

In Sec. 2, we will discuss the locating sets and locating numbers of graphs in general and determine the locating numbers of some well-known graphs. In Sec. 3, we study the locating sets and locating numbers of zero-divisor graphs associated to commutative rings and give some examples. We explore the relationship between

girth and locating numbers in $\Gamma(R)$, particularly when R is a reduced commutative ring. In Sec. 4, we investigate the locating sets and locating numbers of an ideal based zero-divisor graph of R. Further, if I is the intersection of prime ideals of R, we show that the locating number of $\Gamma_I(R)$ is finite. At the end, we mention locating number of $\Gamma(R)$ for some small commutative rings R.

2. Locating Sets and Numbers of Graphs

Definition 2.1. Let G be a connected n vertex graph with $n \geq 2$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of G and a vertex v of G, the locating code (or simply the code) of v with respect to W is the k-vector,

$$c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).$$

The set W is a locating set for G if distinct vertices have distinct codes. A locating set containing a minimum number of vertices is a minimum locating set for G. The locating number denoted by $\mathrm{loc}(G)$ is the number of vertices in the minimum locating set for G. The only vertex of G whose code with respect to W has 0 in its ith coordinate of $c_W(v)$ is $\{w_i\}$. So the vertices of W necessarily have distinct codes. Since only vertices of G that are not in W have coordinates all of which are positive, it is only these vertices that need to be examined to determine if their codes are distinct. This implies that the locating number of G is at most $n-1$. In fact, for every connected graph G of order $n \geq 2$,

$$1 \leq \mathrm{loc}(G) \leq n-1.$$

The locating number of a single vertex graph is 0, since for $w \in V(G)$, $d(w, w) = 0$. The locating number for an empty graph $G = \emptyset$ is not defined. For example, the locating set for the cycle of length four with vertices v_1, v_2, v_3, v_4 is $W = \{v_1, v_2\}$ and the locating number is 2.

Now we have the following lemmas.

Lemma 2.1. *A connected graph G of order n has locating number 1 if and only if $G \cong P_n$, where P_n is a path on n vertices.*

Proof. Suppose $G \cong P_n$, and let $P_n : v_1 v_2 \cdots v_n$ be a path. Since $d(v_i, v_1) = i-1$ for $1 \leq i \leq n$, it follows that v_1 is a minimum locating set of P_n and so $\mathrm{loc}(P_n) = 1$. For the converse, assume that G is a connected graph of order n with locating number 1 and let $W = \{w\}$ be a minimum locating set for G. For each vertex of G, $c_W(v) = d(v, w)$ is a nonnegative integer less than n. Since the codes of the vertices of G with respect to W are distinct, there exists a vertex u of G such that $d(u, w) = n-1$. Consequently, the diameter of G is $n-1$ and since G is connected this implies that $G \cong P_n$. □

Lemma 2.2. *A connected graph G of order $n \geq 2$ has locating number $n-1$ if and only if $G \cong K_n$.*

Proof. First, assume that $G \cong K_n$ and let W be a minimum locating set for G. If $u \notin W$, then every coordinate of $c_W(u)$ is 1. Therefore, every minimum locating set for G must contain all but one vertex of G and so $\mathrm{loc}(G) = n-1$. For the converse, assume that G is not isomorphic to K_n. Then G contains two vertices u and v with $d(u,v) = 2$. Let uxv be a path of length 2 in G, and let $W = V(G) - \{x, v\}$. Since $d(u,v) = 2$ and $d(u,x) = 1$, it follows that $c_W(x) \neq c_W(v)$ and so W is a locating set. Therefore, $\mathrm{loc}(G) \leq n-2$, which is a contradiction. □

Lemma 2.3. *For $n \geq 3$, locating number of a cycle C_n is 2.*

Proof. Let G be a cycle C_n: $v_1v_2 \cdots v_nv_1$ and let W be the minimum locating set of G. We have to show $|W| = 2$. If we remove an edge between any two vertices in G then we have a path. Therefore, $W = \{v_i\}$, $1 \leq i \leq n$, is not the minimum locating set and hence $|W| \geq 2$. Clearly, if we choose any two adjacent vertices v_i and v_{i+1} in G, then codes of each of the remaining vertices of G are different. Therefore, $W = \{v_i, v_{i+1}\}$ is the minimum locating set and $|W| = 2$. □

One can view the locating sets and locating number of a graph as influenced by the symmetry in that graph, using the following definition from [11].

Definition 2.2. Let G be a connected graph with $|V(G)| \geq 2$. Two distinct vertices u and v of G are distance similar, if $d(u,x) = d(v,x)$ for all $x \in V(G) - \{u, v\}$. It can be easily checked that the distance similar relation ($\sim$) is an equivalence relation on $V(G)$ and two distinct vertices are distance similar if either $uv \notin E(G)$ and $N(u) = N(v)$ or $uv \in E(G)$ and $N[u] = N[v]$.

Theorem 2.1. *Let G be a connected graph. Suppose G is partitioned into k distinct distance similar classes $V_1, V_2, \ldots, V_k$ (that is, $x, y \in V_i$ if and only if $d(x,a) = d(y,a)$ for all $a \in V(G) - \{x, y\}$).*

(i) *Any locating set W for G contains all but at most one vertex from each V_i.*
(ii) *Each V_i induces a complete subgraph or a subgraph with no edges.*
(iii) $\mathrm{loc}(G) \geq |V(G)| - k$.
(iv) *There exists a minimal locating set W for G such that if $|V_i| > 1$, at most $|V_i| - 1$ vertices of V_i are elements of W.*
(v) *If m is the number of distance similar classes that consist of a single vertex, then $|V(G)| - k \leq \mathrm{loc}(G) \leq |V(G)| - k + m$.*

Proof. Let $V(G)$ be partitioned into k-distinct distance similar equivalence classes, say $V_1, V_2, \ldots, V_k$. Clearly, each set V_i, $1 \leq i \leq k$, is either an independent set or induces a complete subgraph of G. If W is a subset of $V(G)$ and $u \sim v$, then $c_W(u) = c_W(v)$ whenever both $u, v \notin W$. Hence, each locating set contains all but at most one vertex in each of the equivalence classes V_i. Therefore, $\mathrm{loc}(\Gamma(R)) \geq |V(G)| - k$.

Again let $V_1, V_2, \ldots, V_k$ be a partition of $V(G)$ into distance similar sets. Without loss of generality, suppose $|V_1| > 1$. Let W be a locating set for G such that $V_1 \subseteq W$. Let $x \in V_1$. We will show that either (a) $T = W - \{x\}$ is a locating set for G or (b) there is an element $t \in V(G) - V_1$ such that $T^* = T \cup \{t\} = W \cup \{t\} - \{x\}$ is a locating set of G. That is, if $V_1 \subseteq W$, then there is another locating set of G of cardinality no greater than that of W where V_1 is not a subset. (Note that if $n = 1$, then G is complete and the result is trivial. So suppose $n > 1$.)

Define $T = W - \{x\}$. (Since W is considered in defining an ordered set, without loss of generality take $W = \{x, w_1, w_2, \ldots\}$ and $T = \{w_1, w_2, \ldots\}$.) Let $v_1, v_2 \in V(G)$. If both $v_1, v_2 \in T$, then clearly $c_T(v_1) \neq c_T(v_2)$. If $v_1 \in T$ and $v_2 \notin T$, then $c_T(v_1) \neq c_T(v_2)$ since there is some coordinate of $c_T(v_1)$ that is 0 while the same coordinate of $c_T(v_2)$ is nonzero.

Suppose $v_1, v_2 \notin W$. Then $c_W(v_1) \neq c_W(v_2)$. Therefore, if $c_T(v_1) = c_T(v_2)$, it must be the case that $d(x, v_1) \neq d(x, v_2)$. However, there exists some $z \in T \cap V_1$. Hence, $c_T(v_1) = c_T(v_2)$ would imply $d(z, v_1) = d(z, v_2)$ and thus, since $x, z \in V_1$, $d(x, v_1) = d(z, v_1) = d(z, v_2) = d(x, v_2)$. Therefore, $c_T(v_1) \neq c_T(v_2)$.

If there does not exist any $v_1 \in T$ such that $c_T(x) = c_T(v_1)$, then T is a locating set of G.

So, suppose there does exist some $v_1 \in T$ such that $c_T(x) = c_T(v_1)$. Let $r \in T \cap V_1$. Assume there is some other element $v_2 \in T$ such that $c_T(v_2) = c_T(x) = c_T(v_1)$. Since W is a locating set, $c_W(v_2) \neq c_W(v_1)$. Thus, $d(v_1, x) \neq d(v_2, x)$. However, $c_T(v_2) = c_T(v_1)$ and $x, r \in V_1$ imply $d(v_1, x) = d(v_1, r) = d(v_2, r) = d(v_2, x)$, a contradiction. Hence, if there is an element $v_1 \in T$ such that $c_T(x) = c_T(v_1)$, then v_1 is unique.

Case 1: Suppose $|V_i| > 1$ for each i. Then for any $y \in V(G) - \{x, v_1\}$, choose $q_y \in T$ such that $y \sim q_y$. Then $d(y, v_1) = d(q_y, v_1) = d(q_y, x) = d(y, x)$. Thus, $v_1 \sim x$, which is a contradiction.

Case 2: Suppose $|V_i| = 1$ for some i. Since v_i and x are not distance similar, there is some $s \in V(G) - \{x, v_1\}$ with $d(s, x) \neq d(v_1, x)$. For some j, $s \in V_j$. Note that $V_j = \{s\}$ because if not, there is some $t \in V_j \cap T$ and $d(v_1, s) = d(v_1, t) = d(x, t) = d(x, s)$. Define $T^* = T \cup \{s\}$. Then $|T^*| = |W|$ and $c_T * (x) \neq c_T * (v_1)$. Thus, since $T \subseteq T^*$ and using the argument in the above paragraphs, $c_{T^*}(a) \neq c_{T^*}(b)$ for any two distinct $a, b \in V(G)$. Hence, T^* is a locating set for G.

The last statement of the theorem now follows from parts (i), (iii), and (iv).

□

Corollary 2.1. *For $n \geq 3$, the locating number for the bipartite graph $K_{1,n-1}$ is $n - 2$ and for $r \geq 2$, $n \geq 5$, the locating number for the bipartite graph $K_{r,n-r} = K_{n,m}$ with $n = r$ and $m = n - r$ is $n - 2$.*

Proof. This can be proved by using Theorem 2.1 since in each case the graph is not complete and $V(G)$ will be partitioned into two distance similar equivalence classes. □

Example 2.1. It is possible for a graph on infinitely many vertices to have a finite locating number. Let G_1 be a path of infinite length (that is, $V(G_1) = \{x_1, x_2, \ldots\}$ and the only edges are defined by the rule $x_i x_{i+1}$ for each $i \geq 1$). Define a tree graph G_2 with vertex set $V(G_2) = \{y_1, y_2, \ldots\}$ with the only edges defined by $y_1 y_2$ and $y_i y_{i+2}$ for $i \geq 1$. Then $W_1 = \{x_1\}$ and $W_2 = \{y_1, y_2\}$ are minimal locating sets for G_1 and G_2 respectively, giving $\mathrm{loc}(G_1) = 1$ and $\mathrm{loc}(G_2) = 2$. However, the next result shows that if the diameter of the graph is bounded, then a finite locating number implies the graph is finite.

Theorem 2.2. *Let G be a connected graph with* $\mathrm{diam}(G) = m < \infty$. *If* $\mathrm{loc}(G) = k < \infty$, *then* $|V(G)| \leq (m+1)^k$.

Proof. Let W be a minimal locating set for G with $|W| = k$. Since $\mathrm{diam}(G) = m$, $d(x, y) \in \{0, 1, 2, \ldots, m\}$ for every $x, y \in V(G)$. Then, for each $x \in V(G)$, $c_W(x)$ is a k-coordinate vector where each coordinate is in the set $\{0, 1, 2, \ldots, m\}$. Thus, there are only $(m+1)^k$ possibilities for $c_W(x)$. Since $c_W(x)$ is unique for each $x \in V(G)$, $|V(G)| \leq (m+1)^k$. □

Corollary 2.2. *Let G be a connected graph with finite diameter. Then $|V(G)|$ is finite if and only if* $\mathrm{loc}(G)$ *is finite.*

3. Locating Sets and Numbers of $\Gamma(R)$

In this section, we show how these locating sets and numbers are related to the graph structure $\Gamma(R)$. The following result characterizes those $\Gamma(R)$ which have finite and undefined locating numbers.

Theorem 3.1. *Let R be a commutative ring. Then*

(i) $\mathrm{loc}(\Gamma(R))$ *is finite if and only if R is finite.*
(ii) $\mathrm{loc}(\Gamma(R))$ *is undefined if and only if R is an integral domain.*

Proof. (i) This follows from Theorem 2.2 since the diameter of $\Gamma(R)$ is no more than 3 by [3, Theorem 2.3].
(ii) This follows from the fact that the locating number of $\Gamma(R)$ is undefined if and only if the vertex set of $\Gamma(R)$ is empty. □

Corollary 3.1. *Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where $\mathbb{F}_i$ is an integral domain for each $i = 1, 2, 3, \ldots, n$ (n is a fixed integer, $n \geq 2$). Then* $\mathrm{loc}(\Gamma(R)) = \infty$ *if and only if at least one of $\mathbb{F}_i$, $1 \leq i \leq n$, is infinite.*

Theorem 3.2. *Let R be a commutative ring. Then*

(i) $\mathrm{loc}(\Gamma(R)) = 1$ *if and only if $\Gamma(R)$ is a path.*
(ii) $\mathrm{loc}(\Gamma(R)) = 2$ *if $\Gamma(R)$ is a cycle.*
(iii) $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 1$ *if and only if $\Gamma(R)$ is a complete graph.*

(iv) $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$ *if* $\Gamma(R)$ *is a star graph (other than* $K_{1,1}$*) or a bipartite graph.*

Proof. These statements follow from Lemmas 2.1 and 2.3 and Theorem 2.1. □

Corollary 3.2. *If* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *are two finite fields, then* $\mathrm{loc}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$ *and* $\mathrm{loc}(\Gamma(\mathbb{F}_1 \times \mathbb{F}_2)) = |Z^*(\mathbb{F}_1 \times \mathbb{F}_2)| - 2 = |\mathbb{F}_1| + |\mathbb{F}_2| - 4$ *in all other cases.*

Proof. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a path. In all other cases, $\Gamma(\mathbb{F}_1 \times \mathbb{F}_2)$ is either a star graph or a complete bipartite graph with $|\mathbb{F}_1| + |\mathbb{F}_2| - 2$ vertices. Therefore, the result follows from Theorem 3.2. □

Corollary 3.3. *The graph* $\Gamma(\mathbb{Z}_n)$ *is a Hamiltonian graph if and only if* $\mathrm{loc}(\Gamma \times (\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 1$.

Proof. By [1, Corollary 2], we know that the graph $\Gamma(\mathbb{Z}_n)$ is a Hamiltonian graph if and only if $n = p^2$, where p is a prime larger than 3 and $\Gamma(\mathbb{Z}_n)$ is isomorphic to K_{p-1}. Hence the result. □

Theorem 3.3. *Let* R *be a finite commutative ring with* 1 *(not a domain) with odd characteristic. Suppose* $\Gamma(R)$ *is partitioned into* k *distinct distance similar classes. Then* $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - k$.

Proof. Let $x \in Z^*(R)$. Then $ux \sim x$ for every $u \in U(R)$. Since the characteristic of R is odd, $|U(R)| > 1$ and thus $|V_i| > 1$ for each set V_i in the partition of distance similar classes. Now, by Theorem 2.1, $\mathrm{loc}(\Gamma(R)) = |V(\Gamma(R))| - k = |Z^*(R)| - k$. □

Corollary 3.4. *Let* p *be a prime number.*

(i) *If* $n = 2p$, *then* $\mathrm{loc}(\Gamma(\mathbb{Z}_n)) = p - 2$.
(ii) *If* $n = p^2$ *and* $p > 2$, *then* $\mathrm{loc}(\Gamma(\mathbb{Z}_n)) = p - 2$.
(iii) *If* $n = p^k$ *and* $k \geq 3$, *then* $\mathrm{loc}(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - (k-1) = p^{k-1} - k$.

Proof. (i) The result is clear if $n = 2$ since $\Gamma(\mathbb{Z}_4)$ is a graph with a single vertex. If $n = 2p$ and $p > 2$, then every vertex of $\Gamma(\mathbb{Z}_n)$ is an element of the set $\{2, 2 \cdot 2, \ldots, 2 \cdot (p-1), p\}$. It is clear that $\Gamma(\mathbb{Z}_n)$ is a star graph. Hence by Theorem 3.2, $\mathrm{loc}(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 2 = p - 2$.

(ii) If $n = p^2$ and $p > 2$, then $\Gamma(\mathbb{Z}_n) \cong K_{p-1}$ and the result follows from Theorem 3.2.

(iii) If $n = p^k$ and $k > 3$ and $p > 2$, then $\Gamma(\mathbb{Z}_n)$ will be partitioned into distance similar equivalence classes $V_i = \{up^i \mid u \in U(\mathbb{Z}_n)\}$ for $i = 1, \ldots, k-1$. Thus, by Theorem 3.3, $\mathrm{loc}(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - (k-1) = (p^{k-1} - 1) - (k-1) = p^{k-1} - k$.

If $n = 2^k$ and $k > 3$, then again the graph will be partitioned into distance similar equivalence classes $V_i = \{u2^i \mid u \in U(\mathbb{Z}_n)\}$ for $i = 1, \ldots, k-1$. Note that $|V_i| > 1$ for $i = 1, \ldots, k-2$ and $V_{k-1} = \{2^{k-1}\}$. Thus, by Theorem 2.1, any minimal locating set W contains all but one element of each V_i for $i = 1, \ldots, k-2$. Also, $2^{k-1} \notin W$ since 2^{k-1} is the only possible $x \in V(\Gamma(\mathbb{Z}_n))$ with $c_W(x) = (1, \ldots, 1)$. Hence, $\mathrm{loc}(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - (k-1) = (2^{k-1} - 1) - (k-1) = 2^{k-1} - k$. □

Now, we give some examples.

Example 3.1. For $R = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, we have a single vertex zero-divisor graph for these rings. Therefore, by definition $\mathrm{loc}(\Gamma(R)) = 0$.

Example 3.2. For the rings $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^3)$ and $\mathbb{Z}_3[x]/(x^2)$, we have a path as a zero-divisor graph, hence the locating number of zero-divisor graphs for these rings is 1.

Example 3.3. The zero-divisor graphs for the rings $\mathbb{Z}_{25}$ or $\mathbb{Z}_{25}[x]/(x^2)$ are complete graphs. Therefore, the locating number of the zero-divisor graphs associated to these rings is $|Z^*(R)| - 1$. Here $|Z^*(R)| = 4$, giving 3 as the locating number. Also for the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$, the zero-divisor graph is a cycle on four vertices. Hence $\mathrm{loc}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$.

We now examine the relationship between the girth and locating numbers for $\Gamma(R)$.

Theorem 3.4. *Let R be a finite commutative ring with* $\mathrm{gr}(\Gamma(R)) = \infty$. *Then*

(i) $\mathrm{loc}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \mathrm{loc}(\mathbb{Z}_9) = \mathrm{loc}(\mathbb{Z}_3[X]/(X^3)) = 1$.
(ii) *if R is reduced and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then* $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$.
(iii) *if $R \cong \mathbb{Z}_8$ or $\mathbb{Z}_2[X]/(X^3)$ or $\mathbb{Z}_4[X]/(2X, X^2-2)$, then* $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$.
(iv) *if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^3)$, then* $\mathrm{loc}(\Gamma(R)) = 0$.
(v) *otherwise*, $\mathrm{loc}(\Gamma(R)) = 2$.

Proof. A complete list (up to isomorphism) of all finite commutative rings with infinite girth is given in [8, Theorem 1.7]. Clearly, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a path with two vertices. If R is reduced and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then [8, Theorem 1.7] implies $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Thus $\Gamma(R)$ is complete bipartite and the result follows from Theorem 3.2.

Again appealing to [8, Theorem 1.7], the only nonreduced rings with infinite girth are the rings listed in part (i) (which are paths of length 2), part (iii) (which are isomorphic to $K_{1,2}$), or $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ (which have graphs on 5 vertices with $\mathrm{loc}(\Gamma(R)) = 2$. □

Theorem 3.5. *Let R be a reduced Artinian commutative ring. If* $\mathrm{gr}(\Gamma(R)) = 4$, *then* $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$ *and $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where each $\mathbb{F}_i$ is a field with $|\mathbb{F}_i| \geq 3$.*

Proof. Since R is reduced and Artinian, $R \cong \mathbb{F}_1 \times \cdots \times \mathbb{F}_k$ for some integer $k \geq 2$ and fields $\mathbb{F}_1, \ldots, \mathbb{F}_k$. Since $\mathrm{gr}(\Gamma(R)) = 4$, $k = 2$ or else $\Gamma(R)$ contains a cycle on the vertices $(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, \ldots)$. Thus, $R \cong \mathbb{F}_1 \times \mathbb{F}_2$. If either $\mathbb{F}_1 \cong \mathbb{Z}_2$ or $\mathbb{F}_2 \cong \mathbb{Z}_2$, then $\mathrm{gr}(\Gamma(R)) = \infty$. □

Note that the above theorem cannot be extended to nonreduced rings as $\Gamma(\mathbb{Z}_{12})$ is a graph of girth 4 on 7 vertices but $\mathrm{loc}(\Gamma(\mathbb{Z}_{12})) = 3$.

We now proceed to study the relationship between diameter and locating number of $\Gamma(R)$. By [3, Theorem 2.3], $\mathrm{diam}(\Gamma(R)) \leq 3$. The next result gives an explicit description of the diameter and locating number of $\Gamma(R)$.

Theorem 3.6. *Let R be a commutative ring. Then*

(i) $\mathrm{diam}(\Gamma(R)) = 0$ *if and only if* $\mathrm{loc}(\Gamma(R)) = 0$.
(ii) $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if* $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 1$.

Proof. (i) Suppose $\mathrm{diam}(\Gamma(R)) = 0$. Then $\Gamma(R)$ is a single vertex graph. Therefore, $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Thus, $\mathrm{loc}(\Gamma(R)) = 0$. Conversely, if $\mathrm{loc}(\Gamma(R)) = 0$, then it is clear that $\mathrm{diam}(\Gamma(R)) = 0$.

(ii) Suppose $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 1$. Then by Theorem 3.2, $\Gamma(R)$ is a complete graph. Therefore $\mathrm{diam}(\Gamma(R)) = 1$. Conversely, suppose $\mathrm{diam}(\Gamma(R)) = 1$. This implies $\Gamma(R)$ is a complete graph. Thus, the result follows from Theorem 3.2.

□

Corollary 3.5. *If $R = \mathbb{F}_1 \times \mathbb{F}_2$ is the product of two fields, then* $\mathrm{loc}(\Gamma(R)) = 1$ *or* $|Z^*(R)| - 2$.

Proof. If $\mathbb{F}_1 \cong \mathbb{F}_2 \cong \mathbb{Z}_2$, then clearly $\mathrm{diam}(\Gamma(R)) = 1$. Therefore, by Theorem 3.6, $\mathrm{loc}(\Gamma(R)) = 1$, otherwise, $\mathrm{diam}(\Gamma(R)) = 2$, because in this case $\Gamma(R)$ is either a complete bipartite graph or a star graph. Hence by Theorem 3.2, $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$. □

Corollary 3.6. *If R is a reduced commutative ring which is not a field such that R is a subring of $\mathbb{F}_1 \times \mathbb{F}_2$, where each $\mathbb{F}_i$, $1 \leq i \leq 2$, is a field, then* $\mathrm{loc}(\Gamma(R)) = 1$ *or* $|Z^*(R)| - 2$.

Proof. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{diam}(\Gamma(R)) = 1$. Therefore, by Theorem 3.6, $\mathrm{loc}(\Gamma(R)) = 1$. Otherwise, $\mathbb{F}_1 \times (0)$ and $(0) \times \mathbb{F}_2$ contain the two partite sets of $\Gamma(R)$. So $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$. □

Now we determine the locating number of $\Gamma(R)$, when $\Gamma(R)$ is a regular graph.

Lemma 3.1. *Let R be a finite commutative ring. If $\Gamma(R)$ is regular, then either R is reduced or R is local.*

Proof. By [3, Corollary 2.7], if $\Gamma(R)$ is complete, then either R is local or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is reduced. So, assume $\Gamma(R)$ is regular but not complete and assume there is some $v \in V(\Gamma(R))$ such that $v^2 = 0$. Since $\Gamma(R)$ is regular but not complete, there is some $w \in V(\Gamma(R))$ such that w is not adjacent to v. Thus, $vw \neq 0$. Also $vw \neq w$ since $vw \in \operatorname{ann}(v)$ but $w \notin \operatorname{ann}(v)$. Clearly, $\operatorname{ann}(w) \subseteq \operatorname{ann}(vw)$ and $v \in \operatorname{ann}(vw)$ but $v \notin \operatorname{ann}(w)$. Since $\Gamma(R)$ is regular, $\deg(x) = n$ for some integer n for all $x \in Z^*(R)$ and therefore $|\operatorname{ann}(x)| = n+2$ if $x^2 = 0$ and $|\operatorname{ann}(x)| = n+1$ otherwise. Thus, $n+1 \leq |\operatorname{ann}(w)| < |\operatorname{ann}(vw)| = n+2$. Hence, $\operatorname{ann}(w) = \operatorname{ann}(vw) - \{v\}$. It follows that $vw \in \operatorname{ann}(vw)$ implies $vw \in \operatorname{ann}(w)$. (Consequently, this means $vw \neq v$.) So, if $N(w) = \{vw, t_1, t_2, \ldots, t_{n-1}\}$, then $\{v, w, t_1, t_2, \ldots, t_{n-1}\} = N(vw)$, implying $\deg(vw) > \deg(w)$. This is a contradiction. Hence, if $\Gamma(R)$ is regular but not complete, then R is reduced. □

Theorem 3.7. *Let R be a finite commutative ring. If $\Gamma(R)$ is a regular graph, then* $\operatorname{loc}(\Gamma(R)) = |Z^*(R)| - 1$ *or* $|Z^*(R)| - 2$.

Proof. If R is local, then $\Gamma(R)$ has a vertex adjacent to all others by [3, Corollary 2.7]. If $\Gamma(R)$ is also regular, then all vertices must be adjacent to all others since they all have the same degree. Hence, $\Gamma(R)$ is a complete graph and the result follows from Lemma 2.2.

If R is reduced, we have $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$ for some integer $k \geq 2$ and finite fields $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k$. All these fields must have the same number of elements or the vertices $(0, 1, 1, \ldots, 1), (1, 0, 1, \ldots, 1), \ldots (1, 1, 1, \ldots, 0)$ will have different degrees. Also, $k = 2$ or the vertices $(1, 0, \ldots, 0)$ and $(0, 1, \ldots, 1)$ will have different degrees. Thus, $R \cong \mathbb{F}_1 \times \mathbb{F}_1$ and therefore $\Gamma(R) \cong K_{n,n}$ for some integer n. If $\mathbb{F}_1 \cong \mathbb{Z}_2$, then $\Gamma(R)$ is complete and the result follows as above. Otherwise, the result now follows from Corollary 2.1. □

Now we study locating numbers in graphs determined by equivalence classes of zero-divisors. Spiroff and Wickham in [16] defined the graph of equivalence classes of zero-divisors of a ring R, denoted by $\Gamma_E(R)$, by taking the vertices to be the classes $[x] = \{y \in Z^*(R) \mid \operatorname{ann}(y) = \operatorname{ann}(x)\}$ for $x \in Z^*(R)$ and with each pair of distinct classes $[x]$, $[y]$ joined by an edge if and only if $[x][y] = [0]$. Mulay in [12], first defined $\Gamma_E(R)$ using different terminology and asserted several properties of $\Gamma_E(R)$. In [16, Proposition 1.4], it was proved that $\Gamma_E(R)$ is always connected, so we may investigate the locating number of $\Gamma_E(R)$.

Below, we show by examples that in general $\operatorname{loc}(\Gamma_E(R))$ is not equal to $\operatorname{loc}(\Gamma(R))$. In fact, later we shall prove that $\operatorname{loc}(\Gamma_E(R)) \leq \operatorname{loc}(\Gamma(R))$. Also, we prove that $\operatorname{loc}(\Gamma_E(R)) \neq |V(\Gamma_E(R))| - 1$ when $\Gamma_E(R)$ has at least three vertices, which is true for $\Gamma(R)$ when $\Gamma(R)$ is a complete graph with at least three vertices.

Example 3.4. If $R = \mathbb{Z}_{14}$, then $V(\Gamma(R)) = \{2, 4, 6, 7, 8, 10, 12\}$. Clearly, $\Gamma(R)$ is a star graph and hence by Corollary 2.1, $\operatorname{loc}(\Gamma(R)) = 5$. But $\Gamma_E(R)$ gets reduced to

a path with vertices $\{[7],[2]\}$ whose locating number is 1. Therefore, $\mathrm{loc}(\Gamma_E(R)) < \mathrm{loc}(\Gamma(R))$.

Similarly, if we take $R = \mathbb{Z}_{12}$, then we can find that $\mathrm{loc}(\Gamma(R)) = 3$, where as $\Gamma_E(R)$ gets reduced to a path and hence by Lemma 2.1, $\mathrm{loc}(\Gamma_E(R)) = 1$. Thus, in this case also $\mathrm{loc}(\Gamma_E(R)) < \mathrm{loc}(\Gamma(R))$.

Lemma 3.2. *Let R be a commutative ring. Let $x, y \in V(\Gamma(R))$ with $[x] \neq [y]$. Then $d(x,y) = d([x],[y])$. (That is, the distance from x to y in $\Gamma(R)$ is equal to the distance from $[x]$ to $[y]$ in $\Gamma_E(R)$).*

Proof. Let $P_1 : [x] = [v_0], [v_1], \ldots, [v_n] = [y]$ be a path of shortest length from $[x]$ to $[y]$ in $\Gamma_E(R)$ for some $v_1, \ldots, v_{n-1} \in Z^*(R)$. Then $P_2 : x, v_1, \ldots, v_{n-1}, y$ is a path from x to y in $\Gamma(R)$. Thus, $d(x,y) \leq d([x],[y])$.

Let $P_3 : x = t_0, t_1, \ldots, t_m = y$ be a path of shortest length from x to y in $\Gamma(R)$. Note that $[t_i] \neq [t_j]$ for $i < j$ for otherwise $t_i t_{j+1} = 0$ and $P_4 : x = t_0, t_1, \ldots, t_i, t_{j+1}, \ldots, t_m = y$ is a path from x to y of shorter length (using $P_4 : x = t_0, t_1, \ldots, t_{i-1}, t_j$ if $j = m$). Thus, $d([x],[y]) \leq d(x,y)$. Hence, $d(x,y) = d([x],[y])$. □

Theorem 3.8. *If R is a commutative ring, then* $\mathrm{loc}(\Gamma_E(R)) \leq \mathrm{loc}(\Gamma(R))$.

Proof. If $\Gamma(R)$ is a single vertex graph, then clearly $\Gamma_E(R)$ is a single vertex graph. Thus equality holds in this case.

So, suppose $\Gamma(R)$ has more than one vertex. Let W be a minimal locating set for $\Gamma(R)$. Define $W^* = \{[w] \mid w \in W\}$. Then $W^* \subseteq V(\Gamma_E(R))$ and $|W^*| \leq |W|$. We show that W^* is a locating set for $\Gamma_E(R)$.

If $W^* = V(\Gamma_E(R))$, then trivially W^* is a locating set for $\Gamma_E(R)$. Suppose $[x],[y] \notin W^*$ with $[x] \neq [y]$ for some $x, y \in Z^*(R)$. This would imply $x \notin W$ and $y \notin W$. If $c_{W^*}([x]) = c_{W^*}([y])$, then $d([x],[w]) = d([y],[w])$ for each $[w] \in W^*$. By the above lemma, this implies $d(x,w) = d(y,w)$ for each $w \in W$. Hence, $c_W(x) = c_W(y)$, contradicting that W is a locating set for $\Gamma(R)$. Hence, W^* is a locating set for $\Gamma_E(R)$. Thus, $\mathrm{loc}(\Gamma_E(R)) \leq |W^*| \leq |W| = \mathrm{loc}(\Gamma(R))$. □

Theorem 3.9. *Let R be a commutative ring such that $\Gamma_E(R)$ has at least three vertices.*

(i) *If $\Gamma_E(R)$ is a complete r-partite graph, then $r = 2$ and* $\mathrm{loc}(\Gamma_E(R)) = |V(\Gamma_E(R))| - 2$.
(ii) $\mathrm{loc}(\Gamma_E(R)) \neq n - 1$, *where* $n = |V(\Gamma_E(R))|$.

Proof. (i) By [16, Proposition 1.7], if $\Gamma_E(R)$ is a complete r-partite graph, then $r = 2$ and $\Gamma_E(R)$ is a star graph $K_{1,n}$ with $n \geq 2$. Therefore, by Corollary 2.1, $\mathrm{loc}(\Gamma_E(R)) = |V(\Gamma_E(R))| - 2$.

(ii) Suppose $\mathrm{loc}(\Gamma_E(R)) = n-1$ $(n \geq 3)$, where $n = |V(\Gamma_E(R))|$. Then by Lemma 2.2, $\Gamma_E(R)$ is a complete graph on n vertices which is a contradiction to [16, Proposition 1.5], therefore $\mathrm{loc}(\Gamma_E(R)) \neq n-1$. □

By [16, Theorem 1.8], $\Gamma_E(R)$ is never a cycle and every cycle graph has locating number 2, as seen in Lemma 2.3. However, $\Gamma_E(\mathbb{Z}_4[X]/(X^2)) \cong K_{1,3}$ and therefore $\mathrm{loc}(\mathbb{Z}_4[X]/(X^2)) = 2$.

4. Locating Sets and Numbers of an Ideal-Based Zero-Divisor Graph

In this section, we study the locating sets and numbers of graphs $\Gamma_I(R)$ and $\Gamma(R/I)$, and explore the relationship between locating numbers of $\Gamma_I(R)$ and $\Gamma(R/I)$. Further, we obtain a relationship between the diameter, girth and locating number of $\Gamma_I(R)$. We first have the following definition.

Definition 4.1. Let R be a commutative ring, and let I be an ideal of a ring R. The ideal-based zero-divisor graph [13], denoted $\Gamma_I(R)$, is the graph with vertices $\{x \in R-I \mid xy \in I \text{ for some } y \in R-I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

The following results can be seen in [13].

Lemma 4.1. (i) *If $I = (0)$, then $\Gamma_I(R) = \Gamma(R/I)$.*
(ii) *For a nonzero ideal I of R, $\Gamma_I(R) = \emptyset$ if and only if I is a prime ideal of R.*

Lemma 4.2. *Let I be an ideal of a ring R, and let $x, y \in R - I$.*

(i) *If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.*
(ii) *If x is adjacent to y in $\Gamma_I(R)$ and $x+I \neq y+I$, then $x+I$ is adjacent to $y+I$ in $\Gamma(R/I)$.*
(iii) *If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Remark 4.1. Note that Lemma 4.2 implies that the vertices x and $x + j$ are distance similar for any $x \in V(\Gamma_I(R))$ and $j \in I$. Thus, by Theorem 2.1, $\mathrm{loc}(\Gamma_I(R)) = \infty$ if $|I| = \infty$, even if $|R/I|$ is finite.

The following result characterizes those $\Gamma_I(R)$ which have finite and undefined locating numbers.

Theorem 4.1. *Let I be an ideal of a ring R. Then*

(i) $\mathrm{loc}(\Gamma_I(R))$ *is finite if and only if R is finite.*
(ii) $\mathrm{loc}(\Gamma_I(R))$ *is undefined if and only if I is a prime ideal of ring R.*

Proof. (i) This follows from Corollary 2.2 since the diameter of $\Gamma_I(R)$ is at most three by [13, Theorem 2.4].

(ii) This follows from Lemma 4.1 and the fact that the locating number is undefined if and only if the vertex set is empty. □

We now obtain the lower bounds for the locating numbers of the graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ analogous to the bound obtained in Theorem 3.3.

Theorem 4.2. *Let I be an ideal of a ring R. Then*

(a) $\mathrm{loc}(\Gamma_I(R)) \geq n_1 - k_1$, *where n_1 is the number of vertices in $\Gamma_I(R)$ and k_1 is the distance similar vertices (k_1 is the number of distance similar equivalence classes) in $\Gamma_I(R)$. Furthermore, if I is an nontrivial ideal, then* $\mathrm{loc}(\Gamma_I(R)) = n_1 - k_1$.

(b) $\mathrm{loc}(\Gamma(R/I)) \geq n_2 - k_2$, *where n_2 is the number of vertices and k_2 is the distance similar equivalence classes in $\Gamma(R/I)$.*

Proof. The proof here is similar to the proof of Theorem 2.1. If I is a nontrivial ideal, then, as in Remark 4.1, each distance similar equivalence class has more than one element. □

The following characterization for the graph $\Gamma_I(R)$ is analogous to Theorem 3.2.

Theorem 4.3. *Let I be an ideal of a ring R. Then*

(i) $\mathrm{loc}(\Gamma_I(R)) = 1$ *if and only if $\Gamma_I(R)$ is a path.*

(ii) $\mathrm{loc}(\Gamma_I(R)) = 2$ *if $\Gamma_I(R)$ is a cycle.*

(iii) $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 1$ *if and only if $\Gamma_I(R)$ is a complete graph.*

(iv) $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$ *if $\Gamma(R)$ is a star graph or a bipartite graph (other than $K_{1,1}$).*

Proof. The proof follows from Lemmas 2.1 and 2.3 and Theorem 2.1. □

When $I = \sqrt{I}$, we get an even stronger result.

Corollary 4.1. *Let I be a nontrivial ideal of a finite ring R with $I = \sqrt{I}$. Then* $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$ *if and only if $\Gamma_I(R)$ is a complete bipartite graph other than $K_{1,1}$.*

Proof. Suppose $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$. Then, by Theorem 4.3, $\Gamma_I(R)$ is not complete (and so is not isomorphic to $K_{1,1} \cong K_2$). Let $V_1, V_2, \ldots, V_k$ be the partition of $V(\Gamma_I(R))$ into distance similar equivalence classes. Since $\Gamma_I(R)$ is not complete, $k > 1$. Also, since I is nontrivial, $|V_i| > 1$ for each i by Remark 4.1. So, by Theorem 2.1, $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - k$ and thus $k = 2$.

Now, consider $\Gamma(R/I)$. If $x + I$ is not distance equivalent to $y + I$ in $\Gamma(R/I)$, then there is some $z + I \in V(\Gamma(R/I)) - \{x + I, y + I\}$ such that $d(z + I, x + I) \neq d(z + I, y + I)$. Then $z \in V(\Gamma_I(R)) - \{x, y\}$ with $d(z, x) \neq d(z, y)$. Thus, x is not distance equivalent to y in $\Gamma_I(R)$. Hence, $\Gamma(R/I)$ has two sets in its partition into

distance equivalent sets because $\Gamma_I(R)$ is partitioned into two distance equivalent sets. (Note, if $\Gamma(R/I)$ has only one distance equivalent set, then $\Gamma(R/I)$ is complete and thus $\Gamma_I(R)$ is complete or complete bipartite if $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). Let $V(\Gamma(R/I)) = W_1 \cup W_2$ be the partition of $\Gamma(R/I)$. Note that for $\Gamma(R/I)$ to be connected, some element $a+I \in W_1$ must be adjacent to some element $b+I \in W_2$. Since the elements of W_1 are distance equivalent to $a+I$ and the elements of W_2 are distance equivalent to $b+I$, every element of W_1 is adjacent to every element of W_2. If there is an edge between any two vertices of W_1 (or W_2), then the vertices of W_1 (or W_2) are adjacent to all other vertices of the graph. By [3, Corollary 2.7], this means R/I is local or $R/I \cong \mathbb{Z}_2 \times F$ for some finite field F. However, $I = \sqrt{I}$ means R/I is reduced and thus not local (since clearly R/I is not a field). Thus, in any possible case, $\Gamma(R/I)$ is a complete bipartite graph. Hence, $\Gamma_I(R)$ is complete bipartite by [5, Theorem 4.11].

The other implication is a consequence of Theorem 4.3. □

The above result cannot be extended to nonreduced rings. For example, if $R \cong \mathbb{Z}_{27} \times \mathbb{Z}_2$ and $I = (0) \times \mathbb{Z}_2$, then $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$ but $\Gamma_I(R)$ is not a complete bipartite graph. Also, if $R \cong \mathbb{Z}_9 \times \mathbb{Z}_2$ and $I = (0) \times \mathbb{Z}_2$, then $\Gamma(R/I) \cong K_{1,1}$ and $\Gamma_I(R)$ is a complete graph.

We now show that $\mathrm{loc}(\Gamma(R/I))$ cannot exceed the $\mathrm{loc}(\Gamma_I(R))$ and give some examples.

Theorem 4.4. *If I is an ideal of a ring R, then* $\mathrm{loc}(\Gamma(R/I)) \leq \mathrm{loc}(\Gamma_I(R))$.

Proof. The result is trivial if $I = (0)$. Therefore, suppose I is a nontrivial ideal of R. By Remark 4.1, if $x \in V(\Gamma_I(R))$, then all the elements of $x+I$ are distance similar. Thus, if k_1 is the number of distance similar equivalence classes of $\Gamma(R/I)$ and k_2 is the number of distance similar equivalence classes of $\Gamma_I(R)$, $k_1 \geq k_2$. Hence, $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - k_2 = |I| \cdot |V(\Gamma(R/I))| - k_2 \geq |I| \cdot |V(\Gamma(R/I))| - k_1 \geq |I| \cdot |V(\Gamma(R/I))| - |V(\Gamma(R/I))| \geq (|I| - 1) \cdot |V(\Gamma(R/I))| \geq |V(\Gamma(R/I))| \geq \mathrm{loc}(\Gamma(R/I))$. □

Example 4.1. Let $R = \mathbb{Z}_9 \times \mathbb{Z}_3$ and $I = (0) \times \mathbb{Z}_3$. Then

$\Gamma_I(R) = \{(3,0),(3,1),(3,2),(6,0),(6,1),(6,2)\}$. Clearly, $\Gamma_I(R)$ is the complete graph K_6 on six vertices. Therefore, $\mathrm{loc}(\Gamma_I(R)) = 5$.

Example 4.2. Let $R = \mathbb{Z}_6 \times \mathbb{Z}_2$ and $I = (0) \times \mathbb{Z}_2$. Then

$\Gamma_I(R) = \{(2,0),(2,1),(3,0),(3,1),(4,0),(4,1)\}$, and $\Gamma_I(R)$ is the complete bipartite graph $K_{4,2}$. Therefore, $\mathrm{loc}(\Gamma_I(R)) = 4$.

Example 4.3. As seen in [13], $\Gamma(R/I)$ is a subgraph of $\Gamma_I(R)$. However, in general, if H is a connected subgraph of a graph G, it need not be the case that $\mathrm{loc}(H) \leq \mathrm{loc}(G)$, even among the zero-divisor graphs of rings. For example, if $R \cong \mathbb{Z}_4[X]/(X^2+2X)$, then $\mathrm{loc}(\Gamma(R)) = 3$. As a local ring, this graph has a vertex

adjacent to all others. Thus, since $|V(\Gamma(R))| = 7$, $K_{1,6}$ is a subgraph of $\Gamma(R)$ and $\mathrm{loc}(K_{1,6}) = 5$.

We know by Theorem 4.1, that if I is a prime ideal of a ring R, then $\mathrm{loc}(\Gamma_I(R))$ is undefined. In the following result, if $I = P_1 \cap P_2$, where P_1 and P_2 are prime ideals of a ring R, then $\mathrm{loc}(\Gamma_I(R))$ is finite, in fact it is $|V(\Gamma_I(R))| - 2$.

Theorem 4.5. *Let I be a nonzero ideal of a ring R, then the following hold.*

(i) *If P_1 and P_2 are prime ideals of R and $I = P_1 \cap P_2 \neq 0$, then* $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$.
(ii) *Let I be an ideal of R for which $I = \sqrt{I}$. If* $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$, *then there exist prime ideals P_1 and P_2 such that $I = P_1 \cap P_2$.*

Proof. (i) Let $x, y \in R - I$ with $xy \in I$. Then $xy \in P_1$ and $xy \in P_2$. Since P_1 and P_2 are prime, we have $x \in P_1 - P_2$ and $y \in P_2 - P_1$, or $x \in P_2 - P_1$ and $y \in P_1 - P_2$. Also note that if $z^2 \in I = P_1 \cap P_2$, then $z \in P_1 \cap P_2$. Thus, $\Gamma_I(R)$ is a complete bipartite graph with $P_1 - P_2$ and $P_2 - P_1$ as the two partite sets. So $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$.

(ii) By Corollary 4.1, $\Gamma_I(R)$ is a complete bipartite graph. Therefore, $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$. Suppose the parts of $\Gamma_I(R)$ are X and Y. We set $P_1 = X \cup I$ and $P_2 = Y \cup I$. Then $P_1 \cap P_2 = (X \cup I) \cap (Y \cup I) = (X \cap Y) \cup I = I$. We now prove that P_1 and P_2 are prime ideals of R. To show this let $x, y \in P_1$.

Case 1. If $x, y \in I$, then $x - y \in I$ and so $x - y \in P_1$.

Case 2. If $x, y \in X$, then there is some $z \in Y$ such that $zx \in I$ and $zy \in I$. So $z(x - y) \in I$. If $x - y \in I$, then $x - y \in P_1$. If $x - y \notin I$, then $x - y \in X$, since X and Y are the partite sets of $\Gamma_I(R)$, which implies $x - y \in P_1$.

Case 3. If $x \in X$ and $y \in I$, then $x - y \notin I$. So there is $z \in Y$ such that $z(x - y) \in I$. This implies that $x - y \in X$, and so $x - y \in P_1$.

Thus, it follows that P_1 is a subgroup.

Now, let $r \in R$ and $x \in P_1$. If $x \in I$, then $rx \in I$ and so $rx \in P_1$. If $x \in X$, then there exists $z \in Y$ such that $zx \in I$. So $z(rx) \in I$. If $rx \in I$, then $rx \in P_1$ and if $rx \notin I$, then $rx \in X$ which implies $rx \in P_1$. Therefore, P_1 is an ideal of R.

We now show that P_1 is prime ideal. Let $xy \in P_1$ and suppose $x, y \notin P_1$. Since $P_1 = X \cup I$, this implies $xy \in I$ or $xy \in X$, and so in any case there exists $z \in Y$ such that $z(xy) \in I$. If $zy \in I$, then we have $y \in X$, which is a contradiction. Thus, $zy \notin I$ and so $zy \in X$. Therefore, $z^2y \in I$. Since $I = \sqrt{I}$, then $z^2 \notin I$. Thus, $z^2 \in Y$ and so $y \in X$, which is again a contradiction. Therefore, P_1 is a prime ideal of R. Similarly, it can be shown that P_2 is a prime ideal of R. Hence the result follows.

□

Remark 4.2. In Theorem 4.5(i), if we consider $R = \mathbb{Z}_8$ and $I = (4)$, then $\Gamma_I(R)$ is clearly a path and hence a complete bipartite graph. Therefore, $\mathrm{loc}(\Gamma_I(R)) = 1$,

but I cannot be written as the intersection of two prime ideals. So the converse of Theorem 4.5(i) is not valid in general.

In Sec. 3, we obtained results relating girth and locating number of $\Gamma(R)$. Now we give a relationship between girth and locating number of $\Gamma_I(R)$. We shall also consider the case when I is an intersection of prime ideals.

Theorem 4.6. *Let I be a nonzero ideal of R. If* $\mathrm{gr}(\Gamma_I(R)) = \infty$, *then* $\mathrm{loc}(\Gamma_I \times (R)) = 1$. *If* $\mathrm{gr}(\Gamma_I(R)) = 4$ *and* $\Gamma(R/I)$ *is a complete bipartite graph, then* $\mathrm{loc}(\Gamma_I \times (R)) = |V(\Gamma_I(R))| - 2$.

Proof. Suppose $\mathrm{gr}(\Gamma_I(R)) = \infty$. Then by [13, Theorem 5.5], $\Gamma_I(R)$ is a path on two vertices. Therefore, by Theorem 4.3, $\mathrm{loc}(\Gamma_I(R)) = 1$. Now, suppose $\mathrm{gr}(\Gamma_I(R)) = 4$ and $\Gamma(R/I)$ is a complete bipartite graph. So by [5, Theorem 4.11], $\Gamma_I(R)$ is also a complete bipartite graph. Thus, by Theorem 4.3, $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2$. □

Example 4.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = (0) \times (0) \times \mathbb{Z}_2$. Then $\Gamma(R/I) = \{(0,1,0)+I, (1,0,0)+I\}$ which is a complete bipartite graph on two vertices. Also $\Gamma_I(R) = \{(0,1,0), (1,0,0), (0,1,1), (1,0,1)\}$ is a complete bipartite graph on four vertices, therefore $\mathrm{loc}(\Gamma_I(R)) = 2$. It should be noted here that $\mathrm{gr}(\Gamma_I(R)) = 4$.

By [13, Theorem 2.4], $\mathrm{diam}(\Gamma_I(R)) \leq 3$. The following result gives a relationship between diameter and locating number of $(\Gamma_I(R))$.

Theorem 4.7. *Let I be an ideal of a ring R. Then*

(i) $\mathrm{diam}(\Gamma_I(R)) = 0$ *if and only if* $\mathrm{loc}(\Gamma_I(R)) = 0$.
(ii) $\mathrm{diam}(\Gamma_I(R)) = 1$ *if and only if* $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 1$.

Proof. Similar to the proof of Theorem 3.6. □

Example 4.5. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let I be an ideal $\mathbb{Z}_2 \times (0)$. Then $\mathrm{loc}(\Gamma_I(R)) = 1$ since $(\Gamma_I(R))$ is a path on two vertices. Note that $\mathrm{diam}(\Gamma_I(R)) = 0$ if and only if $\Gamma_I(R)$ is a single vertex if and only if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and $I = (0)$.

We conclude this paper by giving the locating numbers for all possible zero-divisor graphs on 14 or fewer vertices. Following the convention of [15], given a graph G, we define the zero-divisor class of G to be the set of all commutative local rings R with 1 such that $\Gamma(R) \cong G$. To keep this list from becoming too large, for zero-divisor classes with more than one ring, we will give only one representative of that class in the list. For the full list of rings in each class and for illustrations of these graphs, see [14].

(i) For $R \cong \mathbb{Z}_4$, $\mathrm{loc}(\Gamma(R)) = 0$.
(ii) For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3[X]/(X^2)$ or $\mathbb{Z}_6$ or $\mathbb{Z}_8$, $\mathrm{loc}(\Gamma(R)) = 1$.
(iii) For $R \cong \mathbb{Z}_2[X,Y]/(X,Y)^2$ or $\mathbb{Z}_2 \times \mathbb{F}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{loc}(\Gamma(R)) = 2$.

(iv) For $R \cong \mathbb{Z}_{25}$ or $\mathbb{Z}_2 \times \mathbb{Z}_5$ or $\mathbb{Z}_3 \times \mathbb{F}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_4[X]/(X^2 + 2X)$ or $\mathbb{Z}_4[X]/(X^2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{loc}(\Gamma(R)) = 3$.

(v) For $R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_{16}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $\mathrm{loc}(\Gamma(R)) = 4$.

(vi) For $R \cong \mathbb{Z}_{49}$ or $\mathbb{Z}_2 \times \mathbb{Z}_7$ or $\mathbb{F}_4 \times \mathbb{Z}_5$ or $\mathbb{Z}_8[X]/(2X, X^2)$ or $\mathbb{Z}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_2[X]/(X^2) \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 + 2)$, $\mathrm{loc}(\Gamma(R)) = 5$.

(vii) For $R \cong \mathbb{Z}_2[X, Y, Z]/(X, Y, Z)^2$ or $\mathbb{Z}_2 \times \mathbb{F}_8$ or $\mathbb{Z}_3 \times \mathbb{Z}_7$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{Z}_{27}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, $\mathrm{loc}(\Gamma(R)) = 6$.

(viii) For $R \cong \mathbb{Z}_3[X, Y]/(X, Y)^2$ or $\mathbb{Z}_2 \times \mathbb{F}_9$ or $\mathbb{Z}_3 \times \mathbb{F}_8$ or $\mathbb{Z}_7 \times \mathbb{F}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X, Y)^2$ or $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2, X)^2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_9$ or $\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$, $\mathrm{loc}(\Gamma(R)) = 7$.

(ix) For $R \cong \mathbb{Z}_3 \times \mathbb{F}_9$ or $\mathbb{F}_4 \times \mathbb{F}_8$ or $\mathbb{Z}_5 \times \mathbb{Z}_7$, $\mathrm{loc}(\Gamma(R)) = 8$.

(x) For $R \cong \mathbb{Z}_{121}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{11}$ or $\mathbb{F}_4 \times \mathbb{F}_9$ or $\mathbb{Z}_5 \times \mathbb{F}_8$, $\mathrm{loc}(\Gamma(R)) = 9$.

(xi) For $R \cong \mathbb{Z}_3 \times \mathbb{Z}_{11}$ or $\mathbb{Z}_5 \times \mathbb{Z}_9$ or $\mathbb{Z}_7 \times \mathbb{Z}_7$ or $\mathbb{Z}_3 \times \mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$, $\mathrm{loc}(\Gamma(R)) = 10$.

(xii) For $R \cong \mathbb{Z}_{169}$ or $\mathbb{Z}_{13}[x]/(x^2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_{13}$ or $\mathbb{F}_4 \times \mathbb{Z}_{11}$ or $\mathbb{Z}_7 \times \mathbb{F}_8$, $\mathrm{loc} \times (\Gamma(R)) = 11$.

(xiii) For $R \cong \mathbb{Z}_3 \times \mathbb{Z}_{13}$ or $\mathbb{Z}_5 \times \mathbb{Z}_{11}$ or $\mathbb{Z}_7 \times \mathbb{F}_9$ or $\mathbb{F}_8 \times \mathbb{F}_8$, $\mathrm{loc}(\Gamma(R)) = 12$.

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Journal of Algebra and Its Applications
Vol. 15, No. 1 (2016) 1650014 (22 pages)

DOI: 10.1142/S0219498816500146

On locating numbers and codes of zero divisor graphs associated with commutative rings

Rameez Raja* and S. Pirzada†

Department of Mathematics
University of Kashmir, India
**rameeznaqash@gmail.com*
†pirzadasd@kashmiruniversity.ac.in

Shane Redmond

Department of Mathematics and Statistics
Eastern Kentucky University, Richmond, KY 40475, USA
shane.redmond@eku.edu

Received 25 September 2014
Accepted 11 January 2015
Published 31 August 2015

Communicated by S. R. Lopez-Permouth

Let R be a commutative ring with identity and let $G(V, E)$ be a graph. The locating number of the graph $G(V, E)$ denoted by $\mathrm{loc}(G)$ is the cardinality of the minimal locating set $W \subseteq V(G)$. To get the $\mathrm{loc}(G)$, we assign locating codes to the vertices $V(G)\backslash W$ of G in such a way that every two vertices get different codes. In this paper, we consider the ratio of $\mathrm{loc}(G)$ to $|V(G)|$ and show that there is a finite connected graph G with $\mathrm{loc}(G)/|V(G)| = m/n$, where $m < n$ are positive integers. We examine two equivalence relations on the vertices of $\Gamma(R)$ and the relationship between locating sets and the cut vertices of $\Gamma(R)$. Further, we obtain bounds for the locating number in zero-divisor graphs of a commutative ring and discuss the relation between locating number, domination number, clique number and chromatic number of $\Gamma(R)$. We also investigate the locating number in $\Gamma(R)$ when R is a finite product of rings.

Keywords: Locating number; locating code; ring; zero-divisor; zero-divisor graph.

Mathematics Subject Classification: 13A99

1. Introduction

Beck [6] introduced the concept of zero-divisor graph of a commutative ring and defined $\Gamma_B(R)$ to be the graph whose vertices are the elements of R in which two vertices x and y are adjacent if and only if $xy = 0$. Beck was primarily interested in coloring of the graph $\Gamma_B(R)$ and his work was the beginning to study the combinatorial properties of the zero-divisors in a commutative ring R. A different method of associating the graph to a commutative ring was given by Anderson and Livingston

in [3]. They defined $\Gamma(R) = Z^*(R) = Z(R) - \{0\}$ to be the graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y adjacent if and only if $xy = 0$, and they called it as a zero-divisor graph of a commutative ring. Their work contains several fundamental results concerning $\Gamma(R)$. The zero-divisor graph of a commutative ring has also been studied in [1, 4, 7, 9, 10, 15, 14].

This article will continue the investigation that we started in [12] to compute or obtain bounds on locating numbers of $\Gamma(R)$. Section 2 reviews basic definitions and results concerning locating sets and locating numbers, as well as results applicable to graphs in general. The rest of the paper focuses on zero-divisor graphs of commutative rings. Section 3 is an aid in this task by comparing two equivalence relations on the sets of vertices of $\Gamma(R)$. Section 4 relates locating sets to cut vertices of $\Gamma(R)$. Section 5 discusses the relation between chromatic number, clique number, and locating number of $\Gamma(R)$. Section 6 gives some characteristics of $\mathrm{loc}(\Gamma(R))$, when R is a finite product of rings.

Throughout, R is a commutative ring (with unity $1 \neq 0$), unless otherwise stated, $\mathrm{nil}(R)$ its set of nilpotent elements, $U(R)$ its set of units, $Z(R)$ its set of zero-divisors and $Z^*(R)$ its set of nonzero zero-divisors. We say R is reduced if $\mathrm{nil}(R) = 0$. We adopt the approach used by Anderson and Livingston in [3] and consider only the nonzero zero-divisors of R as vertices of the graph. We will denote the ring of integers by $\mathbb{Z}$, the ring of integers modulo n by $\mathbb{Z}_n$ and a finite field on q elements by $\mathbb{F}_q$ respectively. For basic definitions from graph theory we refer to [8, 11, 16], and for commutative ring theory we refer to [5].

2. Definitions and General Results on Locating Numbers

When referring to a graph, we will mean a simple graph $G(V, E)$, which consists of a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of $G(V, E)$ called edges. All graphs G considered here are connected, that is there is a path between every two distinct vertices of G. The distance from a vertex v to u, denoted by $d(v, u)$, is the length of the shortest path from v to u ($d(v, v) = 0$ and $d(v, u) = \infty$, if there is no such path). The diameter of G is $\mathrm{diam}(G) = \sup\{d(v, u) \mid v, u \in V(G)\}$ and $N(v) = \{u \in V(G) \mid d(u, v) = 1\}$.

As introduced independently in [12] for a graph G, an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of G, and a vertex v of G, the locating code (or simply the code) of v with respect to W is the k-vector $c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is a locating set for G if distinct vertices have distinct codes. A locating set containing a minimum number of vertices is a minimal locating set for G. The locating number, denoted by $\mathrm{loc}(G)$, is the number of vertices in the minimal locating set for G. The locating number of a single vertex graph is 0, since for $w \in V(G)$, $d(w, w) = 0$. The locating number for an empty graph is not defined.

Theorems 2.1 through 2.4 can be seen in [12]. Recall that a graph G is said to be a complete if there is an edge between every pair of distinct vertices. A complete

graph with n vertices is denoted by K_n. A graph G is said to be bipartite if its vertex set can be partitioned into two sets $V_1(G)$ and $V_2(G)$ such that every edge of G has one end in $V_1(G)$ and another in $V_2(G)$. A complete bipartite graph is one whose each vertex of one partite set is joined to every vertex of another partite set. We denote complete bipartite graph with partite sets of order m and n by $K_{m,n}$. A complete bipartite graph of the from $K_{1,n}$ is called a star graph.

Theorem 2.1. *Let G be a connected graph. Suppose G is partitioned into k distinct distance similar classes $V_1, V_2, \ldots, V_k$ (that is, $x, y \in V_i$ if and only if $d(x,a) = d(y,a)$ for all $a \in V(G) - \{x,y\}$).*

(i) *Any locating set W for G contains all but at most one vertex from each V_i.*
(ii) *Each V_i induces a complete subgraph or a subgraph with no edges.*
(iii) $\mathrm{loc}(G) \geq |V(G)| - k$.
(iv) *There exists a minimal locating set W for G such that if $|V_i| > 1$, at most $|V_i| - 1$ vertices of V_i are elements of W.*
(v) *If m is the number of distance similar classes that consist of a single vertex, then $|V(G)| - k \leq \mathrm{loc}(G) \leq |V(G)| - k + m$.*

Theorem 2.2. *Let R be a commutative ring. Then*

(i) $\mathrm{loc}(\Gamma(R))$ *is finite if and only if R is finite.*
(ii) $\mathrm{loc}(\Gamma(R))$ *is undefined if and only if R is an integral domain.*

Theorem 2.3. *Let R be a commutative ring. Then*

(i) $\mathrm{loc}(\Gamma(R)) = 1$ *if and only if $\Gamma(R)$ is a path.*
(ii) $\mathrm{loc}(\Gamma(R)) = 2$ *if $\Gamma(R)$ is a cycle.*
(iii) $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 1$ *if and only if $\Gamma(R)$ is a complete graph.*
(iv) $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$ *if $\Gamma(R)$ is a star graph (other than $K_{1,1}$) or a bipartite graph.*

Theorem 2.4. *Let R be a commutative ring. Then*

(i) $\mathrm{diam}(\Gamma(R)) = 0$ *if and only if* $\mathrm{loc}(\Gamma(R)) = 0$.
(ii) $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if* $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 1$.

It is interesting to consider the ratio of $\mathrm{loc}(G)$ to $|V(G)|$. One can think of this as a measure of distance symmetry in G. A ratio that was close to 1 indicates a proportionally larger minimal locating set, which would tend to indicate more distance similar vertices. A ratio that was close to 0 indicates a proportionally smaller minimal locating set, which would tend to indicate fewer distance similar vertices. For general graphs, this ratio can be equal to any rational number in the interval $(0, 1)$.

Theorem 2.5. *Let m and n be positive integers with $m < n$. Then there is a finite connected graph G with* $\mathrm{loc}(G)/|V(G)| = m/n$.

Proof. We first handle several special cases.

Case 1: If $m = 1$, let G be the path P_n.

Case 2: If $m = 2$, let G be the cycle C_n

Case 3: If $m = n - 1$, let G be the complete graph K_n.

Case 4: If $m = n - 2$, let G be the star graph $K_{1,n-1}$.

Case 5: If $3 \leq m \leq n - 3$, then let H be the star graph $K_{1,m+1}$ with vertex v_0 adjacent to the vertices $v_1, v_2, \ldots, v_{m+1}$. Let K be a path on vertices $x_1, x_2, \ldots, x_{n-(m+2)}$ disjoint from $V(H)$. Define G to be the graph with $V(G) = V(H) \cup V(K)$ where H and K are induced subgraphs of G and with one additional edge from v_1 to x_1. Then G is a connected graph on n vertices. Let $W = \{v_1, v_2, \ldots, v_m\}$ and we will show W is a minimal locating set for G. Since $v_i \sim v_j$ for any integers $2 \leq i < j \leq m+1$, $m-1$ vertices from $U = \{v_2, v_3, \ldots, v_{m+1}\}$ must be in any locating set. If $z \in U$, then $T = U - \{z\}$ is not a locating set for G since $c_T(z) = c_T(v_1)$. Thus $\mathrm{loc}(G) > m - 1$. Now, we can see that W is a minimal locating set for G since $c_W(v_0) = (1, 1, \ldots, 1), c_W(v_{m+1}) = (2, 2, \ldots, 2)$, and $c_W(x_i) = (i + 2, i + 2, \ldots, i + 2, i)$ for $i = 1, 2, \ldots, n - (m + 2)$. □

The remainder of this paper will focus on zero-divisor graphs of commutative rings. Note here that the proof of Theorem 2.5 cannot be applied to zero-divisor graphs since the graphs in the proof may have arbitrarily large diameter, while the zero-divisor graphs of commutative rings have diameter at most 3 (as seen in [3, Theorem 2.3]). However, we can find commutative rings R with 1 such that $\mathrm{loc}(\Gamma(R))/|V(\Gamma(R))|$ is arbitrarily close to 1 — for example, take $R \cong \mathbb{Z}_{p^2}$ for a sufficiently large prime number p so that $\mathrm{loc}(\Gamma(R))/|V(\Gamma(R))| = (p-2)/(p-1)$. Also, we can find commutative rings R with 1 such that $\mathrm{loc}(\Gamma(R))/|V(\Gamma(R))|$ is arbitrarily close to 0 — for example, take $R \cong \prod_{i=1}^{n} \mathbb{Z}_2$ for a sufficiently large integer n so that $\mathrm{loc}(\Gamma(R))/|V(\Gamma(R))| \leq n/(2^n - 2)$ (see Sec. 6 for more on $\mathrm{loc}(\prod_{i=1}^{n} \mathbb{Z}_2)$). It is an open question if there are any rational numbers $q \in (0, 1)$ such that $\mathrm{loc}(\Gamma(R))/|V(\Gamma(R))| \neq q$ for any commutative ring R with 1.

3. Equivalence Relations on the Vertices of $\Gamma(R)$

Previous articles have defined two equivalence relations on the vertices of $\Gamma(R)$. The first is from [2]: (A) $x \sim y$ if for all $z \in v(\Gamma(R)) - \{x, y\}$, $d(x, z) = d(y, z)$. The second is from [10] which says (B) x is equivalent to y if $\mathrm{ann}(x) = \mathrm{ann}(y)$. In Lemma 3.1 of [2], it is noted these relations (A) and (B) are equivalent if the ring R is reduced. The following results show these definitions (A) and (B) are equivalent for all rings whose zero-divisor graphs have three or more vertices.

Lemma 3.1. *If R is a finite commutative ring with* 1 *and $x, y \in V(\Gamma(R))$ with $x \sim y$ as in relation* (A), *then either* (1) $x^2 = y^2 = 0$, *or* (2) *both $x^2 \neq 0$ and $y^2 \neq 0$.*

Proof. Assume $x \sim y$, $x^2 = 0$, and $y^2 \neq 0$. We provide a contradiction in every possible case.

Case 1: Suppose $xy = 0$. Then $x + y \notin \{x, y, 0\}$ and $x(x + y) = 0$ while $y(x + y) = y^2 \neq 0$. Hence, $d(x, x + y) = 1$ but $d(y, x + y) > 1$. This is a contradiction since $x \sim y$.

Case 2: Suppose $xy \neq 0$. Note that $xy \neq y$ since $(xy)^2 = 0$. Thus, since $x(xy) = 0$ and $x \sim y$, $y(xy) = 0$. Since $\Gamma(R)$ is connected and $x \sim y$, there is some $w \in V(\Gamma(R)) - \{x, y\}$ such that $d(x, w) = d(y, w) = 1$. That is, $xw = yw = 0$. Then $x(w + x) = 0$ and $y(w + x) = xy \neq 0$. Note that $w + x \neq x$ since $w \neq 0$. Also, $w + x \neq 0$ since otherwise $xy = (-w)y = -(wy) = 0$. If $w + x = y$, then we have a contradiction since $w^2 = w^2 + wx = w(w + x) = wy = 0$ and therefore $y^2 = (w + x)^2 = x^2 + 2xw + w^2 = 0$. Hence, $w + x \in V(\Gamma(R)) - \{x, y\}$ with $d(x, w + x) \neq d(y, w + x)$ a contradiction. □

Corollary 3.2. *Let R be a finite commutative ring with* 1 *and with* $|V(\Gamma(R))| \geq 3$. *If $x, y \in V(\Gamma(R))$ with $x \sim y$ as in relation* (A), *then* $\mathrm{ann}(x) = \mathrm{ann}(y)$. *Furthermore, $xy = 0$ if and only if $x^2 = y^2 = 0$.*

Proof. Suppose $x^2 \neq 0$. Then $y^2 \neq 0$ by the previous lemma. Assume $xy = 0$. Since $|V(\Gamma(R))| \geq 3$, there must be (without loss of generality) some $t \in N(x) - \{y\}$. (Note this implies $t \neq x$). Since $x \sim y$, $t \in N(y)$. Then $y(x+t) = xy+yt = 0+0 = 0$. Thus, since $x \sim y$, either $x + t \in N(x)$ or $x+t = x$ or $x+t = y$ or $x+t = 0$. However, $x + t \notin N(x)$ since $x(x + t) = x^2 \neq 0$. Also, $x + t \neq x$ since $t \neq 0$. Additionally, $x + t \neq y$ since $y \in N(x)$. Lastly, $x + t \neq 0$ since otherwise, $0 = xt = x(-x) = -x^2$. Thus, we have a contradiction. Hence, $xy \neq 0$. Now, it is clear that $x \sim y$ implies $N(x) = N(y)$ and $\mathrm{ann}(x) = N(x) \cup \{0\} = N(y) \cup \{0\} = \mathrm{ann}(y)$.

Now, suppose $x^2 = 0$. Then $y^2 = 0$ by the previous lemma. Assume $\mathrm{ann}(x) \neq \mathrm{ann}(y)$. Then, without loss of generality, there is some $z \in \mathrm{ann}(x)$ with $z \notin \mathrm{ann}(y)$. But $x \sim y$ implies $z \notin V(\Gamma(R)) - \{x, y\}$ and clearly $z \neq 0$. Thus, $z = x$ (since $y \in \mathrm{ann}(y)$), and therefore $xy \neq 0$. Hence, $\mathrm{ann}(x) - \{x\} \subseteq \mathrm{ann}(y)$. Similarly, $\mathrm{ann}(y) - \{y\} \subseteq \mathrm{ann}(x)$. Thus, $\mathrm{ann}(x) - \{x, y\} = \mathrm{ann}(y) - \{x, y\}$.

Now $x + xy \in \mathrm{ann}(x)$. Thus, either $x + xy \in \mathrm{ann}(y)$ or $x + xy = x$ or $x + xy = y$. However, $x + xy \notin \mathrm{ann}(y)$ since $y(x + xy) = xy \neq 0$. Clearly $x + xy \neq x$ since $xy \neq 0$. Lastly, $x + xy \neq y$ because $y \in \mathrm{ann}(y)$. Thus, we have a contradiction. Hence, $xy = 0$. Therefore, since $\{x, y\} \subseteq \mathrm{ann}(x) \cap \mathrm{ann}(y)$, $\mathrm{ann}(x) = N(x) \cup \{0, x\} = N(y) \cup \{0, y\} = \mathrm{ann}(y)$. □

Also, Sec. 3 of [10] shows that the only commutative rings R with 1 where no vertices are distance similar are those which satisfy $|V(\Gamma(R))| = 1$ or $R \cong \prod_{i=1}^{n} \mathbb{Z}_2$.

Proposition 3.3. *Let R be a finite commutative ring and $\Gamma(R)$ be its associated zero-divisor graph. The following conditions hold.*

(i) *For any two vertices $x, y \in Z^*(R)$ of maximum degree in $\Gamma(R)$ either $xy = 0$ or* $\mathrm{ann}(x) = \mathrm{ann}(y)$.

(ii) *If* $x, y \in Z^*(R)$ *and* xy *is a vertex of minimum degree in* $\Gamma(R)$, *then* $\mathrm{ann}(x) = \mathrm{ann}(y)$.

Proof. (i) Let $x, y \in Z^*(R)$. If $x, y \in W$, then $xy = 0$. If $xy \neq 0$, so x, y are not the part of the clique. Clearly $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subseteq \mathrm{ann}(xy)$ which implies $\mathrm{ann}(x) \subseteq \mathrm{ann}(xy)$. Since x, y are vertices with maximum degree, therefore, $\deg(x) \geq \deg(xy)$ which implies $\mathrm{ann}(x) \supseteq \mathrm{ann}(xy)$. Thus, $\mathrm{ann}(x) = \mathrm{ann}(xy)$. Similarly it can be shown that $\mathrm{ann}(y) = \mathrm{ann}(xy)$ which gives $\mathrm{ann}(x) = \mathrm{ann}(y)$.

(ii) Clearly $\mathrm{ann}(x) \cup \mathrm{ann}(y) \subseteq \mathrm{ann}(xy)$ which implies $\mathrm{ann}(x) \subseteq \mathrm{ann}(xy)$. Since $\deg(x) \geq \deg(xy)$, therefore, $\mathrm{ann}(x) \supseteq \mathrm{ann}(xy)$. Thus, x is also a vertex of minimum degree and $\mathrm{ann}(x) = \mathrm{ann}(xy)$. Similarly y is a vertex of minimum degree and $\mathrm{ann}(y) = \mathrm{ann}(xy)$ which gives $\mathrm{ann}(x) = \mathrm{ann}(y)$. □

These results can be applied to ideal-based zero-divisor graphs, first defined in [13]. Given a commutative ring R with 1 and an ideal I of R, the ideal-based zero-divisor graph, denoted $\Gamma_I(R)$, is the graph with vertex set $\{x \in R - I \,|\, xy \in I for\ some\ y \in R - I\}$ and with distinct vertices x and y adjacent whenever $xy \in I$.

Lemma 3.4. *Let* R *be a commutative ring with* 1, *and let* I *be a nontrivial ideal of* R. *Suppose* $|\Gamma(R/I)| \geq 3$. *Let* $x, y \in V(\Gamma_I(R))$ *with* $x + I \neq y + I$ *in* R/I. *Then* $x + I \sim y + I$ *in* $\Gamma(R/I)$ *if and only if* $x \sim y$ *in* $\Gamma_I(R)$ *as in relation* (A).

Proof. Suppose $x+I \sim y+I$. Let $z \in V(\Gamma_I(R)) - \{x, y\}$. If $z+I \neq x+I$ and $z+I \neq y+I$, then $d(x+I, z+I) = d(y+I, z+I)$ in $\Gamma(R/I)$, which implies $d(x, z) = d(y, z)$ in $\Gamma_I(R)$. If, say, $z + I = x + I$, then $d(x, z) = 1$ if $x^2 \in I$ and $d(x, z) = 2$ if $x^2 \notin I$. If $x^2 \in I$, then $(x + I)^2 = 0 + I$ in R/I. Hence, by Corollary 3.2, $(y + I)^2 = 0 + I$ and $(x + I)(y + I) = 0 + I$. Therefore, $(z + I)(y + I) = (x + I)(y + I) = 0 + I$ in R/I implies $yz \in I$. Thus, $d(y, z) = 1$. However, if $x + I = z + I$ and $x^2 \notin I$, let $x - t - z$ be a path of shortest length in $\Gamma_I(R)$. By our assumptions, $t + I \neq x + I$. If $t + I = y + I$, then $(x + I)(y + I) = 0 + I$ in R/I. However, this contradicts the Corollary 3.2, since $(x + I)^2 \neq 0 + I$. Hence, $t + I \neq y + I$. Also, if $yz \in I$, then $0 + I = (y + I)(z + I) = (y + I)(x + I)$, creating a similar contradiction. Now $x + I$ adjacent to $t + I$ in $\Gamma(R/I)$ implies $y + I$ is adjacent to $t + I$, yielding that $y - t - z$ is a path in $\Gamma_I(R)$. Hence, $d(y, z) = 2$.

Suppose $x \sim y$ in $\Gamma_I(R)$. Let $z + I \in V(\Gamma(R/I)) - \{x + I, y + I\}$. Suppose $x + I$ is adjacent to $z + I$ in $\Gamma(R/I)$, then x is adjacent to z in $\Gamma_I(R)$. Thus, $x \sim y$ implies y is also adjacent to z, giving $d(y + I, z + I) = 1$ too. In other words, for $z + I \in V(\Gamma(R/I)) - \{x + I, y + I\}$, $z + I$ is adjacent to $y + I$ if and only if $z + I$ is adjacent to $x + I$. It now follows that $x + I \sim y + I$. □

The next result allows us to determine $\mathrm{loc}(\Gamma_I(R))$ from properties of I and $\Gamma(R/I)$.

Theorem 3.5. *Let R be a commutative ring with 1. Let I be a nontrivial ideal of R.*

(i) *If $|\Gamma(R/I)| \neq 2$, let $V_1, V_2, \ldots, V_n$ be a partition of $V(\Gamma(R/I))$ into distance similar equivalence classes, then $\mathrm{loc}(\Gamma_I(R)) = \sum_{i=1}^{k}(|V_i||I| - 1)$.*
(ii) *If $|\Gamma(R/I)| = 2$, then $\mathrm{loc}(\Gamma_I(R)) = 2|I| - 2$ if $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathrm{loc}(\Gamma_I(R)) = 2|I| - 1$ otherwise.*

Proof. (i) If $|\Gamma(R/I)| = 1$, then $k = 1$ and $\Gamma_I(R)$ is a complete graph on $|I|$ vertices. Thus the result follows from Theorem 2.3.

So, suppose $|\Gamma(R/I)| \geq 3$. Then, Lemma 3.4 implies the number of distance equivalence classes of $\Gamma_I(R)$ equals the number of distance equivalence classes of $\Gamma(R/I)$. The ith class for $V(\Gamma_I(R))$ contains $|V_i||I|$ vertices, all of which but one must be in a minimal locating set. Hence the result follows.

(ii) If $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma_I(R) \cong K_{|I|,|I|}$, giving $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 2 = 2|I| - 2$ as a complete bipartite graph. If $R/I \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma_I(R)$ is a complete graph and thus $\mathrm{loc}(\Gamma_I(R)) = |V(\Gamma_I(R))| - 1 = 2|I| - 1$. □

4. Locating Codes and Cut Vertices of $\Gamma(R)$

In this section, we study the locating codes, locating sets and the cut vertices of $\Gamma(R)$.

Theorem 4.1. *Let R be a finite commutative ring with unity 1 such that $|Z^*(R)| \geq 3$ and let W be a minimal locating set for $\Gamma(R)$. If $\Gamma(R)$ has a cut vertex say x, then either*

(i) *$c_W(x) = (1, 1, \ldots, 1, d(x, v_1), d(x, v_2), \ldots, d(x, v_{l-m+1}))$, where $l = |W|$ and m is the number of degree one vertices incident on x (that is, the locating code for x has at least $m - 1$ coordinates equal to 1), or*
(ii) *$\mathrm{loc}(\Gamma(R)) = 3$.*

Proof. We partition the vertex set $Z^*(R)$ of $\Gamma(R)$ into the k distinct distance similar classes $Z_1^*(R)$, $Z_2^*(R), \ldots, Z_k^*(R)$ (that is, $x, y \in Z_i^*(R)$ if and only if $d(x, a) = d(a, y)$ for all $a \in Z^*(R) - \{x, y\}$).

If x as adjacent to any degree 1 vertices, then clearly these vertices generate one of the distance similar equivalence classes, say $Z_i^*(R)$. Therefore by Theorem 2.1, at least $|Z_i^*(R)| - 1$ vertices are the elements of W.

Thus $c_W(x) = (1, 1, \ldots, 1, d(x, v_1), d(x, v_2), \ldots, d(x, v_{l-m}))$.

By [15, Theorem 3], if $\Gamma(R)$ has a cut vertex but no degree 1 vertices, then R is isomorphic to one of the following seven rings

$\mathbb{Z}_4[X, Y]/(X^2, Y^2, XY - 2, 2X, 2Y)$, $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$, $\mathbb{Z}_4[X]/(X^2)$,
$\mathbb{Z}_4[X]/(X^2 + 2X)$, $\mathbb{Z}_8[X]/(2X, X^2 + 4)$, $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY)$,
$\mathbb{Z}_4[X, Y]/(X^2, Y^2 - XY, XY - 2, 2X, 2Y)$.

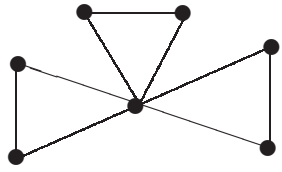

Fig. 1.

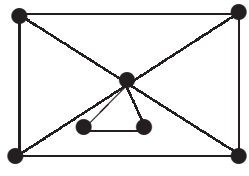

Fig. 2.

It can be easily verified from the graphs (Figs. 1 and 2) associated to these rings that the locating number is 3. □

Theorem 4.2. *Let R be a finite commutative ring with unity* 1 *and $C(\Gamma(R))$ be the set of all cut vertices of $\Gamma(R)$. Then* $\mathrm{loc}(\Gamma'(R)) = |C(\Gamma(R))| - 1$, *where $\Gamma'(R)$ is a subgraph of $\Gamma(R)$ induced by the cut vertices of $\Gamma(R)$.*

Proof. This can be proved by using Corollary 6 of [15] and Theorem 2.3. □

Example 4.3. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, the subgraph $\Gamma'(R)$ induced by three cut vertices is a complete graph. Therefore, by Theorem 2.3, $\mathrm{loc}(\Gamma'(R)) = 2$.

Remark 4.4. The relationship seen in Theorem 4.2 does not hold in general. If we take $R = \mathbb{Z}_2[X, Y]/(X^3, XY, Y^2)$, then $|Z^*(R)| = 7$ (see Fig. 3) and $\Gamma'(R)$ is a complete subgraph induced by four vertices but none of the vertices is a cut vertex of $\Gamma(R)$. Furthermore, the next result shows that cut vertices need not be part of a minimal locating set.

Theorem 4.5. *Let R be a finite commutative ring with* 1 *such that $\Gamma(R)$ has three or more vertices. Then there is a minimal locating set W for $\Gamma(R)$ which contains no cut vertices.*

Proof. By [15, Theorem 3], either $\Gamma(R)$ has a vertex of degree one or R is isomorphic to one of the following seven rings: $\mathbb{Z}_4[X, Y]/(X^2, Y^2, XY - 2, 2X, 2Y), \mathbb{Z}_2[X, Y]/(X^2, Y^2)$, $\mathbb{Z}_4[X]/(X^2)$, $\mathbb{Z}_4[X]/(X^2 + 2X), \mathbb{Z}_8[X]/(2X, X^2 + 4)$,

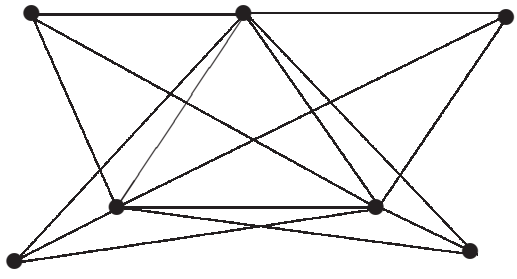

Fig. 3.

$\mathbb{Z}_2[X,Y]/(X^2, Y^2 - XY)$, $\mathbb{Z}_4[X,Y]/(X^2, Y^2 - XY, XY - 2, 2X, 2Y)$. It is a simple exercise to show each of these seven rings has a minimal locating set which does not contain any cut vertices (see Figs. 1 and 2).

Thus, suppose $\Gamma(R)$ has a cut vertex x adjacent to a vertex v with $\deg(v) = 1$. Let $V_1, V_2, \ldots, V_n$ be a partition of the vertex set of $\Gamma(R)$ into classes of distance similar vertices. Without loss of generality, say $x \in V_1$. Then, since x is a cut vertex and, therefore cannot be distance equivalent to any other vertex, $V_1 = \{x\}$. Again without loss of generality, $v \in V_2$, which in fact implies V_2 consists entirely of all the vertices of $\Gamma(R)$ of degree 1 adjacent to x. By Theorem 2.1, we can find a minimal locating set W^* for $\Gamma(R)$ such that W^* contains $|V_i| - 1$ vertices of V_i whenever $|V_i| \geq 2$.

Case 1: $V_2 = \{v\}$. First, we show x and v cannot both be elements of W^*. It is easy to verify this result is true if $\Gamma(R)$ is a star-graph (see Theorem 2.3), so assume $\Gamma(R)$ is not a star-graph. If $x \in W^*$ and $v \in W^*$, then $d(t,v) = d(t,x) + 1$ for all $t \in V(\Gamma(R)) - \{v, x\}$. Thus, coordinate x of $C_{W^*}(t)$ is always one less than coordinate v of $C_{W^*}(t)$. Hence, $W^* - \{v\}$ would be a minimal locating set for $\Gamma(R)$ (noting that $C_{W^*-\{v\}}(v)$ is unique since it is the only vertex with 1 in coordinate x and entries greater than 1 in all other coordinates). This would be a contradiction.

So, if $x \notin W^*$, then there is nothing to prove. If $x \in W^*$, then we will show $W = (W^* - \{x\}) \cup \{v\}$ is a minimal locating set. For $v_1, v_2 \in V(\Gamma(R)) - \{x, v\}$, coordinate v of $C_W(v_i)$ is one more than coordinate x of $C_{W^*}(v_i)$. Thus $C_{W^*}(v_1) \neq C_{W^*}(v_2)$ implies $C_W(v_1) \neq C_W(v_2)$. Furthermore, $C_W(x)$ is unique since it is the only vertex with 1 in coordinate v. Hence, W will be a minimal locating set for $\Gamma(R)$.

Case 2: $|V_2| \geq 2$. Again, if $x \notin W^*$, there is nothing to prove. So, suppose $x \in W^*$. Then there is some vertex $z \in V_2 - W^*$. Let $W = (W^* - \{x\}) \cup \{z\}$. Then, by an argument analogous to that of Case 1, W will be a minimal locating set for $\Gamma(R)$.

For each additional cut vertex y of $\Gamma(R)$, we may continue to refine W as in Cases 1 and 2 until we obtain a locating set not containing any cut vertices. □

5. Locating, Clique and Chromatic Numbers of $\Gamma(R)$

As discussed in [12], for a zero-divisor graph of order 2, we have $1 \leq \mathrm{loc}(\Gamma(R)) \leq |Z^*(R)| - 1$. Furthermore, we know exactly the graphs for which the extreme value are attainable. If in addition to the order of $\Gamma(R)$, we also know the diameter and the maximum degree of $\Gamma(R)$, the bounds for the locating number of $\Gamma(R)$ can be improved. We have the following observation.

Lemma 5.1. *Let R be a commutative ring and let x, y be adjacent vertices of $\Gamma(R)$. Then $|d(x,z) - d(y,z)| \leq 1$, for every vertex z of $\Gamma(R)$.*

Proof. By triangular inequality, we have $d(x,z) \leq d(x,y) + d(y,z) \leq 1 + d(y,z)$ which implies $d(x,z) - d(y,z) \leq 1$. Similarly $d(y,z) - d(x,z) \leq 1$. Therefore $|d(x, z) - d(y,z)| \leq 1$, for every vertex z of $\Gamma(R)$. □

The following result relates the maximum degree, diameter and the locating number of $\Gamma(R)$.

Theorem 5.2. *Let R be a commutative ring and let $\Gamma(R)$ be the corresponding zero-divisor graph of R such that $|Z^*(R)| \geq 2$. Then*

$$\lceil \log_3(\Delta + 1) \rceil \leq \mathrm{loc}(\Gamma(R)) \leq |Z^*(R)| - d,$$

where Δ is the maximum degree and d is the diameter of $\Gamma(R)$.

Proof. First, we establish the upper bound. Let u and v be the vertices for which $d(u,v) = \sup\{d(x,y) \,|\, x, y \in Z^*(R)\} = d$, that is, $d(u,v)$ is the diameter of $\Gamma(R)$ and let $u = v_0, v_1, \ldots, v_d = v$ be $u - v$ path of length d. Let $W = V(\Gamma(R)) - \{v_1, v_2, \ldots, v_d\}$. Since $u \in W$ and $d(u, v_i) = i$ for $1 \leq i \leq d$, it follows that W is a locating set of cardinality $|Z^*(R)| - d$ for $\Gamma(R)$. Thus, $\mathrm{loc}(\Gamma(R)) \leq |Z^*(R)| - d$.

Now, to obtain the lower bound, let $\mathrm{loc}(\Gamma(R)) = k$ and $v \in V(\Gamma(R))$ with $\deg(v) = \Delta$. Moreover, let $N(v)$ be the neighborhood of v and let $W_1 = \{w_1, w_2, \ldots, w_k\}$ be a minimum locating set of $\Gamma(R)$. Observe that if $u \in N(v)$, then by Lemma 5.1, for each i, $(1 \leq i \leq k)$, the distance $d(u, w_i)$ is one of the numbers $d(v, w_i)$, or $d(v, w_i) + 1$ or $d(v, w_i) - 1$. Since W_1 is a minimum locating set, therefore $c_{W_1}(u) \neq c_{W_1}(v)$ for all $u \in N(v)$. Thus, there are three possible numbers for each of the k coordinates of $c_{W_1}(u)$. On the other hand, since it cannot occur that $d(u, w_i) = d(v, w_i)$ for all i $(1 \leq i \leq k)$, it follows that there are at most $3^k - 1$ distinct codes of the vertices in $N(v)$ with respect to W_1. Therefore, $|N(v)| = \Delta \leq 3^k - 1$, which implies that $\mathrm{loc}(\Gamma(R)) = k \geq \log_3(\Delta + 1)$. Since $\mathrm{loc}(\Gamma(R))$ is an integer, therefore, $\mathrm{loc}(\Gamma(R)) \geq \lceil \log_3(\Delta + 1) \rceil$. □

Example 5.3. Let $R = \mathbb{Z}_6$, $\mathbb{Z}_8$ or $\mathbb{Z}_3[X]/(X^2)$. The zero-divisor graphs for these rings is a path $P_{|Z^*(R)|}$ with $|Z^*(R)| \geq 2$, the maximum degree is $\Delta = 2$ and diameter $d = |Z^*(R)| - 1$. The inequalities in Theorem 5.2 imply that $\lceil \log_3(2+1) \rceil = 1 \leq P_{|Z^*(R)|} \leq 1$ and both bounds are sharp for $P_{|Z^*(R)|}$. Furthermore, for the ring

$\mathbb{Z}_{p^2}$, where p is a prime larger than 3, the zero-divisor graph is a complete graph $K_{|Z^*(R)|}$ which has diameter 1 and the locating number $|Z^*(R)| - 1$, so the upper bound is attainable in this case. Each of the two bounds is sharp for other graphs as well.

Recall that the chromatic number of the graph denoted by $\chi(G)$ is defined to be the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. An independent set of the graph G is a set of vertices where no two vertices are adjacent. The independence number denoted by $\alpha(G)$ is the size of the maximum independent set in the graph G. A clique in a graph G is an induced complete subgraph. The clique number denoted by $\omega(G)$ is the greatest integer $n \geq 1$ such that $K_n \subseteq G$ and $\omega(G)$ is infinite if $K_n \subseteq G$ for all $n \geq 1$. Now we determine the relationship between the numbers $\mathrm{loc}(\Gamma(R))$, $\omega(\Gamma(R))$ and $\chi(\Gamma(R))$ for the graph $\Gamma(R)$. In the following results we also exploit the information gained from the degree of vertices.

Lemma 5.4. *The only finite commutative rings R with 1 such that $\Gamma(R)$ has a maximal clique W where the degree of each vertex of W is $\omega(\Gamma(R))$ are (up to isomorphism) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Assume R is local. Then, R has a vertex z adjacent to all other vertices. Clearly, z would be an element of any largest clique for $\Gamma(R)$. But then $|V(\Gamma(R))| - 1 = \deg(z) = \deg(w) = \omega(\Gamma(R))$ for every $w \in W$. That is, every vertex of W is adjacent to all other vertices of $\Gamma(R)$. Now, $\omega(\Gamma(R)) = |V(\Gamma(R))| - 1$ implies there is a unique $t \in V(\Gamma(R))$ with $t \notin W$. However, since every element of W is adjacent to t, $W \cup \{t\}$ is a larger clique than W a contradiction.

So, R is not local. We can write $R \cong R_1 \times R_2 \times \cdots \times R_n$ for some $n \geq 2$ where each R_i is a local ring.

Assume R is not reduced. That is, without loss of generality, R_1 is not a field. Then, there is some $x_1 \in R_1$ such that x_1 is adjacent to all other vertices of $\Gamma(R_1)$.

Suppose there is some $w = (w_1, w_2, \ldots, w_n) \in W$ with $w_1 \in U(R_1)$. Since the vertices of W form a clique, the first coordinate of any other vertex in W must be 0. But then, each of these other elements is adjacent to at least two elements not in W, namely $(x_1, 0, \ldots, 0)$ and $(x_1 + 1, 0, \ldots, 0)$. This is a contradiction.

Thus, for each $w \in W$, the first coordinate must be an element of $Z(R_1)$. Then, $x' = (x_1, 0, \ldots, 0)$ is adjacent to each element of W. Since W is a maximal clique, $x' \in W$.

Assume $n \geq 3$. Note that if $(z, 1, \ldots, 1) \in W$ for any $z \in Z(R_1)$, then coordinates 2 through n of every element of W must be 0. However, then $W \cup \{(x_1, 1, 0, \ldots, 0)\}$ would be a larger clique of $\Gamma(R)$. So, $(z, 1, \ldots, 1) \neq W$ for $z \in Z(R_1)$. However, this gives at least two elements of $V(\Gamma(R))$ adjacent to x' but not in W (namely, $(x_1, 1, \ldots, 1)$ and $(0, 1, \ldots, 1)$). This is a contradiction.

Assume $n = 2$. There can be at most one element of W of the form (z, u_2) for $z \in Z(R_1)$ and $u_2 \in U(R_2)$ since all elements of W are adjacent. However, then x' is

adjacent to any element of this form. Thus, $|U(R_2)| = 1$ or else x' is adjacent to more than one vertex not in W (namely any elements of the form $(0, u_2)$ and (x_1, u_2)), implying $R_2 \cong \mathbb{Z}_2$. Similarly, $|Z(R_1)| \leq 2$ (or else x' is adjacent to more than one vertex not in W). This implies R_1 is a field or $R_1 \cong \mathbb{Z}_4$ or $R_1 \cong \mathbb{Z}_2[X]/(X^2)$. In all three cases, $\Gamma(R) = \Gamma(R_1 \times \mathbb{Z}_2)$ yields a star-graph, which is a contradiction since a star-graph cannot meet the conditions of the hypothesis.

Hence, we must conclude R is reduced. Then clearly each R_i is a field and $W = \{(u_1, 0, \ldots, 0), (0, u_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, u_n)\}$ where $u_i \in U(R_i)$. Therefore $n \leq 3$ or else each element of W is adjacent to more than one element not in W (for example, $(u_1, 0, \ldots, 0)$ would be adjacent to $(0, 1, 1, \ldots, 1)$ and $(0, 0, 1, \ldots, 1)$).

If $n = 2$, the graph is complete bipartite with K_2 the largest complete subgraph. Hence, each vertex must have degree 2 (i.e. $\Gamma(R) \cong K_{2,2}$). Hence, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

If $n = 3$, we must have each $R_i \cong \mathbb{Z}_2$ since otherwise (for example) $(u_1, 0, 0)$ would be adjacent to more than one element not in W. Hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Proposition 5.5. *Let R be a finite commutative ring, $\Gamma(R)$ be its associated zero-divisor graph and W be the maximal clique of $\Gamma(R)$. The following conditions hold.*

(i) *If the degree of each vertex in W is $\omega(\Gamma(R))$, then $\mathrm{loc}(\Gamma(R)) \leq \omega(\Gamma(R)) = \chi(\Gamma(R))$.*
(ii) *If $\Gamma(R)$ is a single vertex graph or a path or a cycle or a complete graph, then $\mathrm{loc}(\Gamma(R)) \leq \omega(\Gamma(R)) \leq \chi(\Gamma(R))$.*

Proof. (i) This is a consequence of Lemma 5.4, since $\mathrm{loc}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$ and $\omega(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 3 = \chi(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$, and $\mathrm{loc}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$ and $\omega(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2 = \chi(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$.

(ii) By Theorem 2.3, we have

$$\mathrm{loc}(\Gamma(R)) = \begin{cases} 0 & \text{if } \Gamma(R) \text{ is a single vertex graph,} \\ 1 & \text{if } \Gamma(R) \text{ is a path,} \\ 2 & \text{if } \Gamma(R) \text{ is a cycle,} \\ |Z^*(R)| - 1 & \text{if } \Gamma(R) \text{ is a complete graph.} \end{cases}$$

Further,

$$\chi(\Gamma(R)) = \begin{cases} 1 & \text{if } \Gamma(R) \text{ is a single vertex graph,} \\ 2 & \text{if } \Gamma(R) \text{ is a path,} \\ 2 & \text{if } \Gamma(R) \text{ is a cycle } C_{|Z^*(R)|}, (|Z^*(R)| \text{ is even}), \\ 3 & \text{if } \Gamma(R) \text{ is a cycle } C_{|Z^*(R)|}, (|Z^*(R)| \text{ is odd}), \\ |Z^*(R)| & \text{if } \Gamma(R) \text{ is a complete graph.} \end{cases}$$

Moreover, it can be easily seen that $\omega(\Gamma(R)) \geq \mathrm{loc}(\Gamma(R))$ and $\chi(\Gamma(R)) \geq \omega(\Gamma(R))$. Therefore, from the above observations, it follows that $\mathrm{loc}(\Gamma(R)) \leq \omega(\Gamma(R)) \leq \chi(\Gamma(R))$. □

Corollary 5.6. *If $\Gamma(R)$ is a single vertex graph or a path or a cycle or a complete graph, then*

(i) $\mathrm{loc}(\Gamma(R)) \leq 1 + \Delta(\Gamma(R))$, *where $\Delta(\Gamma(R))$ is the maximum degree of $\Gamma(R)$,*
(ii) $\mathrm{loc}(\Gamma(R)) \leq 1 + \max(\delta(\Gamma'(R)))$, *where $\delta(\Gamma'(R))$ is the minimum degree for any subgraph $\Gamma'(R)$ of $\Gamma(R)$,*
(iii) $\mathrm{loc}(\Gamma(R)) \leq 1 + \min\{\max(i-1, \deg(x_i))\} = \min\{\max(i, 1+\deg(x_i))\}$, *where $x_i \in Z^*(R)$ for $1 \leq i \leq |Z^*(R)|$,*
(iv) $\mathrm{loc}(\Gamma(R)) \leq \lfloor \frac{|Z^*(R)|+\omega(\Gamma(R))}{2} \rfloor$, *where $\omega(\Gamma(R))$ is a clique number for $\Gamma(R)$,*
(v) $\mathrm{loc}(\Gamma(R)) \leq \frac{|Z^*(R)|+1+\omega(\Gamma(R))-\alpha(\Gamma(R))}{2}$, *where $\alpha(\Gamma(R))$ is the independence number for $\Gamma(R)$.*

Using Proposition 5.5, we have the following observation in general simple graphs connecting the automorphism group of a graph and the symmetric group S_n.

Corollary 5.7. *Let G_n $(n \geq 2)$ be the graph with $2n$ vertices obtained from K_n by attaching a single pendant vertex to each vertex in the clique. Then $\mathrm{loc}(G_n) \leq \omega(G_n) = \chi(G_n)$ and $\mathrm{Aut}(G_n) \cong S_n$.*

Remark 5.8. Clearly (i) of Proposition 5.5 does not hold for all the zero-divisor graphs. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_4$. From the zero-divisor graph of R (Fig. 4) we see $\mathrm{loc}(\Gamma(R)) = 3$ where as $\chi(\Gamma(R)) = 2$.

Proposition 5.9. *The only finite reduced commutative rings R with 1 where $\mathrm{loc}(\Gamma(R)) \leq \omega(\Gamma(R))$ are (up to isomorphism) $\mathbb{F}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\prod_{i=1}^{n} \mathbb{Z}_2$ for $n \geq 2$.*

Proof. If R is reduced, then $R \cong \mathbb{F}_1 \times \cdots \times \mathbb{F}_n$ for some fields $\mathbb{F}_1, \ldots, \mathbb{F}_n$ and for some integer $n \geq 2$. Then, $W = \{(u_1, 0, \ldots, 0), (0, u_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, u_n)\}$ where $u_i \in \mathbb{F}_i$ with $u_i \neq 0$ is a maximal clique for R. That is, $\omega(\Gamma(R)) = n$.

Suppose $n \geq 4$. Assume $\mathbb{F}_1 \ncong \mathbb{Z}_2$. Then, $\Gamma(R)$ can be partitioned into distance similar equivalence classes where $2^{n-1} - 1$ of these have more than one element namely the distance equivalence classes of $(1, v_2, \ldots, v_n)$ where at least one $v_i = 0$ and all other $v_j = 1$. Thus, a minimal locating set for $\Gamma(R)$ has at least $2^{n-1} - 1 > n$ vertices. Therefore, since the same argument can be applied to $R_2, \ldots, R_n$, if $n \geq 4$, $R \cong \prod_{i=1}^{n} \mathbb{Z}_2$ and $\mathrm{loc}(\prod_{i=1}^{n} \mathbb{Z}_2) \leq n = \omega(\prod_{i=1}^{n} \mathbb{Z}_2)$.

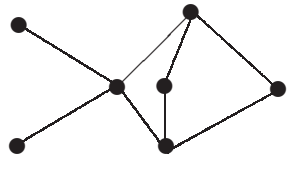

Fig. 4.

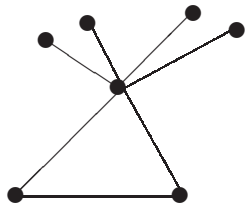

Fig. 5.

Suppose $n = 3$. There are six distance similar equivalence classes of vertices of $\Gamma(R)$, $(1,1,0), (1,0,1), (0,1,1), (1,0,0), (0,1,0), (0,0,1)$. If $|\mathbb{F}_1| \geq 3$, then there are three equivalence classes that have more than one element. To achieve $\mathrm{loc}(\Gamma(R)) \leq 3$, we then must have each of these three equivalence classes with two elements and the other equivalence classes with one element. That is, $\mathbb{F}_1 \cong \mathbb{Z}_3$, $\mathbb{F}_2 \cong \mathbb{F}_3 \cong \mathbb{Z}_2$. It is routine to verify that indeed $\mathrm{loc}(\Gamma(R)) \leq \omega(\Gamma(R))$ for $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $n = 2$. Then, $\Gamma(R)$ is a complete bipartite graph with K_2 the largest complete subgraph. There are two distance similar equivalence classes of vertices of $\Gamma(R)$, the equivalence classes of $(1,0)$ and of $(0,1)$. If $|\mathbb{F}_i| \geq 5$, then one of these equivalence classes has at least four elements, which would yield at least three elements in any minimal locating set. Checking all cases for $\mathbb{F}_1 \times \mathbb{F}_2$ with $|\mathbb{F}_i| \leq 4$ yields the only the rings $\mathbb{F}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_3 \times \mathbb{Z}_3$ having locating number at most 2. □

Example 5.10. Let $R = \mathbb{Z}_4[X]/(X^2+2)$. Here $|R| = 16$, the vertex set is $Z^*(R) = \{X + (X^2+2), 2X + (X^2+2), 3X + (X^2+2), 2 + 3X + (X^2+2), 2 + 2X + (X^2+2), 2 + (X^2+2), 2 + X + (X^2+2)\}$. By drawing the graph (Fig. 5) we see $\mathrm{loc}(\Gamma(R)) = 4$, where as the clique number and the chromatic number for $\Gamma(R)$ is 3.

6. Locating Numbers of $\Gamma(R)$ When R is a Finite Product of Rings

In this section, we give some locating bounds for R when $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i, $1 \leq i \leq k$ is an integral domain. We further give the bound for the clique number of R when R is the finite product of local rings. We also show that $\mathrm{loc}(\Gamma(R))$ is greater than the dominating and clique numbers.

Theorem 6.1. *Let $R_1, R_2, \ldots, R_k$ be integral domains with 1. Suppose $|R_i| > 2$ for some i. Then* $\mathrm{loc}(\Gamma(R_1 \times \cdots \times R_k)) = |Z(R)| - 2^k + 2$.

Proof. Note that $(a_1, a_2, \ldots, a_k) = a \sim b = (b_1, b_2, \ldots, b_k)$ if and only if it is the case $a_i = 0$ if and only if $b_i = 0$ for each i (in other words, $a \sim b$ if and only if both a and b have the same number of coordinates equal to zero in the same coordinate positions). Using this, we can count the number of distance similar vertices of $\Gamma(R)$,

and determine the size of the minimal locating set (since all but one of the vertices in each distance similar equivalence class must be members of any locating set, as seen in Theorem 2.1).

For the equivalence class of $(a_1, 0, \ldots, 0)$ $(a_1 \neq 0)$, the cardinality of the equivalence class is $|R_1| - 1$. Therefore, $|R_1| - 2$ vertices must be in each minimal locating set. Doing the same for equivalence classes of vectors with one nonzero coordinate, we see that a locating set W must contain at least $\sum_{i=1}^{k}(|R_i| - 2)$ vertices.

For the equivalence class of $(a_1, a_2, 0, \ldots, 0)$ $(a_1 \neq 0, a_2 \neq 0)$, the cardinality of the equivalence class is $(|R_1| - 1)(|R_2| - 1)$. Therefore, $(|R_1| - 1)(|R_2| - 1) - 1$ vertices must be in each locating set. Doing the same for equivalence classes of vectors with two nonzero coordinates, we see that a locating set W must contain at least $\sum_{i_1 < i_2}[(|R_{i_1}| - 1)(|R_{i_2}| - 1) - 1]$.

Using the same method for vectors with n nonzero coordinates for $1 \leq n \leq k - 1$, we see for a minimal locating set W, $|W| \geq \sum_{i=1}^{k}(|R_i| - 2) + \sum_{i_1 < i_2}[(|R_{i_1}| - 1)(|R_{i_2}| - 1) - 1] + \sum_{i_1 < i_2 < i_3}[(|R_{i_1}| - 1)(|R_{i_2}| - 1)(|R_{i_3}| - 1) - 1] + \cdots + \sum_{i_1 < i_2 < \cdots < i_{k-1}}[(|R_{i_1}| - 1)(|R_{i_2}| - 1)(|R_{i_{k-1}}| - 1) - 1]$. (Alternatively, if M is the number of equivalence classes of vertices of $\Gamma(R)$, then $|W| \leq |Z(R)^*| - M$ and $M = \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-1} = 2^k - 2$.)

So, this is a lower bound for $|W|$ in all cases. We will show that as long as $R \ncong \prod_{i=1}^{k} \mathbb{Z}_2$, then $|W|$ is equal to this sum. Note that if each $R_i \cong \mathbb{Z}_2$, then each term in the above sum is 0. So, suppose at least one $R_i \ncong \mathbb{Z}_2$. Without loss of generality, $R_1 \ncong \mathbb{Z}_2$. Then the above sum is greater than 0. Let W be a subset of the vertices of $\Gamma(R)$ consisting of all but one vertex from each distance equivalence class of vertices. Note $W \neq \emptyset$ (since not all $R_i \cong \mathbb{Z}_2$, some equivalence class has at least two elements).

For some distinct $x, y \in V(\Gamma(R)) - W$, assume $C_W(x) = C_W(y)$ and we provide a contradiction. Note that by the way W is constructed, $x \not\sim y$. We will denote $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Also, let u_i be the vector with 1 in coordinate i and 0 in all other coordinates and v_i be the vector with 0 in coordinate i and 1 in all other coordinates.

Case 1: Suppose $x_1 = y_1 = 0$. For $m \geq 2$, define $t_m = u_1 + u_m$. Since $R_1 \ncong \mathbb{Z}_2$, for each m there is some $w_m \in W$ with $w_m \sim t_m$. Note that $d(x, w_m) = 1$ if and only if $x_m = 0$. Thus, $C_W(x) = C_W(y)$ implies $x_i = 0$ if and only if $y_i = 0$. This would imply $x \sim y$, a contradiction.

Case 2: Suppose $x_1 \neq 0$ and $y_1 \neq 0$. Note that for each $m \geq 2$, there is some $w_m \in W$ with $w_m \sim v_m$. Also, $d(x, w_m) = 2$ if and only if $x_m = 0$. Thus, $C_W(x) = C_W(y)$ implies $x_i = 0$ if and only if $y_i = 0$. This would imply $x \sim y$, a contradiction.

Case 3: Suppose (without loss of generality) $x_1 = 0$ and $y_1 \neq 0$. Then, there is some $w \in W$ such that $w \sim u_1$. Thus, $d(x, w) = 1$ but $d(y, w) > 1$. This contradicts $C_W(x) = C_W(y)$.

Hence, we have a contradiction in all possible cases. Thus, W is a locating set for $\Gamma(R)$. □

In light of Theorem 6.1, one should consider the locating number of $\prod_{i=1}^{k} \mathbb{Z}_2$. By inspection of the graphs, one can verify $\mathrm{loc}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$, $\mathrm{loc}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$, and $\mathrm{loc}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$. However, the same is not true if $k > 4$.

Proposition 6.2. *For any $k \geq 2$, $\mathrm{loc}(\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)) \leq k$.*

Proof. Let u_i be the vector with 1 in coordinate i and 0 in all other coordinates, and let v_i be the vector with 0 in coordinate i and 1 in all other coordinates.

Let $U = \{u_1, u_2, u_3, u_4, \ldots, u_k\}$. Note that for any $t \in V(\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)) - U$, $d(t, u_i) = 1$ if and only if coordinate i of t is 0, and $d(t, u_i) = 2$ otherwise. Thus, $c_U(t) = t + (1, 1, 1, 1, \ldots, 1)$ (as vectors in $\mathbb{Z}^k$). Hence, U is a locating set for $\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)$ and $\mathrm{loc}(\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)) \leq k$. □

Theorem 6.3. $\mathrm{loc}(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)) = 5$.

Proof. As in Proposition 6.2, let u_i be the vector with 1 in coordinate i and 0 in all other coordinates, and let v_i be the vector with 0 in coordinate i and 1 in all other coordinates.

Assume $\mathrm{loc}(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)) \leq 4$ and we will provide a contradiction. By assumption, there is some locating set $W = \{x_1, x_2, x_3, x_4\}$ for $\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)$. By Theorem 4.5, we may assume W is chosen so that no x_i is a cut vertex. Since the only cut vertices of $\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)$ are u_1, u_2, u_3, u_4, u_5, $u_i \notin W$ for $i = 1, \ldots, 5$.

Part One: Suppose W contains some vector with only one coordinate equal to 0. Without loss of generality, say $x_1 = v_1$.

Case 1: Suppose that x_2 has three zero coordinates. Then, without loss of generality, there are two cases to consider.

Subcase A: If $x_2 = (1, 1, 0, 0, 0)$, then $c_W(t) = (3, 2, a, b)$ for the following vertices: $u_1 + u_3, u_1 + u_4, u_1 + u_5, u_1 + u_2 + u_3, u_1 + u_2 + u_4, u_1 + u_2 + u_5, u_1 + u_3 + u_4, u_1 + u_3 + u_5, u_1 + u_4 + u_5, v_3, v_4, v_5$. Note that $a, b \in \{1, 2, 3\}$ for any vector in $V(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)) - W$. So, even if x_3 and x_4 are in the list of vectors given here, there are at least 10 vertices whose codes have the form $(3, 2, a, b)$ while there are only nine possible codes of this form. Hence, each vertex will not get a unique code. This contradicts the claim that W is a locating set.

Subcase B: If $x_2 = (0, 1, 1, 0, 0)$, then $c_W(t) = (3, 2, a, b)$ for the following vertices: $u_1 + u_2, u_1 + u_3, u_1 + u_2 + u_3, u_1 + u_2 + u_4, u_1 + u_2 + u_5, u_1 + u_3 + u_4, u_1 + u_3 + u_5, v_4, v_5$. There are nine vertices in this list. If one of these vertices is x_3, then $a \neq 1$ since then x_3 and any remaining vector from the list both have first coordinate equal to 1. This would give either seven or eight vertices remaining in this list not in W but with a maximum of six distinct codes of the form $(3, 2, a, b)$. Similarly, x_4 cannot be in this list. Thus, with nine vertices in this list but not in W and only nine possible codes of the from $(3, 2, a, b)$, we must have the first coordinate of x_3 equal to 0 (or $a = 1$ will never be possible) and the first coordinate of x_4 equal to 0 (or $b = 1$ will

never be possible). Since $x_3 \neq u_i$ for any i, x_3 must have two nonzero coordinates in coordinates 2 through 5. The same is true for x_4. Thus, $c_W(t) = (2, 2, 2, 2)$ for the following vectors: $(0, 1, 1, 1, 0), (0, 1, 1, 0, 1), (0, 1, 0, 1, 1), (0, 0, 1, 1, 1)$. Even if x_3 or x_4 (or both) are in this list, there will be two vectors with the same code. This is a contradiction.

Case 2: Suppose that x_2 has two zero coordinates. Then, without loss of generality, there are two cases to consider.

Subcase A: If $x_2 = (0, 0, 1, 1, 1)$, then $c_W(t) = (2, 2, a, b)$ for the following vertices: $u_3, u_4, u_5, u_3 + u_4, u_3 + u_5, u_4 + u_5, u_2 + u_3, u_2 + u_4, u_2 + u_5, u_2 + u_3 + u_4, u_2 + u_3 + u_5, u_2 + u_4 + u_5$. Even if x_3 and x_4 are in this list, there are at least 10 vertices whose codes have the from $(2, 2, a, b)$ while there are only nine possible codes of this form. This is a contradiction.

Subcase B: If $x_2 = (1, 1, 1, 0, 0)$, then $c_W(t) = (3, 2, a, b)$ for the following vertices: $u_1+u_2, u_1+u_3, u_1+u_4, u_1+u_5, u_1+u_2+u_4, u_1+u_2+u_5, u_1+u_3+u_4, u_1+u_3+u_5, v_4, v_5$. There are 10 vertices in this list. If none of these vertices are in W, then we have 10 vertices for nine possible codes of the form $(3, 2, a, b)$. If x_3 or x_4 is in this list, then $a \neq 1$ or $b \neq 1$, giving either eight or nine vertices from this list with codes of the form $(3, 2, a, b)$ but at most six possible codes of this form. This is a contradiction.

Case 3. By Cases 1 and 2, it must be the case that all elements of W have only one zero coordinate (since Cases 1 and 2 may be applied to x_3 and x_4 just as they were to x_2). So, without loss of generality, $W = \{v_1, v_2, v_3, v_4\}$. But then $c_W((0, 0, 1, 1, 1)) = (2, 2, 3, 3) = c_W((0, 0, 1, 1, 0))$. This is a contradiction.

Hence, W contains no vectors with only one zero coordinate (i.e. $v_i \notin W$ for any i).

Part Two: Suppose W contains a vector with three coordinates equal to 0. Without loss of generality, $x_1 = (0, 0, 0, 1, 1)$.

Case 1: Suppose $d(x_1, x_2) = 1$. Since $u_i \notin W$ for each i, we must have (without loss of generality) $x_2 = (1, 1, 1, 0, 0)$ or $x_2 = (1, 1, 0, 0, 0)$ (i.e. either two or three of the first three coordinates of x_2 must equal 1).

Subcase A: If $x_2 = (1, 1, 1, 0, 0)$, then $c_W(t) = (2, 2, a, b)$ for the following vertices: $u_1 + u_4, u_2 + u_4, u_3 + u_4, u_1 + u_5, u_2 + u_5, u_3 + u_5, u_1 + u_2 + u_4, u_1 + u_3 + u_4, u_2 + u_3 + u_4, u_1 + u_2 + u_5, u_1 + u_3 + u_5, u_2 + u_3 + u_5$. So, even if x_3 and x_4 are in the list of vectors given here, there are at least 10 vertices whose codes have the form $(2, 2, a, b)$ while there are only nine possible codes of this form. This is a contradiction.

Subcase B: If $x_2 = (1, 1, 0, 0, 0)$, then $c_W(t) = (2, 2, a, b)$ for the following vertices: $u_1 + u_4, u_1 + u_5, u_2 + u_4, u_2 + u_5, u_1 + u_2 + u_4, u_1 + u_2 + u_5, u_1 + u_3 + u_4, u_1 + u_3 + u_5, u_2 + u_3 + u_4, u_2 + u_3 + u_5, u_1 + u_4 + u_5, u_2 + u_4 + u_5, v_3$. So, even if x_3 and x_4 are in this list, there are at least 11 vertices for nine codes of the form $(2, 2, a, b)$. This is a contradiction.

Case 2: Suppose $d(x_1, x_2) = 3$. Then $x_2 = v_4$ or $x_2 = v_5$. As seen in Part One, this is not possible.

Case 3: By Cases 1 and 2, we must have $d(x_1, x_2) = d(x_1, x_3) = d(x_1, x_4) = 2$ (as Cases 1 and 2 can be applied to x_3 and x_4 just as they were to x_2). Any vertex that is distance 2 from x_1 must have at least one zero in the first three coordinates and either one or two ones in the last two coordinates.

Subcase A: Without loss of generality, $x_2 = (c, d, 0, 0, 1)$ for some $c, d \in \{0, 1\}$. Then, $c_W(t) = (2, 2, a, b)$ for the following vertices: $u_5, u_1 + u_5, u_2, +u_5, u_3 + u_5, u_1 + u_2 + u_5, u_1 + u_3 + u_5, u_2 + u_3 + u_5, u_1 + u_4 + u_5, u_2 + u_4 + u_5, v_3, v_4$. There are 11 vertices in this list. If 0 or 1 of these vertices are in W, there at least 10 vertices of the form $(2, 2, a, b)$ for the nine possible codes. If two of these vertices are in W (say, without loss of generality, one of the vertices is x_3), then $a \neq 1$ since x_3 and each vertex in this list have 1 in the fifth coordinate. Thus, there are nine vertices with code $(2, 2, a, b)$ but at most six such possible codes. Finally, if three of these vertices are in W, then $a \neq 1$ and $b \neq 1$, leaving eight vertices but only four codes of the form $(2, 2, a, b)$. Thus, we have a contradiction in all possible cases.

Subcase B: Without loss of generality, $x_2 = (c, d, 0, 1, 1)$. Since $x_2 \neq v_i$ for any i, we may assume $d = 0$ (without loss of generality). Then, $c_W(t) = (2, 2, a, b)$ for the following vertices: $u_4, u_5, u_1 + u_4, u_2 + u_4, u_3 + u_4, u_1 + u_5, u_2 + u_5, u_3 + u_5, u_1 + u_4 + u_5, u_2 + u_4 + u_5, u_3 + u_4 + u_5, u_1 + u_2 + u_4, u_1 + u_2 + u_5, v_2, v_3$. Even if x_2, x_3 and x_4 are in this list, there are at least 12 vertices for nine codes of the form $(2, 2, a, b)$. This is a contradiction.

Hence, W cannot contain any vectors with three coordinates equal to 0.

Part Three: From Parts One and Two, we see that each vector of W must have two coordinates equal to 0. Note that this implies $d(x_i, x_j) \neq 1$ whenever $i \neq j$. Without loss of generality, let $x_1 = (0, 0, 1, 1, 1)$.

Case 1: Suppose $d(x_1, x_j) = 3$ for $j = 2, 3, 4$. There are only three vectors in $\prod_{i=1}^{5} \mathbb{Z}_2$ with two zero coordinates that are distance 3 from x_1. Thus, (without loss of generality), $x_2 = (1, 1, 1, 0, 0)$, $x_3 = (1, 1, 0, 0, 1)$, $x_4 = (1, 1, 0, 1, 0)$. However, then $c_W(u_1) = c_W(u_2) = (1, 2, 2, 2)$, a contradiction.

Case 2: Suppose $d(x_1, x_2) = d(x_1, x_3) = 3$ and $d(x_1, x_4) = 2$. Again, since there are only three vectors with two zero coordinates that are distance 3 from x_1, we can say $x_2 = (1, 1, 1, 0, 0)$ and $x_3 = (1, 1, 0, 0, 1)$ without loss of generality. Then, $c_W(t) = (2, 2, 2, b)$ for the following vertices: $u_1 + u_3, u_1 + u_4, u_1 + u_5, u_2 + u_3, u_2 + u_4, u_2 + u_5$. Even if one of these vertices is x_4, there are at least five vertices for the three codes of the form $(2, 2, 2, b)$. This is a contradiction.

Case 3: Suppose $d(x_1, x_2) = 3, d(x_1, x_3) = d(x_1, x_4) = 2$. Then, without loss of generality, $x_2 = (1, 1, 1, 0, 0)$. Thus, $c_W(t) = (2, 2, a, b)$ for the following vectors: $u_1 + u_4, u_1 + u_5, u_2 + u_4, u_2 + u_5, u_3, u_1 + u_3, u_2 + u_3, u_3 + u_4, u_3 + u_5, u_1 + u_3 + u_4, u_1 + u_3 + u_5, u_2 + u_3 + u_4, u_2 + u_3 + u_5$. Even if x_3 and x_4 are in this list, there are at least 11 vertices for the nine codes of the form $(2, 2, a, b)$. This is a contradiction.

Case 4: Suppose $d(x_i, x_j) = 2$ whenever $i \neq j$. Then there are only six possibilities for x_2, x_3, x_4: $(0,1,1,0,1)$, $(0,1,0,1,1)$, $(0,1,1,1,0)$, $(1,0,1,0,1)$, $(1,0,0,1,1)$, $(1,0,1,1,0)$. Without loss of generality, $x_2 = (0,1,1,0,1)$. To keep the distance between all vertices in W equal to 2, the only possibilities for x_3 and x_4 are now: $(0,1,0,1,1)$, $(0,1,1,1,0)$, $(1,0,1,0,1)$. Since $d(x_3, x_4) = 2$, we must choose (without loss of generality) $x_3 = (0,1,0,1,1)$ and $x_4(0,1,1,1,0)$. However, $c_W(v_1) = c_W(u_4 + u_5) = (2,2,2,2)$. This is a contradiction.

Having exhausted all possible cases, we must conclude $\mathrm{loc}(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)) > 4$. Hence, $\mathrm{loc}(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2)) = 5$. □

Recall that a set S of vertices of G is a dominating set of G if every vertex of G is either in S or adjacent to an element of S. A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$.

Remark 6.4. (a) In Theorem 6.1, if $k = 2$, $|R_1| = 2$ and $|R_2| \geq 3$, clearly $\Gamma(R)$ is a star graph. Therefore, by Theorem 2.3, $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$. The vertex adjacent to all others in the graph is a singleton dominating set. So, $\gamma(\Gamma(R)) = 1$.

Further, for $n = 2$, if $|R_1| \geq 3$ and $|R_2| \geq 3$, then $\Gamma(R)$ is a complete bipartite graph (but not a star graph). By Theorem 2.3, $\mathrm{loc}(\Gamma(R)) = |Z^*(R)| - 2$. The set $D = \{(a_1, 0, \ldots, 0), \ldots, (0, 0, \ldots, a_k)\}$ is a minimum dominating set of $\Gamma(R)$, since there is no other dominating set whose cardinality is less than k. So, $\gamma(\Gamma(R)) = k = \omega(\Gamma(R)) \neq \mathrm{loc}(\Gamma(R)) = |Z(R)| - 2^k + 2$. Therefore, when $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i, $(1 \leq i \leq k)$ is an integral domain, we conclude that domination and the clique number are equal for the zero-divisor graph $\Gamma(R_1 \times R_2 \times \cdots \times R_k)$ unless the graph is a star graph (that is, unless $k = 2$ and either $R_1 \cong \mathbb{Z}_2$ or $R_2 \cong \mathbb{Z}_2$).

(b) For $n = 2$, $|R_1| = 2$ and $|R_2| \geq 3$, the zero-divisor graph $\Gamma(R)$ is a star graph and therefore, $\gamma(\Gamma(R)) = 1 < 2 = \omega(\Gamma(R)) \leq |Z^*(R)| - 2 = \mathrm{loc}(\Gamma(R))$ for $|Z^*(R)| \geq 4$. When $|Z^*(R)| = 3$, $\gamma(\Gamma(R)) = \mathrm{loc}(\Gamma(R)) = 1$. When $|Z^*(R)| = 4$, $\omega(\Gamma(R)) = \mathrm{loc}(\Gamma(R)) = 2$. However, for $|Z^*(R)| = 3$, $2 = \omega(\Gamma(R)) > \mathrm{loc}(\Gamma(R)) = 1$.

(c) For $n = 2$, $|R_1| \geq 3$ and $|R_2| \geq 3$, the zero-divisor graph $\Gamma(R)$ is a complete bipartite graph with $\gamma(\Gamma(R)) = 2$. Therefore, for $|Z^*(R)| \geq 4$, $\gamma(\Gamma(R)) = 2 = \omega(\Gamma(R)) \leq |Z^*(R)| - 2 = \mathrm{loc}(\Gamma(R))$.

(d) If $\Gamma(R)$ is a cycle, then $|Z^*(R)| = 3$ or 4 (since $\mathrm{diam}(\Gamma(R)) \leq 3$ and no other cycles appear in the list of graphs realizable as zero-divisor graphs given in [14]). For $|Z^*(R)| = 3$, $1 = \gamma(\Gamma(R)) < \mathrm{loc}(\Gamma(R)) = \omega(\Gamma(R)) = 2$. For $|Z^*(R)| = 4$, $\gamma(\Gamma(R)) = \mathrm{loc}(\Gamma(R)) = \omega(\Gamma(R)) = 2$.

Corollary 6.5. *Let $R = R_1 \times R_2 \times \cdots \times R_k$, where R_i for each i, $(1 \leq i \leq k)$, is a local ring. Then*

(i) $\gamma(\Gamma(R)) = k$ *unless* $\Gamma(R)$ *is a star-graph (in which case* $\gamma(\Gamma(R)) = 1$*).*

(ii) *Without loss of generality, let* $R_1, \ldots, R_m$ *be non-fields and* $R_{m+1}, \ldots, R_k$ *be fields. Let* $N_1, \ldots, N_m$ *be the clique numbers of* $\Gamma(R_1), \ldots, \Gamma(R_m)$. *Then* $\omega(\Gamma(R)) \geq (N_1+1)(N_2+1)\cdots(N_m+1)+(k-m)-1$.

Proof. (i) Since the zero-divisor graph of a local ring that is not a field has a vertex that is adjacent to all others, the result is clear if $k=1$. The result is also clear if $\Gamma(R)$ is a star-graph.

Suppose $n \geq 2$, For each i, define $x_i = 1 \in R_i$ if R_i is a field or $R_i \cong \mathbb{Z}_4$ or $R_i \cong \mathbb{Z}_2[X]/(X^2)$; and otherwise choose $x_i \in V(\Gamma(R_i))$ such that x_i is adjacent to all other vertices. Let $D = \{(x_1, 0, \ldots, 0), (0, x_2, \ldots, 0), \ldots, (0, 0, \ldots, x_k)\}$. Clearly the set D forms a dominating set of $\Gamma(R)$ of minimum cardinality.

(ii) Let $C_i \subseteq V(\Gamma(R_i))$ be a set of vertices compromising a maximal clique of $\Gamma(R_i)$ for $i = 1, \ldots, m$. Define $W = \{(c_1, c_2, \ldots, c_m, 0, \ldots, 0) : c_i \in C_i \cup \{0\}\} - \{(0, \ldots, 0)\}$. Let $T = \{u_{m+1}, \ldots, u_k\}$ (where again we define u_i be the vector with 1 in coordinate i and 0 in all other coordinates). Then, the induced subgraph of $\Gamma(R)$ with vertex set $W \cup T$ is a complete subgraph of $\Gamma(R)$. □

The next result shows that $\omega(\Gamma(R))$ exceeds the number $\omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$ while as $\chi(\Gamma(R))$ exceeds the number $\chi(\Gamma(R_1)) + \chi(\Gamma(R_2))$. Further it shows that the equality between these numbers holds only if R_2 is reduced.

Theorem 6.6. *Let* R_1 *and* R_2 *be two rings that are not integral domains and let* $R \cong R_1 \times R_2$. *The following conditions hold.*

(i) $\omega(\Gamma(R)) \geq \omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$ *and* $\chi(\Gamma(R)) \geq \chi(\Gamma(R_1)) + \chi(\Gamma(R_2))$,
(ii) *If* R_2 *is reduced, then* $\omega(\Gamma(R)) = \omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$ *and* $\chi(\Gamma(R)) = \chi(\Gamma(R_1)) + \chi(\Gamma(R_2))$.

Proof. It can be easily proved that $\omega(\Gamma(R)) \geq \omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$. To prove that $\chi(\Gamma(R)) \geq \chi(\Gamma(R_1)) + \chi(\Gamma(R_2))$, we assume $s_1 = \chi(\Gamma(R_1)) < \infty$ and $s_2 = \chi(\Gamma(R_2)) < \infty$. If $\chi(\Gamma(R_1))$ or $\chi(\Gamma(R_2))$ is infinite, then the inequality is trivially satisfied. Thus, we require s_1 colors to color $\Gamma(R_1)$ and s_2 colors to color $\Gamma(R_2)$. Since each vertex of $\Gamma(R_1) = \Gamma(R_1) \times \{0\}$ is adjacent to every vertex of $\Gamma(R_2) = \{0\} \times \Gamma(R_2)$, therefore, we require at least $s_1 + s_2$ colors to color $\Gamma(R)$.

Now suppose R_2 is reduced. We first prove $\omega(\Gamma(R)) = \omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$. Let $w_1 = \omega(\Gamma(R_1)) < \infty$ and $w_2 = \omega(\Gamma(R_2)) < \infty$ and let

$$S = \{(a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t)\}$$

be a maximal clique of $\Gamma(R)$. Without loss of generality, suppose $b_i \neq 0$ for $i = 1, \ldots, s$ and $b_i = 0$ for $i > s$ (where $0 \leq s \leq t$). Then, $\{b_1, b_2, \ldots, b_s\}$ is a clique of $\Gamma(R_2)$, and so it follows $s \leq w_2$. On the other hand, since $b_i = 0$ implies $a_i \neq 0$ for $i > s$, $\{a_{s+1}, \ldots, a_t\}$ is a clique of $\Gamma(R_1)$. Therefore, $t - s \leq w_1$ and hence $|S| = t = s + (t-s) \leq w_1 + w_2$, that is, $\omega(\Gamma(R)) \leq \omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$. Therefore, by (i), $\omega(\Gamma(R)) = \omega(\Gamma(R_1)) + \omega(\Gamma(R_2))$.

Moreover, Let $c_1 = \chi(\Gamma(R_1)) < \infty$ and $c_2 = \chi(\Gamma(R_2)) < \infty$. Clearly there are independent sets $W_1, W_2, \ldots, W_{c_1}$ of the graph $\Gamma(R_1)$ and independent sets S_1, $S_2, \ldots, S_{c_2}$ of the graph $\Gamma(R_2)$. So, $\Gamma(R_1)$ is a disjoint union of sets $W_1, W_2, \ldots, W_{c_1}$ and $\Gamma(R_2)$ is a disjoint union of $S_1, S_2, \ldots, S_{c_2}$.

To finish the proof we need to color the vertices of $\Gamma(R)$. First observe that the sets $(\Gamma(R_1) \times S_i)$ are independent for every i, $(1 \leq i \leq c_2)$, because for $(a_j, b_j) \in (\Gamma(R_1) \times S_i)$, $j = 1, 2$, if $(a_1, b_1)(a_2, b_2) = (0, 0)$ then $a_1 a_2 = 0$ and $b_1 b_2 = 0$. So for $b_1 = b_2, b_1^2 = 0$, which implies $b_1 = 0$, a contradiction (as R_2 is reduced). Further, it can be easily checked that the sets $U(R_1) \times Z(R_2)$, $Z(R_1) \times U(R_2)$ and $\{0\} \times S_1, \{0\} \times S_2, \ldots, \{0\} \times S_{c_2}$ are independent. Thus, $\Gamma(R)$ is a disjoint union of sets $W_1 \times \{0\}, W_2 \times \{0\}, \ldots, W_{c_1} \times \{0\}, (\Gamma(R_1) \times S_1), (\Gamma(R_1) \times S_2), \ldots, (\Gamma(R_1) \times S_{c_2})$, $U(R_1) \times Z(R_2)$, $Z(R_1) \times U(R_2)$ and $\{0\} \times S_1, \{0\} \times S_2, \ldots, \{0\} \times S_{c_2}$. Clearly no vertex of $U(R_1) \times Z(R_2)$ is adjacent to any vertex of $W_1 \times \{0\}$. So, we can use the same color for the vertices of sets $U(R_1) \times Z(R_2)$ and $W_1 \times \{0\}$. We can say same for the vertices of $Z(R_1) \times U(R_2)$ and $(\Gamma(R_1) \times S_1)$. Also no vertex of $\{0\} \times S_1$ is adjacent to any vertex of $(\Gamma(R_1) \times S_1)$. Therefore, we can use the same color for the vertices of $\{0\} \times S_1$ and $(\Gamma(R_1) \times S_1)$. Thus, it follows that we can use at most $c_1 + c_2$ colors such that every two adjacent vertices have different colors. Therefore, $\chi(\Gamma(R)) \leq \chi(\Gamma(R_1)) + \chi(\Gamma(R_2))$. □

Corollary 6.7. *Let $R \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is an integral domain for each i $(1 \leq i \leq k)$. Then $\chi(\Gamma(R)) = \omega(\Gamma(R)$.*

Acknowledgments

We thank the anonymous referee for his useful comments. This research is supported by the University Grants Commission (Govt. of India) research project MRP-MAJOR-MATH-2013-8034.

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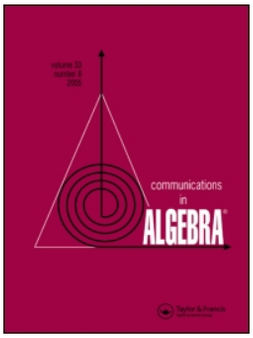

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

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To cite this article: S. Pirzada & Rameez Raja (2017) On the metric dimension of a zero-divisor graph, Communications in Algebra, 45:4, 1399-1408, DOI: 10.1080/00927872.2016.1175602

To link to this article: http://dx.doi.org/10.1080/00927872.2016.1175602

Accepted author version posted online: 07 Oct 2016.
Published online: 07 Oct 2016.

COMMUNICATIONS IN ALGEBRA®
2017, VOL. 45, NO. 4, 1399–1408
http://dx.doi.org/10.1080/00927872.2016.1175602

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On the metric dimension of a zero-divisor graph

S. Pirzada and Rameez Raja

Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India

ABSTRACT
Let R be a commutative ring with unity 1 and let $G(V, E)$ be a simple graph. In this research article, we study the metric dimension in zero-divisor graphs associated with commutative rings. We show that for a given rational $q \in (0, 1)$, there exists a finite graph G such that the ratio $\frac{dim_M(G)}{|V(H)|} = q$, where H is any induced connected subgraph of G. We provide a metric dimension formula for a zero-divisor graph $\Gamma(R \times \mathbb{F}_q)$ and give metric dimension of the zero-divisor graph $\Gamma(R_1 \times R_2 \times \cdots \times R_n)$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings with each having unity 1 and none of R_i, $1 \leq i \leq n$, being isomorphic to the Boolean ring $\prod_{i=1}^{n} \mathbb{Z}_2$. We discuss the metric dimension of Cartesian product of zero-divisor graphs and show that there exists a zero-divisor graph $\Gamma(R_1) \times \Gamma(R_2)$ such that $dim_M(\Gamma(R_1) \times \Gamma(R_2))$ lies between the numbers $dim_M(\Gamma(R_1))$ and $dim_M(\Gamma(R_1)) + 1$, where R_1 and R_2 are any two finite commutative rings (not domains) with each having unity 1.

ARTICLE HISTORY
Received 19 September 2015
Revised 18 March 2016
Communicated by
J. L. Gomez Pardo

KEYWORDS
Metric dimension; ring; zero-divisor; zero-divisor graph

AMS MATHEMATICS SUBJECT CLASSIFICATION
13A99; 05C78; 05C12

1. Introduction

The metric dimension has been extensively studied by various authors especially in the last decade. The problem of finding the metric dimension of a graph was first studied by Harary and Melter [13]. The metric dimension has been discovered or invented in different forms and has appeared in different applications including combinatorial optimization [22], strategies for the master mind game [9], network discovery and verification [8], robot navigation, [15] and so on. It is noted in [12, 15] that determining the metric dimension of a graph is an NP-complete problem.

The concept of undirected zero-divisor graph of a commutative ring was first introduced by Beck [7]. In his work, all the elements of a ring R were the vertices of the graph, and he was mainly interested in coloring. This work was further studied by Anderson and Naseer [5]. A different approach of associating a graph $\Gamma(R)$ to R with vertices as $Z^*(R) = Z(R)\backslash\{0\}$ was given by Anderson and Livingston in [4]. Two vertices $x, y \in Z^*(R)$ of $\Gamma(R)$ are adjacent if and only if $xy = 0$. They believed that this better illustrates the zero-divisor structure of the ring. The zero-divisor graph of a commutative ring has also been studied in [1–3, 5, 16, 18, 19] and was extended by Redmond [20] to noncommutative rings. Redmond [21] also extended the zero-divisor of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$ with vertex set $\{x \in R - I \mid xy \in I \textit{ for some } y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

We adopt the approach used by Anderson and Livingston in [4] and consider only nonzero zero-divisors as the vertices of the graph $\Gamma(R)$. We denote the ring of integers by $\mathbb{Z}$, ring of integers modulo n by $\mathbb{Z}_n$ and the field on q number of elements by $\mathbb{F}_q$. For ring theory, we refer to [6, 14], and for graph theory, we refer to [11, 17, 23].

CONTACT S. Pirzada pirzadasd@kashmiruniversity.ac.in Department of Mathematics, University of Kashmir, Srinagar 190006, Kashmir, India.

2. Metric dimension of simple graphs

A simple graph $G(V, E)$ consists of a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. A graph G is connected if there is a path between every two distinct vertices of G. The distance from a vertex v to u denoted by $d(v, u)$ is the length of the shortest path from v to u ($d(v, v) = 0$ and $d(v, u) = \infty$, if there is no such path). The diameter of G is $diam(G) = sup\{d(v, u) \mid v, u \in V(G)\}$. The neighborhood $N(v)$ of a vertex v denotes the set all vertices of G adjacent to the vertex v and $N[v] = N(v) \cup \{v\}$.

Below, we study the concept of metric dimension in simple connected graphs. We give an example which illustrates the concept in G. We consider the ratio $\frac{dim_M(G)}{|V(H)|}$, where H is an induced connected subgraph of G. We show that there exists a finite graph G such that the ratio $\frac{dim_M(G)}{|V(H)|}$ is arbitrary small which would indicate graphs with smaller metric basis and which in turn would tend to indicate fewer distance similar vertices. We also explore the connection between clique number and metric dimension in the graph G.

Definition 2.1. A set of vertices $S \subseteq V(G)$ resolves a graph G, and S is a resolving set of G, if every vertex is uniquely determined by its vector of distances to the vertices of S. In general, for an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of vertices in a connected graph G and a vertex $v \in V(G)\backslash S$ of G, the metric representation of v with respect to S is the k-vector $D(v|S) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. The set S is a resolving set for G if $D(v|S) = D(u|S)$ implies that $v = u$, for all pair of vertices, $v, u \in V(G)\backslash S$. Equivalently, S is a resolving set for G if $D(v|S) \neq D(u|S)$, for all pair of distinct vertices $v, u \in V(G)\backslash S$. A resolving set S of minimum cardinality is the metric basis for G, and the number of elements in the resolving set of minimum cardinality is the metric dimension of G. We denote the metric dimension of a graph G by $dim_M(G)$.

The resolving set is also called the locating set, metric representation of a vertex is also called the locating code of a vertex and the metric dimension of a graph is also called the locating number of a graph.

Clearly, the study of the metric dimension in a graph G is influenced by the symmetry in that graph, and in this connection, we have the following definition.

Definition 2.2. Let G be a connected graph with order $n \geq 2$. Two distinct vertices are distance similar, if $d(u, x) = d(v, x)$, for all $x \in V(G)\backslash\{u, v\}$. It can be easily checked that the distance relation ($\sim$) is an equivalence relation on $V(G)$ and two vertices are distance similar if either $uv \notin E(G)$ and $N(u) = N(v)$ or $uv \in E(G)$ and $N[u] = N[v]$.

Consider the graph G of order $n = 11$, shown in Fig. 1. Three of the distance similar equivalence classes are $V_1 = \{u_1, w_1\}$, $V_2 = \{u_2, w_2\}$, and $V_3 = \{u_3, w_3\}$. Each of the remaining five classes consists of a single vertex. Thus, there are $k = 8$ equivalence classes and so by [Theorem 2.1, [18]], $dim_M(G) \geq n-k \geq 3$. We may assume that any resolving set for G contains w_1, w_2, and w_3. In fact, $S = \{w_1, w_2, w_3\}$ is a resolving set and consequently the metric basis for the graph G. Therefore, $dim_M(G) = n-k = 3$. The metric representations for the vertices of $V(G)\backslash S$ with respect to S are: $D(u_1|S) = (1, 3, 4)$, $D(u_2|S) = (3, 2, 3)$, $D(u_3|S) = (4, 3, 1)$, $D(u_4|S) = (1, 2, 3)$, $D(u_5|S) = (1, 3, 3)$, $D(u_6|S) = (2, 1, 2)$, $D(u_7|S) = (2, 2, 2)$, and $D(u_8|S) = (3, 2, 1)$.

A graph G is said to be *complete* if there is an edge between every pair of distinct vertices. A complete graph with n vertices is denoted by K_n. A graph G is said to be bipartite if its vertex set can be partitioned into two sets $V_1(G)$ and $V_2(G)$ such that every edge of G has one end in $V_1(G)$ and another in $V_2(G)$. A complete bipartite graph is one whose each vertex of one partite set is joined to every vertex of another partite set. We denote a complete bipartite graph with partite sets of order m and n by $K_{m,n}$. A complete bipartite graph of the from $K_{1,n}$ is called a star graph.

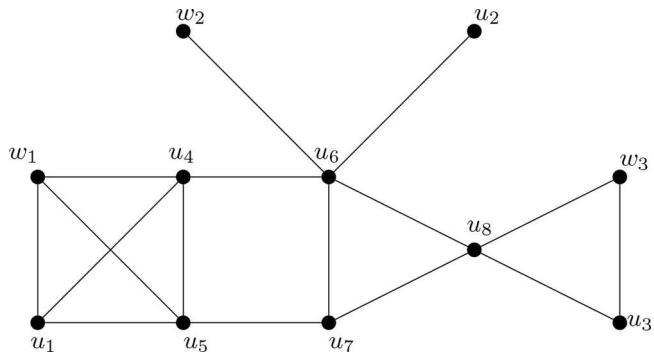

Figure 1. The graph of order 11 with 8 distance similar equivalence classes.

The ratio $\frac{dim_M(G)}{|V(G)|}$ gives the measure of distance symmetry in a graph G. If the ratio is close to 1, this indicates the larger metric basis for the graph G, which would tend to indicate more distance similar classes. If the ratio is close to 0, this indicates a proportionally smaller metric basis for the graph G which would indicate a fewer distance similar vertices. In the following result, we show that there exists a finite graph G such that $\frac{dim_M(G)}{|V(G)|}$ is a rational in $(0, 1)$. In fact, for every $\epsilon > 0$, we show that there exists a finite graph G such that $\frac{dim_M(G)}{|V(G)|} \leq \epsilon$.

Proposition 2.3. *Given a rational $q \in (0, 1)$, there exists a connected graph G such that $\frac{dim_M(G)}{|V(H)|} = q$, where H is a connected induced subgraph of G. In particular, if H is a star graph $K_{1,2n}$, then $\frac{dim_M(G)}{|V(H)|} = \frac{2n}{2n+1}$.*

Proof. The result is trivial if G is a path P_n or a cycle C_n or a complete graph K_n or a complete bipartite graph $K_{m,n}$.

Let H be a star graph $K_{1,2n}$ $(n \geq 2)$, with vertex v_0 adjacent to the vertices $v_1, v_2, \ldots, v_{2n}$. Let K be another graph on $2n$ vertices $v_1^*, v_2^*, \ldots, v_{2n}^*$ distinct from the vertices $v_1, v_2, \ldots, v_{2n}$ with v_0 adjacent to v_i^*, for all $1 \leq i \leq 2n$, and no vertex v_i^* is adjacent to v_j^*, for all $1 \leq i, j \leq 2n$. Add two new vertices u and u^* and the edges uv_i and $u^*v_i^*$ for all i, $1 \leq i \leq 2n$.

Further, we add two sets of vertices $S = \{s_1, s_2, \ldots, s_n\}$ and $S^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$ together with the edges s_iu and $s_i^*u^*$, for $1 \leq i \leq n$. Moreover, we add edges between S and $v_1, v_2, \ldots, v_{2n}$, so that each of the $2n$ possible n-vectors of $1's$ and $2's$ appear exactly once as $D(v_i|S)$, for $1 \leq i \leq 2n$. Similarly, edges are added between S^* and $v_1^*, v_2^*, \ldots, v_{2n}^*$, so that the metric representation $D(v_i^*|S)$ are distinct for all i, $1 \leq i \leq n$. Denote the resulting graph by G.

Now, we proceed to determine the resolving set for the graph G.

Claim: $S \bigcup S^*$ is a resolving set for G.

Observe that

$$D\left(v_0 \mid S \bigcup S^*\right) = (2, 2, \ldots, 2, 2 \ldots, 2),$$

$$D\left(v_i \mid S \bigcup S^*\right) = (-, -, \ldots, -, 3, 3, \ldots, 3), \quad 1 \leq i \leq 2n,$$

$$D\left(v_i^* \mid S \bigcup S^*\right) = (3, 3, \ldots, 3, -, -, \ldots, -), \quad 1 \leq i \leq 2n,$$

$$D\left(u \mid S \bigcup S^*\right) = (1, 1, \ldots, 1, 4, \ldots, 4) \quad \text{and}$$

$$D\left(u^* \mid S \bigcup S^*\right) = (4, 4, \ldots, 4, 1, \ldots, 1).$$

Thus, all the vertices of $V(G)\backslash S\bigcup S^*$ have distinct metric representations with respect to $S\bigcup S^*$. This establishes our claim and therefore $S\bigcup S^*$ is a resolving set of cardinality $2n$ for G. Since G contains $H = K_{1,2n}$ as an induced connected subgraph, it follows that

$$\frac{dim_M(G)}{|V(H)|} = \frac{2n}{2n+1} \qquad \text{and also } \lim_{n\to\infty} \frac{2n}{2n+1} = 1. \qquad \square$$

Corollary 2.4. *For every $\epsilon > 0$, there exists a connected graph G such that $\frac{dim_M(G)}{|V(H)|} \leq \epsilon$, where H is a connected induced subgraph of G.*

Proof. Replace the star graph $K_{1,2n}$ $(n \geq 2)$, by the star graph $K_{1,2^n}$ $(n \geq 3)$, and follow the same procedure as in Proposition 2.3, we eventually get $K_{1,2^n}$ as an induced connected subgraph of the resulting graph G. Therefore, we have

$$\frac{dim_M(G)}{|V(H)|} = \frac{2n}{2^n+1} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{2n}{2^n+1} = 0. \qquad \square$$

Recall that a clique in a graph G is an induced complete subgraph. The clique number denoted by $\omega(G)$ is the greatest integer $n \geq 1$ such that $K_n \subseteq G$. The following result relates clique number, diameter, and the metric dimension of a simple connected graph G.

Theorem 2.5. *Let m be a positive integer. Then there is some positive integer n with $2^{n-1} < m < 2^n$ and a finite connected graph G with $\mid V(G) \mid = m + n$, $diam(G) \leq 3$, $\omega(G) = m$, and $dim_M(G) \leq n$.*

Proof. Let H be a complete graph on m vertices $\{v_1, v_2, \ldots, v_m\}$. Assign to each vertex v_i a unique n-digit binary number $k_i = a_{i,1}a_{i,2}\cdots a_{i,n}$. Construct a graph with vertex set $V(G) = V(H)\bigcup\{w_1, w_2, \ldots, w_n\}$ with H as an induced subgraph and where w_i, $(1 \leq i \leq n)$, is adjacent to v_j, $(1 \leq j \leq m)$, if and only if $a_{j,i} = 0$. Thus, $S = \{w_1, w_2, \ldots, w_n\}$ is the resolving set for G, since $D(v_j|S) = (a_{i,1}a_{i,2}\cdots a_{i,n}) + (1, 1, \ldots, 1)$. The other assertions of the result can be easily verified. $\square$

The following example illustrates Theorem 2.5. Consider the graph G shown in Fig. 2 which has been obtained by assigning each vertex v_1, v_2, v_3, v_4 a unique binary two-digit number $a_{i,1}a_{i,2}$ with $a_{i,1}a_{i,2} \in \mathbb{Z}_2$, for all i, $1 \leq i \leq 4$. The vertex set of G is $V(G) = V(H)\bigcup\{w_1, w_2\}$, where H is an induced complete subgraph of G on the four vertices v_1, v_2, v_3, v_4 and vertices w_1, w_2 are adjacent to the vertices v_1, v_2, v_3 since $a_{11} = a_{12} = a_{22} = a_{31} = 0$. Clearly, G is a connected graph with $|V(G)| = 4 + 2 = 6$, $diam(G) = 2$, $\omega(G) = 4$, and $dim_M(G) = 2$. The metric basis for the graph G is $S = \{w_1, w_2\}$, and the metric representations of vertices are: $D(v_1|S) = (1, 1)$, $D(v_2|S) = (2, 1)$, $D(v_3|S) = (1, 2)$, and $D(v_4|S) = (2, 2)$.

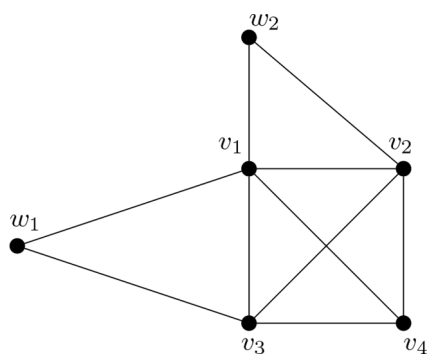

Figure 2. A graph with diam $(G) = 2$, $\omega(G) = 4$, and $\dim_M(G) = 2$.

3. Metric dimension of zero-divisor graphs

In this section, we show that metric dimension of the graph $\Gamma(R)$ is finite if and only if R is finite. We give an upper bound for the number of zero-divisors in a finite commutative ring with unity 1 whose metric dimension is any non-negative integer. We study metric dimension in terms of distance similar equivalence classes and provide a combinatorial formula for metric dimension of the zero-divisor graph $\Gamma(R \times \mathbb{F}_q)$. We investigate metric dimension in $\Gamma(R_1 \times R_2 \times \cdots \times R_n)$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings each having unity 1 and none of R_i, $(1 \leq i \leq n)$, being isomorphic to the Boolean ring $\prod_{i=1}^{n} \mathbb{Z}_2$. We discuss metric dimension of the Cartesian product $\Gamma(R_1) \times \Gamma(R_2)$ of graphs $\Gamma(R_1)$ and $\Gamma(R_2)$, where R_1 and R_2 are any two finite commutative rings (not domains) with each having unity 1.

The concept of the resolving set, the metric representation and the metric dimension in terms of the locating set, the locating code, and the locating number in a zero-divisor graph associated with a commutative ring with unity was introduced in [18] and had been further studied in [19]. The authors in [18, 19] have discussed various properties of the locating set and the locating number which includes the characterization of all finite commutative rings with unity, examination of two equivalence relations on the vertices of $\Gamma(R)$, relationship between the locating set and the cut vertices of $\Gamma(R)$, investigation of the locating number in $\Gamma(R)$ when R is a finite product of the integral domains and so on.

Let G_k be a graph on infinite number of vertices (e.g., an infinite tree) with vertex set $V(G_k) = \{v\} \cup \{v_1^{(1)}, v_2^{(1)}, \ldots, v_k^{(1)}\} \cup \{v_1^{(2)}, v_2^{(2)}, \ldots, v_k^{(2)}\} \cup \cdots \cup \{v_1^{(i)}, v_2^{(i)}, \ldots, v_k^{(i)}\} \cup \cdots$ for $i \geq 1$, and the edges are defined by the rule $vv_t^{(1)}$, $(1 \leq t \leq k)$, and $v_1^{(j)}v_1^{(j+1)}$, $v_2^{(j)}v_2^{(j+1)}$, ..., $v_k^{(j)}v_k^{(j+1)}$ for all $j = 1, 2, 3, \ldots$. For $k = 1$, G_1 is an infinite tree with vertex set $V(G_1) = \{v\} \cup \{v_1^{(1)}\} \cup \{v_1^{(2)}\} \cup \cdots \cup \{v_1^{(i)}\} \cup \cdots$ for $i \geq 1$, and the edges are defined by $vv_1^{(1)}$ and $v_1^{(j)}v_1^{(j+1)}$ for all $j = 1, 2, 3, \ldots$. Notice, here that the infinite tree G_1 is rooted at the vertex v. For $k = 2$, G_2 is a graph with vertex set $V(G_2) = \{v\} \cup \{v_1^{(1)}, v_2^{(1)}\} \cup \{v_1^{(2)}, v_2^{(2)}\} \cup \cdots \cup \{v_1^{(i)}, v_2^{(i)}\} \cup \cdots$ for $i \geq 1$, and the edges are defined by $vv_t^{(1)}$, $(1 \leq t \leq 2)$, and $v_1^{(j)}v_1^{(j+1)}$, $v_2^{(j)}v_2^{(j+1)}$ for all $j = 1, 2, 3, \ldots$. Clearly, G_2 is a tree rooted at the vertex v with two infinite branches. The infinite trees G_k are often denoted by $P_{k,1}$ to indicate trees rooted in v with k infinite branches. It is straight forward to prove that $dim_M(P_{1,1}) = 1$ and $dim_M(P_{2,1}) = 2$. Thus, it is possible for a graph on infinite number of vertices to have finite metric dimension. However, it has been proved in [18] that if both diameter and metric dimension of a graph G are finite, then the graph G is also finite. The following result characterizes finite commutative rings with unity 1 and gives an upper bound for the number of zero-divisors in a finite commutative ring R with unity 1, of course with R not being an integral domain, since the zero-divisor graph of an integral domain is empty.

Proposition 3.1. *Let R be a commutative ring with unity 1 (not a domain). Then $dim_M(\Gamma(R))$ is finite if and only if R is finite.*

Proof. Suppose R is finite. Then, it is clear that $dim_M(\Gamma(R))$ is finite.

Now, suppose $dim_M(\Gamma(R))$ is finite. Let S be the metric basis for $\Gamma(R)$ with $|S| = k$, where k is some non-negative integer. By [Theorem 2.3, [4]], the diameter of $\Gamma(R)$ is not more than 3. Therefore, $d(x, y) \in \{0, 1, 2, 3\}$ for every $x, y \in Z^*(R)$. For each $x \in Z^*(R)$, the metric representation $D(x|S)$ is the k-coordinate vector, where each coordinate is in the set $\{0, 1, 2, 3\}$. Thus there are only $(3+1)^k$ possibilities for $D(x|S)$. Since $D(x|S)$ is unique for each $x \in Z^*(R)$, so $|Z^*(R)| \leq 4^k$. This implies that $Z^*(R)$ is finite and hence R is finite. □

Corollary 3.2. *Let R be a commutative ring with unity 1 (not a domain) such that $dim_M(\Gamma(R)) = k$, where k is any non-negative integer. Then $|Z(R)| \leq 4^k + 1$.*

Theorem 6.1 [19] gives the metric dimension for the zero-divisor graph $\Gamma(R_1 \times R_2 \times \ldots \times R_n)$, where $R_1, R_2, \ldots, R_n$ are integral domains, and also gives bounds for the metric dimension of

the zero-divisor graph $\Gamma(\prod_{i=1}^{n}\mathbb{Z}_2)$. Special emphasis has been given to the graph $\Gamma(\prod_{i=1}^{n}\mathbb{Z}_2)$ of a finite Boolean ring, and it is shown that $dim_M(\Gamma(\prod_{i=1}^{n}\mathbb{Z}_2)) \leq n$, $dim_M(\Gamma(\prod_{i=1}^{n}\mathbb{Z}_2)) = n-1$, for $n = 2, 3, 4$, and $dim_M(\Gamma(\prod_{i=1}^{n}\mathbb{Z}_2)) = n$, for $n = 5$ [19]. We need to know as how the metric dimension behaves with respect to the product $R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings with each having unity 1. This is addressed in the following result.

Lemma 3.3. *A finite commutative ring R with unity 1 has exactly one unit if and only if $R \cong \prod_{i=1}^{n}\mathbb{Z}_2$ for some positive integer n.*

Proof. Clearly, the ring listed has only one unit.

Suppose R has exactly one unit. If R is a local ring with maximal ideal M, then $|R| = p^k$ and $|M| = p^m$ for some prime p and integers $0 \leq m < k$. Then, $1 = |U(R)| = |R| - |M|$ only when $|R| = 2$ and $|M| = 1$. If R is not local, then R can be written as the finite product of local rings that is $R \cong R_1 \times R_2 \times ... \times R_n$, where $R_1, R_2, \ldots, R_n$ are finite local rings. If any R_i has more than one unit, then R would have more than one unit. Hence, the result follows. □

Lemma 3.4. *Let R be a finite commutative local ring with unity 1 and for prime q let $\mathbb{F}_q$ be a finite field. Then $|Z^*(R \times \mathbb{F}_q)| = |U(R)| + (|Z^*(R)| + 1)q - 1$.*

Proof. Let R be a finite commutative local ring with unity 1. We consider three cases.

Case 1. $R \cong \mathbb{F}_p$, for some prime p. Then, we have $|Z^*(R)| = 0$ and $|U(R)| = p - 1$. Therefore, $|Z^*(R \times \mathbb{F}_q)| = p + q - 2$.

Case 2. $R \cong \mathbb{Z}_{p^k} (k \geq 2)$. Then, for every $a \in U(R)$, we have $\{(a, 0)\} \subset Z^*(R \times \mathbb{F}_q)$. Also, $\{(0, x^i)\} \subset Z^*(R \times \mathbb{F}_q)$ for all $x^i \in \mathbb{F}_q$ with $0 \leq i \leq q-2$. Moreover, for any unit u in R, the nonzero zero-divisors of $R \times \mathbb{F}_q$ other than the above are given by the sets,

$$\begin{aligned}
Z_1 &= \{(up, 0), (up^2, 0), \ldots, (up^{k-1}, 0)\},\\
Z_2 &= \{(up, 1), (up, x), \ldots, (up, x^{q-2})\},\\
Z_3 &= \{(up^2, 1), (up^2, x), \ldots, (up^2, x^{q-2})\},\\
&\ldots,\\
Z_k &= \{(up^{k-1}, 1), (up^{k-1}, x), \ldots, (up^{k-1}, x^{q-2})\}.
\end{aligned}$$

Thus, $Z^*(R \times \mathbb{F}_q) = \{(a, 0)\} \bigcup \{(0, x^i)\} \bigcup Z_1 \bigcup Z_2 \bigcup \cdots \bigcup Z_k$, which implies that

$$\begin{aligned}
|Z^*(R \times \mathbb{F}_q)| &= |U(R)| + q - 1 + |Z^*(R)|q\\
&= p^k - p^{k-1} + (p^{k-1} - 1 + 1)q - 1\\
&= p^k + (q-1)p^{k-1} - 1.
\end{aligned}$$

Case 3. *R is a local ring other than $\mathbb{Z}_{p^k}$ and $\mathbb{F}_p$.* Corbas and Williams in [10] proved that the rings of orders p^2 and p^3 are precisely the rings: $\mathbb{F}_{p^2}, F_p[X]/(X^2)$, and $\mathbb{Z}_{p^2}$ and $\mathbb{F}_{p^3}, \mathbb{F}_p[X, Y]/(X, Y)^2, \mathbb{F}_p[X]/(X^3)$ $\mathbb{Z}_p[X]/(pX, X^2)$, and $\mathbb{Z}_{p^2}[X]/(pX, X^2 - sp)$, where s is a non-square in $\mathbb{F}_p \backslash \{0\}$. For full characterizations of local rings of order p^k see [10]. We show that the result holds for any local ring R with $|R| = p^k$ $(k \geq 2)$. For each $a \in U(R)$, clearly $\{(a, 0)\} \subset Z^*(R \times \mathbb{F}_q)$ and $\{(0, x^i)\} \subset Z^*(R \times \mathbb{F}_q)$ for all $x^i \in \mathbb{F}_q$ with $0 \leq i \leq q-2$. Let $x_1, x_2, \ldots, x_r$ be all nonzero zero-divisors of R. Then the nonzero zero-divisors of $R \times \mathbb{F}_q$ other than the above are given by the sets,

$$\begin{aligned}
Z_1^\star &= \{(x_1, 0), (x_2, 0), \ldots, (x_r, 0)\},\\
Z_2^\star &= \{(x_1, 1), (x_1, x), \ldots, (x_1, x^{q-2})\},\\
Z_3^\star &= \{(x_2, 1), (x_2, x), \ldots, (x_2, x^{q-2})\},\\
&\ldots,\\
Z_t^\star &= \{(x_r, 1), (x_r, x), \ldots, (x_r, x^{q-2})\}.
\end{aligned}$$

Now, proceeding as in case 2 above, we have

$$|Z^*(R \times \mathbb{F}_q)| = |U(R)| + (|Z^*(R)| + 1)q - 1. \qquad \square$$

Example 3.5. Let $R = \mathbb{F}_2[X]/(X^3, XY, Y^2) = \{aX^2 + bX + cY + d \mid a, b, c, d \in \mathbb{F}_2\}$ and let $\mathbb{F}_q = \mathbb{Z}_5$. Clearly, R is a local ring of order 16 with $Z(R) = \{0, x, y, x^2, x+y, x+x^2, y+x^2, x+x^2+y\}$. Therefore, by Lemma 3.4, $|Z^*(R \times \mathbb{F}_q)| = |U(R)| + (|Z^*(R)| + 1)q - 1 = 8 + (8)5 - 1 = 47$.

Theorem 3.6. *Let $R_1, R_2, \ldots, R_n$ be n finite commutative rings (not domains) each having unity 1 with none of R_i, $1 \le i \le n$, being isomorphic to $\prod_{i=1}^{n} \mathbb{Z}_2$ for any positive integer n. Then for a finite commutative ring R with unity 1 and for a finite field $\mathbb{F}_q$ on prime q number of elements,*
(a) $dim_M(\Gamma(R_1 \times R_2 \times \cdots \times R_n)) \ge \sum_{i=1}^{n} dim_M(\Gamma(R_i))$,
(b) $dim_M(\Gamma(R \times \mathbb{F}_q)) = |Z^*(R \times \mathbb{F}_q)| - 2^{n+1} + 2$ *or* $|Z^*(R \times \mathbb{F}_q)| - 2$ *or at least* $|U(R)| + (|Z^*(R)| + 1)q - t - 3$, *where t is any positive integer.*

Proof. To prove (a), we take $n = 2$, and the general result follows by induction. Let $V_1, V_2, \ldots, V_k$ be the partition of $V(\Gamma(R_1))$ into distance similar equivalence classes and let $W_1, W_2, \ldots, W_m$ be the partition of $V(\Gamma(R_2))$ into distance similar equivalence classes. Further, let $T_1, T_2, \ldots, T_r$ be the partition of $V(\Gamma(R_1 \times R_2))$ into distance similar equivalence classes. Let $x \in V_1$. We consider two cases.

Case 1. $|V_1| > 1$. Then, for $t \in V_1 \backslash \{x\}$, clearly $x \sim t$ implies $(x, 0) \sim (t, 0)$ since for $a \in R_1 \backslash \{x, t, 0\}$, $(x, 0)(a, b) = (0, 0) \Leftrightarrow xa = 0 \Leftrightarrow at = 0 \Leftrightarrow (t, 0)(a, b) = (0, 0)$. Also, $(x, 1) \sim (t, 1)$ since for $a \in R_1 \backslash \{x, t, 0\}$, $(x, 1)(a, b) = (0, 0)) \Leftrightarrow xa = 0$ and $b = 0 \Leftrightarrow at = 0$ and $b = 0 \Leftrightarrow (t, 1)(a, b) = (0, 0)$.

Case 2. $|V_1| = 1$. Then, for every $u_1, u_2 \in U(R_2)$, $(x, u_1) \sim (x, u_2)$ since $(x, u_1)(a, b) = (0, 0) \Leftrightarrow xa = 0$ and $b = 0 \Leftrightarrow (x, u_2)(a, b) = (0, 0)$.

An analogous argument to that in Cases 1 and 2 can be applied to any V_i for $1 \le i \le k$, or W_j for $1 \le j \le m$.

Cases 1 and 2 show that for any V_i, there is some T_l for $1 \le l \le r$, with $|V_i| < |T_l|$. Similarly, for any W_j, there is some T_s with $|W_j| < |T_s|$. Thus, by [Theorem 2.1, [18]], $dim_M(\Gamma(R_1 \times R_2)) \ge dim_M(\Gamma(R_1)) + dim_M(\Gamma(R_2))$.

To see (b), let R be a finite commutative ring with unity 1. We consider four cases.
Case 1. *R is a reduced ring.* Then, $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \ldots \times \mathbb{F}_n$. Therefore, by [Theorem 6.1, [19]], $dim_M(\Gamma(R \times \mathbb{F}_q)) = |Z^*(R \times \mathbb{F}_q)| - 2^{n+1} + 2$.

Case 2. *R is a field.* Then we have $n = 1$ in Case 1. Clearly, $\Gamma(R \times \mathbb{F}_q)$ is either a star graph or a complete bipartite graph on $|R| + |\mathbb{F}_q| - 2$ number of vertices. Therefore, by [Theorem 3.2, [18]], the metric dimension of the graph $\Gamma(R \times \mathbb{F}_q)$ is $|Z^*(R \times \mathbb{F}_q)| - 2$. If $R \cong \mathbb{Z}_2 \cong \mathbb{F}_q$, then $dim_M(\Gamma(R \times \mathbb{F}_q)) = 1$.

Case 3. $R \cong \mathbb{Z}_{p^k} (k \ge 2)$. In this case, we partition the graph into the distance similar equivalence classes. By Lemma 3.4, the distance similar classes in $\Gamma(R \times \mathbb{F}_q)$ for any unit $u \in U(R)$ with $1 \ne u$ are given by sets,

$$\begin{aligned}
&\{(a, 0) | a \in U(R)\},\\
&\{(0, x^i) | x^i \in \mathbb{F}_q \textit{ for all } i,\ 0 \le i \le q-2\},\\
&Z'_1 = \{(p, 0), (up, 0)\},\\
&Z'_2 = \{(p^2, 0), (up^2, 0)\},\\
&\ldots,\\
&Z'_l = \{(p^{k-1}, 0), (up^{k-1}, 0)\},\\
&Z'_{l+1} = \{(p, 1), (p, x), \ldots, (p, x^{q-2}), (up, 1), (up, x), \ldots, (up, x^{q-2})\},\\
&\ldots,\\
&Z'_t = \{(p^{k-1}, 1), (p^{k-1}, x), \ldots, (p^{k-1}, x^{q-2}), (up^{k-1}, 1), (up^{k-1}, x), \ldots,\\
&(up^{k-1}, x^{q-2})\}.
\end{aligned}$$

For $u = 1$,

$$\begin{aligned}
&Z'_1 = \{(p,0)\},\\
&Z'_2 = \{(p^2,0)\},\\
&\ldots,\\
&Z'_l = \{(p^{k-1},0)\},\\
&Z'_{l+1} = \{(p,1),(p,x),\ldots,(p,x^{q-2})\},\\
&\ldots,\\
&Z'_t = \{(p^{k-1},1),(p^{k-1},x),\ldots,(p^{k-1},x^{q-2})\}.
\end{aligned}$$

Therefore, by [Theorem 2.1, [18]], $dim_M(\Gamma(R \times \mathbb{F}_q)) \geq |Z^*(R \times \mathbb{F}_q)| - (2+t)$. Hence, by Lemma 3.4 (Case 2),

$$\begin{aligned}
dim_M(\Gamma(R \times \mathbb{F}_q)) &\geq p^k + (q-1)p^{k-1} - 1 - (2+t)\\
&= p^k + (q-1)p^{k-1} - t - 3,
\end{aligned}$$

where t is the number of distance similar equivalence classes of the form $Z'_1, Z'_2, \ldots, Z'_t$.

Case 4. *R is a local ring other than $\mathbb{Z}_{p^k}$ and $\mathbb{F}_{p^k}(k \geq 1)$*. Then, by Case 3 of Lemma 3.4, $dim_M(\Gamma(R \times \mathbb{F}_q)) \geq |U(R)| + (|Z^*(R)| + 1)q - t - 3$, where t is the number of distance similar equivalence classes.

If R is any finite commutative ring with unity 1, then it is clear that $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are finite local rings. Using again Lemma 3.4, one can obtain the metric dimension formula for the graph $\Gamma(R \times \mathbb{F}_q)$.

Having exhausted all the possible cases for all finite commutative rings, we conclude that $dim_M(\Gamma(R \times \mathbb{F}_q)) = |Z^*(R \times \mathbb{F}_q)| - 2^{n+1} + 2$ or $|Z^*(R \times \mathbb{F}_q)| - 2$ or at least $|U(R)| + (|Z^*(R)| + 1)q - t - 3$. □

Example 3.7. Consider the zero-divisor graph of Fig. 3 associated with the ring $\mathbb{Z}_8 \times \mathbb{Z}_2$. Partition the graph $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)$ into the distance similar equivalence classes which are given by sets, $\{(1,0),(3,0),(5,0),(7,0)\}$, $\{(2,1),(6,1)\}$, $\{(0,1)\}$, $\{(2,0),(6,0)\}$, $\{(4,0)\}$, and $\{(4,1)\}$. Therefore, by Theorem 3.6, $dim_M(\mathbb{Z}_8 \times \mathbb{Z}_2) \geq 4 + (4)2 - 4 - 3 = 5$. From Fig. 3, it can be easily verified that $\{(1,0),(3,0),(5,0),(7,0),(2,0),(6,1)\}$ is the metric basis for the zero-divisor graph $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)$ and consequently $dim_M(\mathbb{Z}_8 \times \mathbb{Z}_2) = 6$.

Now, we study the metric dimension of the graph $\Gamma(R_1) \times \Gamma(R_2)$, where R_1 and R_2 are any two finite commutative ring (not domains) with each having unity 1. We define the Cartesian product $\Gamma(R_1) \times \Gamma(R_2)$ of two zero-divisor graphs $\Gamma(R_1)$ and $\Gamma(R_2)$, with vertex set as $Z^*(R_1) \times Z^*(R_2)$ and the edge set $\{(x_1,x_2)(y_1,y_2) \mid x_1 = y_1 \text{ and } x_2y_2 \in E(\Gamma(R_2)),\ \text{or } x_2 = y_2 \text{ and } x_1y_1 \in E(\Gamma(R_1))\}$. We show that there exists a zero-divisor graph $\Gamma(R_1) \times \Gamma(R_2)$ such that the metric dimension of this graph is any non-negative integer between the numbers $dim_M(\Gamma(R_1))$ and $dim_M(\Gamma(R_1)) + 1$.

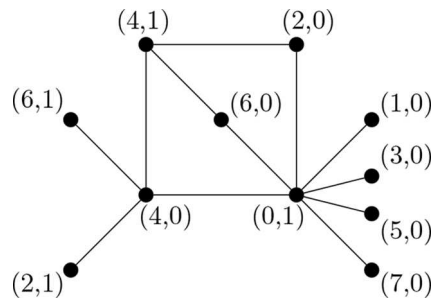

Figure 3. $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)$.

Example 3.8. Let $R_1 = \mathbb{Z}_6$ and $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $Z^*(\mathbb{Z}_6) = \{2, 3, 4\}$ and $Z^*(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 1), (1, 0)\}$. Let $\Gamma(\mathbb{Z}_6) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ be the resulting graph with vertex set $\{[2, (1, 0)], [2, (0, 1)], [3, (1, 0)], [3, (0, 1)], [4, (1, 0)], [4, (0, 1)]\}$ and the edge set $\{[2, (1, 0)][2, (0, 1)], [2, (1, 0)][3, (1, 0)], [2, (0, 1)][3, (0, 1)], [3, (1, 0)][4, (1, 0)], [3, (0, 1)], [4, (0, 1)], [3, (0, 1)][3, (1, 0)], [4, (1, 0)][4, (0, 1)]\}$. It can be seen by drawing the graph $\Gamma(\mathbb{Z}_6) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ that $dim_M(\Gamma(\mathbb{Z}_6) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 2 = dim_M(\Gamma(\mathbb{Z}_6)) + 1$.

Theorem 3.9. *If R is a finite commutative ring with unity 1 (not a domain), then*

$$\dim_M(\Gamma(R)) \leq dim_M(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) \leq dim_M(\Gamma(R)) + 1.$$

Proof. We partition the graph $\Gamma^*(R) = \Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ into the distance similar equivalence classes. Since the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ consists of only one unit, therefore each of the distance similar class consists of only one vertex. Let $\Gamma_1(R)$ and $\Gamma_2(R)$ be two isomorphic copies of $\Gamma(R)$ in $\Gamma^*(R)$. Let S be the metric bases for $\Gamma(R)$ and let $S_1 = \{s_1, s_2, \ldots, s_k\}$ and $S_2 = \{w_1, w_2, \ldots, w_k\}$ be metric bases of $\Gamma_1(R)$ and $\Gamma_2(R)$ corresponding to metric basis S of $\Gamma^*(R)$.

Claim: $W = S_1 \bigcup \{w_1\}$ is a resolving set for $\Gamma^*(R)$.

Let u and v be two distinct vertices of $\Gamma^*(R)$. We show that $D(u|W) \neq D(v|W)$. This is certainly the case if $u, v \in W$. Thus, we may assume that $u, v \notin W$. We have four cases to consider.

Case 1. *u and v are both in $\Gamma_1(R)$.* Then, $d_{\Gamma^*(R)}(u, s_i) = d_{\Gamma_1(R)}(u, s_i)$ and $d_{\Gamma^*(R)}(v, s_i) = d_{\Gamma_1(R)}(v, s_i)$ for $1 \leq i \leq k$. Suppose that $D(u|W) = D(v|W)$. Since S_1 is a resolving set for $\Gamma_1(R)$, it follows that $d_{\Gamma_1(R)}(u, s_i) \neq d_{\Gamma_1(R)}(v, s_i)$ for some i, $1 \leq i \leq k$. Hence, $d_{\Gamma^*(R)}(u, s_i) \neq d_{\Gamma^*(R)}(v, s_i)$, which contradicts our supposition that $D(u|W) = D(v|W)$. Therefore, $D(u|W) \neq D(v|W)$.

Case 2. *$u \in \Gamma_1(R)$ and $v \in \Gamma_2(R)$.* We have $d_{\Gamma^*(R)}(u, w_1) \neq d_{\Gamma^*(R)}(u, s_i) + 1$ and $d_{\Gamma^*(R)}(v, w_1) \neq d_{\Gamma^*(R)}(v, s_1) - 1$ for all i, $1 \leq i \leq k$. This implies either $d_{\Gamma^*(R)}(u, w_1) \neq d_{\Gamma^*(R)}(v, w_1)$ or $d_{\Gamma^*(R)}(u, s_1) \neq d_{\Gamma^*(R)}(v, s_1)$, contradicting again our supposition that $D(u|W) = D(v|W)$. Therefore, $D(u|W) \neq D(v|W)$.

Case 3. *$v \in \Gamma_1(R)$ and $u \in \Gamma_2(R)$.* This follows from Case 2.

Case 4. *u or v are both in $\Gamma_2(R)$.* In this case, we first suppose that at least one of u and v belongs to S, say $u \in S$. Then, $u = w_i$ for some i, $1 \leq i \leq k$, and so $d_{\Gamma^*(R)}(u, s_i) = 1$. Since $D(u|W) = D(v|W)$, therefore it follows that $u = w_i = v$, because the only vertex in $\Gamma_2(R)$ which is adjacent to s_i is w_i.

Thus, W is a resolving set for $\Gamma^*(R)$.

Now, we prove that $dim_M(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) \geq dim_M(\Gamma(R))$. Let $V_1 = V(\Gamma_1(R))$ and $V_2 = V(\Gamma_2(R))$. Then, $V(\Gamma^*(R)) = V_1 \bigcup V_2$. Clearly, $S_1 = S \bigcap V_1$ and $S_2 = S \bigcap V_2$. Let $W_1 \subseteq V_1$ be the disjoint union of S_1 and the set S_2^*, where S_2^* consists of those vertices of V_1 which correspond to S_2.

Therefore,

$$|W_1| = \left|S_1 \bigcup S_2^*\right| \leq |S_1| + |S_2^*| = |S|.$$

We show that W_1 is a resolving set for $\Gamma_1(R)$.

Let x and y be distinct vertices. We show that $D(x|W_1) \neq D(y|W_1)$. If either x or y belongs to S_1, then this is certainly the case. Otherwise, there exists a vertex $w \in S$ such that $d_{\Gamma^*(R)}(x, w) \neq d_{\Gamma^*(R)}(y, w)$. If $w \in V_1$, then $d_{\Gamma_1(R)}(x, w) = d_{\Gamma^*(R)}(x, w) \neq d_{\Gamma^*(R)}(y, w) = d_{\Gamma_1(R)}(y, w)$. If $w \in V_2$, then let w^* be the vertex of S_1 corresponding to w. So, $d_{\Gamma_1(R)}(x, w^*) = d_{\Gamma^*(R)}(x, w) - 1 \neq d_{\Gamma^*(R)}(y, w) - 1 = d_{\Gamma_1(R)}(y, w^*)$. Thus, in either case $D(x|W_1) \neq D(y|W_1)$. □

Remark 3.10. The equality for either of the bound in Theorem 3.9 is possible. Example 3.8 gives the equality for the upper bound, while if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then it can be easily seen that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a cycle on four vertices. Therefore, by [Theorem 3.2, [18]], $dim_M(R \cong \mathbb{Z}_3 \times \mathbb{Z}_3) = 2$, where as the $dim_M(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$.

We conclude this paper with some discussion on the ratio $\frac{dim_M(\Gamma(R))}{|V(\Gamma(R))|}$ which we introduced for the simple graphs in Section 2. This ratio has significance in zero-divisor graphs as well. Note here that

Proposition 2.3 can also be applied to zero-divisor graphs. Let $R \cong \prod_{i=1}^{n} \mathbb{Z}_2$ for a sufficiently large integer n. Then, by [Proposition 6.2, [19]], $\frac{dim_M(\Gamma(R))}{|V(\Gamma(R))|} \leq \frac{n}{(2^n-2)}$. It can be easily seen that $\lim_{n\to\infty} \frac{n}{(2^n-2)} = 0$. Also, for a larger prime p, there exists a zero-divisor graph $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_p)$ such that the ratio $\frac{dim_M(\Gamma(\mathbb{Z}_2\times\mathbb{F}_p))}{|V(\Gamma(\mathbb{Z}_2\times\mathbb{F}_p))|}$ is $\frac{p-2}{p}$, showing that $\frac{dim_M(\Gamma(\mathbb{Z}_2\times\mathbb{F}_p))}{|V(\Gamma(\mathbb{Z}_2\times\mathbb{F}_p))|}$ approaches to 1.

Acknowledgments

The authors thank the anonymous referees for their valuable suggestions and comments which improved the presentation of the paper. This research is supported by the University Grants Commission, New Delhi with research project number MRP-MAJOR-MATH-2013-8034.

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J. Korean Math. Soc. **53** (2016), No. 5, pp. 1167–1182
http://dx.doi.org/10.4134/JKMS.j150457
pISSN: 0304-9914 / eISSN: 2234-3008

ON GRAPHS ASSOCIATED WITH MODULES OVER COMMUTATIVE RINGS

Shariefuddin Pirzada and Rameez Raja

Abstract. Let M be an R-module, where R is a commutative ring with identity 1 and let $G(V, E)$ be a graph. In this paper, we study the graphs associated with modules over commutative rings. We associate three simple graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ to M called full annihilating, semi-annihilating and star-annihilating graph. When M is finite over R, we investigate metric dimensions in $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$. We show that M over R is finite if and only if the metric dimension of the graph $ann_f(\Gamma(M_R))$ is finite. We further show that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if M is a prime-multiplication-like R-module. We investigate the case when M is a free R-module, where R is an integral domain and show that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if $M \cong R$. Finally, we characterize all the non-simple weakly virtually divisible modules M for which $Ann(M)$ is a prime ideal and $Soc(M) = 0$.

1. Introduction

The subject of associating a graph to an algebraic structure has become an exciting research topic and has attracted considerable attention over the last two decades, see for instance [1, 3, 4, 11, 21, 22, 27, 28]. Associating a graph to a commutative ring R was introduced by Beck in [10] and was further studied by D. D. Anderson and Naseer in [3]. A different approach of associating a graph $\Gamma(R)$ to R with vertices as $Z^*(R) = Z(R)\backslash\{0\}$, where $Z(R)$ is the set of all zero-divisors of R was given by D. F. Anderson and Livingston in [5]. Two vertices $x, y \in Z^*(R)$ of $\Gamma(R)$ are adjacent if and only if $xy = 0$. Redmond in [28] extended the zero-divisor graph of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$ with vertex set $\{x \in R - I \,|\, xy \in I$ for some $y \in$

Received August 2, 2015; Revised January 16, 2016.
2010 *Mathematics Subject Classification.* Primary 13A99, 05C99, 13C99.
Key words and phrases. module, zero-divisor graph, ring, metric dimension, multiplication-like module.
This work was financially supported by MRP-MAJOR-MATH-2013-8034 of UGC New Delhi.

$R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. The concept of zero-divisor graphs has been also extended to modules over rings. Ghalandarzadeh and Malakooti Rad in [16] extended the notion of zero-divisor graph to the torsion graph associated with a module M over a ring R, whose vertices are the nonzero torsion elements of M such that two distinct vertices a and b are adjacent if and only if $(a : M)(b : M)M = 0$. Recent generalizations of zero-divisor graphs to module theory can be found in [9, 29].

On the other hand, the problem of metric dimension in graphs was first introduced in 1975 by Harary and Melter [18]. However, the metric dimension problem for hypercube was studied much earlier in 1963 by Erdos and Renyi [14]. The metric dimension in graphs has been extensively studied by various authors for many particular classes of graphs such as trees, cycles, complete graphs, grids, wheels, fans, unicyclic graphs, honeycombs and circulant graphs. Bailey and Cameron [7] established a relationship between the base size of automorphism group of a graph and its metric dimension. The relationship in [7] then motivated authors in [6, 8, 15] to study metric dimensions of distance regular graphs, such as Grassman graphs, Johnson and Kneser graphs and also bilinear form graphs. Recently in [25, 26], the concept of metric dimension in terms of locating number was introduced in zero-divisor graphs associated with commutative rings. The authors in [25, 26] have discussed various properties of locating numbers (metric dimensions) which includes the characterization of all finite rings, examination of two equivalence relations on the vertices of $\Gamma(R)$, relationship between the locating set (resolving set) and cut vertices of $\Gamma(R)$, investigation of metric dimension in $\Gamma(R)$ when R is a finite product of integral domains and so on. It is shown in [12, 19, 20] that determining the metric dimension of an arbitrary graph is an NP-complete problem. The problem is still NP-complete even if we consider some specific families of graphs, such as planar graphs [12] or Gabriel unit disk graphs [19].

Throughout, R is a commutative ring (with 1) and all modules are unitary unless otherwise stated. The symbols $\subseteq$ and $\subset$, has usual set theoretic meaning as containment and proper containment of sets. We will denote the ring of integers by $\mathbb{Z}$, the ring of integers modulo n by $\mathbb{Z}_n$ and a finite field on q elements by $\mathbb{F}_q$ respectively. For basic definitions from graph theory we refer to [13, 23, 30], and for module theory we refer to [2, 31].

2. Definitions and preliminaries

A simple graph $G(V, E)$ consists of a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. A graph G is connected if there is a path between every two distinct vertices of G. The distance from a vertex v to u denoted by $d(v, u)$ is the length of the shortest path from v to u ($d(v, v) = 0$ and $d(v, u) = \infty$, if there is no such path). The diameter of G is $diam(G) = \sup\{d(v, u) \mid v, u \in V(G)\}$. A graph G is said to be complete if there is an

edge between every pair of distinct vertices. A complete graph with n vertices is denoted by K_n. A graph G is said to be bipartite if its vertex set can be partitioned into two sets $V_1(G)$ and $V_2(G)$ such that every edge of G has one end in $V_1(G)$ and another in $V_2(G)$. A complete bipartite graph is one in which each vertex of one partite set is joined to every vertex of another partite set. We denote complete bipartite graph with partite sets of order m and n by $K_{m,n}$. A complete bipartite graph of the from $K_{1,n}$ is called a star graph. A graph G is Hamiltonian if it has a cycle which contains every vertex of the graph. Moreover, $N(v)$ denotes the set all vertices of G adjacent to the vertex v and $N[v] = N(v) \cup \{v\}$.

A set of vertices $S \subseteq V(G)$ resolves a graph G, and S is a resolving set of G, if every vertex is uniquely determined by its vector of distances to the vertices of S. More generally, for an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of vertices in a connected graph G and a vertex $v \in V(G) \setminus S$ of G, the metric representation of v with respect to S is the k-vector $D(v|S) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. The set S is resolving set for G if $D(v|S) = D(u|S)$ implies that $u = v$ for all pair of vertices in $V(G) \setminus S$. Equivalently, S is a resolving set for G if $D(v|S) \neq D(u|S)$ for all pair of distinct vertices $u, v \in V(G) \setminus S$. A resolving set S of minimum cardinality is the metric basis for G, and the number of elements in the resolving set of minimum cardinality is the metric dimension of G. The metric dimension of a graph G is denoted by $\dim(G)$. Note here that by Definition 2.1 of [25] the metric dimension of an empty graph is not defined.

The resolving set is also called the locating set, metric representation of a vertex is also called the locating code of a vertex and the metric dimension of a graph is also called the locating number of a graph.

The concept of resolving set, metric representation and metric dimension in terms of locating set, locating code and locating number in zero-divisor graphs associated with commutative rings was introduced in [25] and has been further studied in [24, 26]. The authors in [24, 25, 26] have discussed various properties of metric dimensions which includes the characterization of all finite rings, examination of two equivalence relations on the vertices of $\Gamma(R)$, relationship between the resolving set and cut vertices of $\Gamma(R)$, investigation of metric dimension in $\Gamma(R)$ when R is a finite product of integral domains, when R is the finite product $R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are n finite commutative rings with none of them being isomorphic to the Boolean ring $\prod_{i=1}^{n} \mathbb{Z}_2$, provided a combinatorial formula for computing the metric dimension of a zero-divisor graph $\Gamma(R \times \mathbb{F}_q)$ and so on.

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two finite fields. Then the zero-divisor graph $\Gamma(\mathbb{F}_1 \times \mathbb{F}_2)$ associated with $\mathbb{F}_1 \times \mathbb{F}_2$ is either a star graph or a complete bipartite graph. Therefore, from Corollary 2.1 of [25], the metric dimension of $\Gamma(\mathbb{F}_1 \times \mathbb{F}_2)$ is $|\mathbb{F}_1| + |\mathbb{F}_2| - 4$. However, for $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{Z}_2$ the metric dimension is 1 because $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is a path and the metric dimension of all finite paths by Lemma 2.1 of [25] is 1. For the rings $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_5$ and $\mathbb{Z}_3 \times \mathbb{F}_4$, it can be easily seen that the zero-divisor graphs associated with these rings

are complete bipartite graphs $K_{1,6}$, $K_{3,4}$, $K_{1,4}$ and $K_{2,3}$. Therefore, from Corollary 2.1 of [25], the metric dimensions of these graphs are 5 and 3. Further, for the rings $\mathbb{Z}_2[x,y,z]/(x,y,z)^2$, $\mathbb{Z}_4[x,y]/(x^2,y^2,xy,2x,2y)$, $\mathbb{F}_8[x]/(x^2)$ and $\mathbb{Z}_4[x]/(x^3+x+1)$, the associated zero-divisor graph is a complete graph K_7. Therefore, from Lemma 2.2 of [25], the metric dimension is 6. For the rings $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^2}[x]/(x^2)$, ($p \geq 2$ is a prime number), the associated zero-divisor graph is a complete graph on $p-1$ number of vertices. Therefore the metric dimension of the associated graph is $p-2$.

If $I = (0) \times \mathbb{Z}_3$ is an ideal of ring $R = \mathbb{Z}_9 \times \mathbb{Z}_3$, then the ideal based zero-divisor graph $\Gamma_I(R)$ defined in [28] with vertex set $V(\Gamma_I(R)) = \{(3,0),(3,1),(3,2),(6,0),(6,1),(6,2)\}$ is a complete graph K_6 on six vertices. Therefore metric dimension of $\Gamma_I(R)$ is 5. If I is a prime ideal of a ring R, then $\dim(\Gamma_I(R))$ is undefined. However, if $I = P_1 \cap P_2$, where P_1 and P_2 are prime ideals of a ring R, then $\dim(\Gamma_I(R))$ is finite which is in fact equal to $|V(\Gamma_I(R))| - 2$.

For more on the metric dimension of zero-divisor graphs, graphs determined by the equivalence classes of zero-divisors and ideal based zero-divisor graphs associated with commutative rings see [24, 25, 26]. In the remaining paper, we discuss the nature of graphs associated with modules and also determine the metric dimensions of these graphs, when M is finite over R. First we have the following definition.

Definition 2.1. Let M be an R-module. For an element $x \in M$, we define a set $[x : M_R] = \{r \in R : rM \subseteq Rx\}$, which clearly is an annihilator of the factor module M/Rx. The annihilator of a module M is defined as $Ann(M) = \{s \in R|\ sm = 0 \text{ for all } m \in M\}$. Clearly, for each $x \in M$, $Ann(M) \subseteq [x : M_R]$. Further, $Rx = M$ if and only if $[x : M_R] = R$. Since $Ann(M) \subseteq [x : M_R]$, based on the above definition we now classify the elements of M into three categories. An element $x \in M$ is a

(i) *full-annihilator*, if either $x = 0$ or $[x : M_R][y : M_R]M = 0$ for some nonzero $y \in M$ with $[y : M_R] \neq R$.

(ii) *semi-annihilator*, if either $x = 0$ or $[x : M_R] \neq 0$ and $[x : M_R][y : M_R]M = 0$ for some nonzero $y \in M$ with $0 \neq [y : M_R] \neq R$.

(iii) *star-annihilator*, if either $x = 0$ or $Ann(M) \subset [x : M_R]$ and $[x : M_R][y : M_R]M = 0$ for some nonzero $y \in M$ with $Ann(M) \subset [y : M_R] \neq R$.

We denote by the sets $A_f(M)$, $A_s(M)$ and $A_t(M)$ respectively the *full-annihilators*, *semi-annihilators* and *star-annihilators* for any module M over R. The name given to these sets is because of the containment $A_t(M) \subseteq A_s(M) \subseteq A_f(M)$. If $M = R$, then for each $x \in R$, $[x : R_R] = Ann(R/Rx) = Rx$. So $[x : R_R][y : R_R] = 0$ if and only if $xy = 0$. Therefore, x is a zero-divisor in R if and only if $[x : R_R][y : R_R]R = 0$ for some $y \neq 0 \in R$. Thus, we have the usual zero-divisors for R. So, for $M = R$ the full-annihilators, semi-annihilators and star-annihilators of M coincides with the zero-divisors of R.

Moreover, we let $\widehat{A_f(M)} = A_f(M)\backslash\{0\}$, $\widehat{A_s(M)} = A_s(M)\backslash\{0\}$ and $\widehat{A_t(M)} = A_t(M)\backslash\{0\}$ and associate three simple graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$

and $ann_t(\Gamma(M_R))$ to M over R called as full-annihilating, semi-annihilating and star-annihilating graphs of M over R and the vertices x and y are adjacent if and only if $[x : M_R][y : M_R]M = 0$. It is clear that $ann_t(\Gamma(M_R)) \subseteq ann_s(\Gamma(M_R)) \subseteq ann_f(\Gamma(M_R))$ as induced subgraphs. We will call all these graphs as annihilating graphs of M over R. It can be easily seen that for $M = R$, all the annihilating graphs are the zero-divisor graph of a commutative ring introduced by Anderson and Livingston in [5].

If M is finite over R, then $ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R))$, whereas the graph $ann_t(\Gamma(M_R))$ can be different from $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$. In the following example, we show that for a finite module $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ over $\mathbb{Z}$, $ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = K_7$ and $ann_t(\Gamma(M_R)) = K_5$.

Example 2.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Then, M over $\mathbb{Z}$ consists of eight elements as $\{(0,0),(1,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3)\}$. Let $m_1 = (1,0)$, $m_2 = (0,1)$, $m_3 = (0,2)$, $m_4 = (0,3)$, $m_5 = (1,1)$, $m_6 = (1,2)$, and $m_7 = (1,3)$ be nonzero elements of M. It can be easily verified that $[m_2 : M_R] = [m_3 : M_R] = [m_4 : M_R] = [m_5 : M_R] = [m_7 : M_R] = 2\mathbb{Z}$ and $[m_1 : M_R] = [m_6 : M_R] = 4\mathbb{Z} = Ann(M)$. Thus, $\widehat{A_f(M)} = \widehat{A_s(M)} = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\}$ and $\widehat{A_t(M)} = \{m_2, m_3, m_4, m_5, m_7\}$. Since $[m_i : M_R][m_j : M_R]M = 0$, for all $1 \leq i, j \leq 7$, it follows that $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are complete graphs with seven vertices but $ann_t(\Gamma(M_R))$ is a complete graph with five vertices.

The above examples lead to a natural question: what is the nature of graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ when M is infinite over R.

The following example illustrates that the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are different when M is infinite over R.

Example 2.3. Let $M = \oplus_{i=1}^{n}\mathbb{Z}$ and $R = \mathbb{Z}$. Then, for all non-zero $x, y \in M$, $[x : M_R][y : M_R]M = 0$ with $[y : M_R] \neq R$. So, the graph $ann_f(\Gamma(M_R))$ is complete with vertices as $\widehat{M}$ and by definition it follows that the graph $ann_s(\Gamma(M_R))$ is empty. Thus in general for infinite modules over commutative rings the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are different.

3. Graphs associated with multiplication-like modules over R

In this section, we characterize all the finite modules over commutative rings. Moreover, we characterize all the graphs associated with multiplication-like modules, prime multiplication modules and indecomposable modules.

The following observation shows that the graph $ann_f(\Gamma(M_R))$ is connected and has exceedingly small diameter which is analogous to the case for graphs $\Gamma(R)$ and $\Gamma_I(R)$ found in [[5], [28], Theorem 2.3 and Theorem 2.4].

Lemma 3.1. *Let M be an R-module. Then $ann_f(\Gamma(M_R))$ is a connected graph and $diam(ann_f(\Gamma(M_R))) \leq 3$.*

Proof. Let $x, y \in \widehat{A_f(M)}$ with $x \neq y$. We have the following cases.

Case 1. $[x : M_R][y : M_R]M = 0$. Then, $x - y$ is a path.

Case 2. $[x : M_R][y : M_R]M \neq 0$. If $[x : M_R]^2M = 0$ and $[y : M_R]^2M = 0$ then, $x - z - y$ is a path of length 2, for each $0 \neq z \in Rx \cap Ry$, $[x : M_R] \subseteq [x : M_R] \cap [y : M_R]$.

Case 3. $[x : M_R][y : M_R]M \neq 0$, $[y : M_R]^2M \neq 0$ and $[x : M_R]^2M = 0$. Then, there exists $b \in \widehat{A_f(M)} \backslash \{x, y\}$ such that $[b : M_R][y : M_R]M = 0$. If $[b : M_R][x : M_R]M = 0$, then $x - b - y$ is a path of length 2. If $[b : M_R][x : M_R]M \neq 0$, then for each $0 \neq c \in Rb \cap Rx$, $[c : M_R] \subseteq [b : M_R] \cap [x : M_R]$, $x - c - y$ is a path of length 2.

Case 4. $[x : M_R][y : M_R]M \neq 0$, $[x : M_R]^2M \neq 0$ and $[y : M_R]^2M = 0$. The proof follows from Case 3.

Case 5. $[x : M_R][y : M_R]M \neq 0$, $[x : M_R] \neq 0$ and $[y : M_R] \neq 0$. Then, there exists $a \in \widehat{A_f(M)} \backslash \{x, y\}$ with $[a : M_R][x : M_R]M = 0$ and $b \in \widehat{A_f(M)} \backslash \{x, y\}$ such that $[b : M_R][y : M_R]M = 0$.

Subcase 1. $[a : M_R] = [b : M_R]$. Then, $x - a - y$ is a path of length 2.

Subcase 2. $[a : M_R] \neq [b : M_R]$ and $[a : M_R][b : M_R]M = 0$, then $x - a - b - y$ is a path of length 3 and hence $d(x, y) \leq 3$. If $[a : M_R][b : M_R]M \neq 0$, then there exists $0 \neq d \in Ra \cap Rb$ such that $x - d - y$ is a path of length 2.

Thus $ann_f(\Gamma(M_R))$ is connected and $diam(ann_f(\Gamma(M_R))) \leq 3$. □

Let M be a nonzero R-module. Then M is a *prime module* if whenever N is a nonzero submodule of M and A is an ideal of R such that $NA = 0$, then $MA = 0$. That is, $Ann(M) = Ann(N)$ for all nonzero submodules N of M. Also, an R-module M is called a *multiplication module* if each submodule of M is of the form IM, where I is an ideal of R. A multiplication module M is *multiplication-like module* if for each nonzero submodule N of M, $Ann(M) \subset Ann(M/N)$. It is clear that for each nonzero submodule N of a multiplication module M, $N = Ann(M/N)M$ and $Ann(M) \subset Ann(M/N)$. Thus, it follows that every multiplication module is a multiplication-like module.

We have the following observation.

Lemma 3.2. *Let M be an R-module. Then M is multiplication-like if and only if $Ann(M) \subset [m : M_R]$ for each $0 \neq m \in M$.*

Proof. Suppose M is a multiplication-like module. Then

$$Ann(M) \subset Ann(M/N)$$

for each submodule N of M. Now using Definition 2.1, it follows $Ann(M) \subset [m : M_R]$ because for $m \in M$, we have $[m : M_R] = Ann(M/Rm)$ and clearly Rm is a submodule of M by taking the action of R on Rm.

Conversely, suppose that for each $0 \neq m \in M$, we have $Ann(M) \subset [m : M_R]$. Let N be a submodule of M. We show that M is a multiplication module. For each $0 \neq x \in N$, there exists an ideal $[x : N_R]$ of R such that $[x : N_R]M \subseteq Rx$.

Let $I = \sum_{0 \neq x \in N} [x : N_R]$. Then, $0 \neq IM = N$. Thus, it follows that M is a multiplication module and hence a multiplication-like module. $\square$

Now, we characterize all the finite modules over commutative rings. We show that M is finite over R if and only if metric dimension of the graph $ann_f(\Gamma(M_R))$ is finite. In fact the following result is the generalization of Theorem 3.1 of [25].

Theorem 3.3. $\dim(ann_f(\Gamma(M_R)))$ *is finite if and only if M is finite over R.*

Proof. Suppose M is finite over R. Then, clearly the set $A_f(M)$ is finite. It follows that the graph $ann_f(\Gamma(M_R))$ is finite, which implies that the number $\dim(ann_f(\Gamma(M_R)))$ is finite.

Conversely, suppose $ann_f(\Gamma(M_R))$ is finite. Then, $\dim(ann_f(\Gamma(M_R)))$ is finite. If there exists $0 \neq x \in M$ such that $[x : M_R] = Ann(M)$, then $[x : M_R][y : M_R]M = 0$ for all $0 \neq y \in M$, that is, $\widehat{A_f(M)} = \widehat{M}$, where $\widehat{M}$ is a set of nonzero elements of M. Therefore, M is finite. If $[x : M_R] \neq Ann(M)$ for any $0 \neq x \in M$, then by Lemma 3.2, M is a multiplication-like module.

Suppose M is infinite. Since M is a multiplication-like module, for each nonzero submodule N of M, $Ann(M/N) \neq 0$, that is, $[x : M_R] \neq 0$ for all $x \in M$. Since $ann_f(\Gamma(M_R))$ is finite and nonempty, there exists some $x, y \in \widehat{M}$ such that $[x : M_R][y : M_R]M = 0$. It follows that there is a path $x - y$ in $ann_f(\Gamma(M_R))$. Let $r \in R$ and assume $ry \neq 0$. It is clear that $[ry : M_R] \subseteq [y : M_R]$. So, $[x : M_R][ry : M_R]M \subseteq [x : M_R][y : M_R]M = 0$. Thus $x - ry$ is a path in $ann_f(\Gamma(M_R))$ and therefore $Rx \subseteq A_f(M)$ is finite, since $0 \neq [x : M_R] \subseteq Rx$, $[x : M_R]$ is also finite. Let $z \in [x : M_R]$ such that $0 \neq zM$. Then, zM is finite and there exists an ideal A of R such that $0 \neq AM \subseteq zM$. If M is not finite, then there is an element $m_1 \in M$ such that $T = \{m \in M : zm_1 = zm\}$ is infinite. Clearly, $N = \{m \in M : zm = 0\}$ is a nonzero submodule and is infinite. Since M is multiplication, there exists an ideal B of R such that $0 \neq BM \subseteq N$. Let $jm^* \in JM = \{\sum_{\text{finite}} j_i m_i : j_i \in J, m_i \in M\}$. Then, $[jm^* : M_R] \subseteq Rjm^* \subseteq JM$. So, for each $0 \neq m \in N, [m : M_R][jm^* : M_R]M \subseteq [m : M_R]JM \subseteq [m : M_R]zM \subseteq z[m : M_R]M \subseteq zN$ (because N is a submodule of a multiplication module). Therefore, $N \subseteq \widehat{A_f(M)}$, a contradiction. Thus M must be finite. $\square$

Remark 3.4. For a finite module M over R, we have

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)).$$

So, if $ann_f(\Gamma(M_R))$ is finite, then clearly $\dim(ann_s(\Gamma(M_R)))$ is finite. Now, if $\dim(ann_s(\Gamma(M_R)))$ is finite, then by Lemma 3.1, the diameter of $ann_s(\Gamma(M_R))$ is not more than 3. Therefore, by [25, Theorem 2.2] the number of vertices of $ann_s(\Gamma(M_R))$ is finite, which implies that the graph $ann_s(\Gamma(M_R))$ is also finite.

Remark 3.5. (i) By [25, Lemma 2.1],

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))) = 1$$

if and only if the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are paths on $|\widehat{A_f(M)}|$, $|\widehat{A_s(M)}|$ and $|\widehat{A_t(M)}|$ number of vertices.

(ii) If $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are cycles on $|\widehat{A_f(M)}|$, $|\widehat{A_s(M)}|$ and $|\widehat{A_t(M)}|$ number of vertices, then, by [25, Lemma 2.3],

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))) = 2.$$

(iii) By [25, Lemma 2.2], $\dim(ann_f(\Gamma(M_R))) = |\widehat{A_f(M)}| - 1$, $\dim(ann_s(\Gamma(M_R))) = |\widehat{A_s(M)}| - 1$ and $\dim(ann_t(\Gamma(M_R))) = |\widehat{A_t(M)}| - 1$ if and only if the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are complete on $|\widehat{A_f(M)}|$, $|\widehat{A_s(M)}|$ and $|\widehat{A_t(M)}|$ number of vertices.

(iv) If $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are complete bipartite graphs or star graphs (other than $K_{1,1}$), then by [25, Corollary 2.1] $\dim(ann_f(\Gamma(M_R))) = |\widehat{A_f(M)}| - 2$, $\dim(ann_s(\Gamma(M_R))) = |\widehat{A_s(M)}| - 2$ and $\dim(ann_t(\Gamma(M_R))) = |\widehat{A_t(M)}| - 2$.

Example 3.6. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_6$. Then, $[2 : M_R] = 2\mathbb{Z}$, $[3 : M_R] = 3\mathbb{Z}$ and $[4 : M_R] = 4\mathbb{Z}$ with $Ann(M) = 6\mathbb{Z}$. Thus, $\widehat{A_f(M)} = \widehat{A_s(M)} = \widehat{A(M)} = \{2, 3, 4\}$. Clearly $[2 : M_R][3 : M_R]M = 0$ and $[3 : M_R][4 : M_R]M = 0$. Therefore, we have $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ as paths on three vertices. Thus we conclude that $\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))) = 1$.

Remark 3.7. Let M be an R-module and $M = M_1 \oplus M_2$, where M_1 and M_2 are non-isomorphic simple submodules of M. Then, for $M_1 = M_2 = \mathbb{Z}_2$, $ann_f(\Gamma(M_R))$ is a path. In all other cases, $ann_f(\Gamma(M_R))$ is a complete bipartite graph. Therefore, by [25, Corollary 2.1], $\dim(ann_f(\Gamma(M_R))) = |\widehat{A_f(M)}| - 2$.

Remark 3.8. Let $p \geq 3$ be a prime number. Clearly, $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is a complete graph on $p^2 - 1$ vertices. It follows that $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is isomorphic to K_{p^2-1}. Therefore, from [25, Lemma 2.2], it follows that the graph $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))$ is Hamiltonian if and only if $\dim(ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p))) = p^2 - 2$.

From Example 2.2, we have $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ as complete graphs on seven vertices, where as $ann_t(\Gamma(M_R))$ is a complete graph on five vertices. Therefore, by Lemma 2.2 of [25],

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = 6 \neq 4 = \dim(ann_t(\Gamma(M_R))).$$

If M is a multiplication-like module over R, then all the annihilating graphs associated with M have same metric dimension, as can be seen below.

Theorem 3.9. *Let M be a multiplication-like R-module. Then*

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$

Proof. Suppose M is a multiplication-like module. If $ann_f(\Gamma(M_R)) = \phi$, then clearly $ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)) = \phi$. Assume that $ann_f(\Gamma(M_R)) \neq \phi$ and fix a vertex x in $ann_f(\Gamma(M_R))$. Then, there exists $0 \neq y \in M$ such that $[x : M_R][y : M_R]M = 0$, (that is the vertices x and y are connected by a path $x - y$ in $ann_f(\Gamma(M_R))$). Since for each $0 \neq m \in M$, $Ann(M) \subset [m : M_R]$, so $x \in ann_t(\Gamma(M_R))$. It follows that $x - y$ is a path in $ann_t(\Gamma(M_R))$. Thus,

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$ □

Corollary 3.10. *If M is a multiplication R-module, then*

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$

Proof. Since every multiplication module is a multiplication-like module, the result follows from Theorem 3.9. □

Remark 3.11. From Theorem 3.9, for a multiplication-like R-module M, it follows that all the annihilating graphs coincide. Thus the metric dimensions of all these graphs are the same, that is,

$$\dim(ann_f(\Gamma(M_R))) = \dim(ann_s(\Gamma(M_R))) = \dim(ann_t(\Gamma(M_R))).$$

So is the case for the multiplication R-module.

By [25, Theorem 3.1], it is clear that for a commutative ring R, the metric dimension of graph $\Gamma(R)$ is undefined if and only if R is an integral domain. That is, the graph $\Gamma(R)$ is empty if and only if R is an integral domain.

In the following result, we see that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if M is a prime multiplication-like module.

Theorem 3.12. *Let M be an R-module. Then M is a prime multiplication-like module if and only if the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty.*

Proof. Suppose M is a prime multiplication-like module. Then for every $0 \neq x \in M$, we have $Ann(M) \subset [x : M_R]$. It follows that $[x : M_R][y : M_R]M \neq 0$ for each $0 \neq x, y \in M$. So,

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)) = \phi.$$

Conversely, suppose that graphs

$$ann_f(\Gamma(M_R)),\ ann_s(\Gamma(M_R)) \text{ and } ann_t(\Gamma(M_R))$$

are empty. Then, by Theorem 3.9, M is a multiplication-like module. Assume that M is not prime. Then, by [17, Corollary 1.6], $Ann(M)$ is not a prime ideal. Therefore, $ABM = 0$, for some ideals A and B with $Ann(M) \subset A, B$. Since $AM \neq 0$ and $BM \neq 0$, there exist $0 \neq x \in AM$ and $0 \neq y \in BM$ such

that $[x : M_R] \subseteq Rx \subseteq AM$ and $[y : M_R] \subseteq Ry \subseteq BM$. Then, $[x : M_R][y : M_R]M \subseteq ABM = 0$. Therefore, $ann_f(\Gamma(M_R)) \neq \phi$, a contradiction. Thus M is a prime multiplication-like module. □

Remark 3.13. From Theorem 3.12, it follows that metric dimension of all the annihilating graphs associated with M over R are undefined if and only if M over R is a prime multiplication-like module.

The following is an immediate consequence of Theorem 3.12.

Corollary 3.14. *Let M be an R-module. Then M is a prime multiplication module if and only if M is a multiplication-like module for which $Ann(M)$ is a prime ideal.*

A nonzero R-module M is called an *indecomposable* if M cannot be written as a direct sum of nonzero submodules.

We have the following observation regarding decomposable modules.

Lemma 3.15. *Let $M = M_1 \oplus M_2$ be decomposable R-module, where M_1 and M_2 are nonzero R-modules. If $ann_f(\Gamma(M_{1R}))$ is a complete graph, then $ann_f(\Gamma(M_R))$ is also a complete graph.*

Proof. Let $0 \neq x \in \widehat{A_f(M_1)}$. Then there exists $0 \neq y \in M_1$ such that $[x : M_{1R}][y : M_{1R}]M_1 = 0$, where $[x : M_{1R}] = Ann(M_1/Rx)$ and $[y : M_{1R}] = Ann(M_1/Ry)$. Clearly,

$$[(x,0) : M_R] = Ann(\frac{M_1 \oplus M_2}{R(x,0)}) = Ann(\frac{M_1}{Rx} \oplus M_2), \tag{1}$$

$$[(y,0) : M_R] = Ann(\frac{M_1 \oplus M_2}{R(y,0)}) = Ann(\frac{M_1}{Ry} \oplus M_2). \tag{2}$$

Moreover,

$$[x : M_R] = Ann(\frac{M_1 \oplus M_2}{Rx \oplus (0)}) = Ann(\frac{M_1}{Rx} \oplus M_2), \tag{3}$$

$$[y : M_R] = Ann(\frac{M_1 \oplus M_2}{(0) \oplus Ry}) = Ann(M_1 \oplus \frac{M_2}{Ry}). \tag{4}$$

It follows that $[x : M_R] \subseteq Ann(M_2)$ and $[y : M_R] \subseteq Ann(M_1)$. Thus, $[x : M_R][y : M_R]M = 0$.

Using (1), (2), (3) and (4), we have $[(x,0) : M_R] \subseteq [x : M_R]$, $[(y,0) : M_R] \subseteq [y : M_R]$ and $[(x,0) : M_R]M_2 = [(y,0) : M_R]M_2 = 0$. Therefore, $[(x,0) : M_R][(y,0) : M_R]M = 0$, which implies $(x,0) \in \widehat{A_f(M)}$ and the vertices $(x,0)$, $(y,0)$ are adjacent in $ann_f(\Gamma(M_R))$, which further implies that if the graph $ann_f(\Gamma(M_{1R}))$ is complete, then the graph $ann_f(\Gamma(M_R))$ is also complete. □

The following result shows that if the graph $ann_f(\Gamma(M_R))$ is empty, then M is always indecomposable.

Theorem 3.16. *Let M be an R-module. If* $\dim(ann_f(\Gamma(M_R)))$ *is undefined, then M is indecomposable.*

Proof. Suppose $\dim(ann_f(\Gamma(M_R)))$ is undefined. Then $ann_f(\Gamma(M_R)) = \phi$. Therefore, by Theorem 3.12, M is a prime multiplication-like module. If $M = M_1 \oplus M_2$, where M_1 and M_2 are nonzero R-modules, then by Lemma 3.15, $ann_f(\Gamma(M_R)) \neq \phi$, a contradiction. □

The following is a consequence of Theorem 3.16.

Corollary 3.17. *Every prime multiplication-like R-module is an indecomposable R-module.*

Let M be an R-module. If for some ideal I of R, $am = 0$ for all $a \in I$, $m \in M$, then we say M is annihilated by I. In this situation we can make M into an R/I-module by defining an action of the quotient ring R/I on M.

In the following result, we show that the graph $ann_t(\Gamma(M_R)))$ coincides with the graph $ann_t(\Gamma(M_{R/I}))$ while the graph $ann_f(\Gamma(M_R))$ coincides with $ann_f(\Gamma(M_{R/I}))$.

Proposition 3.18. *Let M be an R-module with $I = Ann(M)$. Then*

$$ann_t(\Gamma(M_R))) = ann_t(\Gamma(M_{R/I})),$$

and

$$ann_f(\Gamma(M_R))) = ann_f(\Gamma(M_{R/I})).$$

Proof. To prove the result, it is enough to show that the graphs $ann_t(\Gamma(M_R))$ and $ann_t(\Gamma(M_{R/I}))$ coincide. That is, we show that the vertices x and y are adjacent in $ann_t(\Gamma(M_R))$ if and only if they are adjacent in $ann_t(\Gamma(M_{R/I}))$. Let $x \in \widehat{A_t(M)}$. Then, there exists $0 \neq y \in M$ such that $Ann(M) \subset [x : M_R]$ and $[x : M_R][y : M_R]M = 0$ with $Ann(M) \subset [y : M_R] \subset R$. It is clear here that $I = Ann(M) \subseteq [x : M_R] \cap [y : M_R]$. Thus, $([x : M_R]/I)([y : M_R]/I)M = 0$ (because $Ann(M/Rx)$ over R/I is $[x : M_R]/I$ and $Ann(M/Ry)$ over R/I is $[y : M_R]/I$). It follows that $x \in \widehat{A_t(M)}$ if and only if $x \in A(\widehat{M_t(R/I)})$ and the vertices x and y are adjacent in $ann_t(\Gamma(M_R))$ if and only if they are adjacent in $ann_t(\Gamma(M_{R/I}))$. Thus, $ann_t(\Gamma(M_R))$ and $ann_t(\Gamma(M_{R/I}))$ are equal. Similarly it can be proved that $ann_f(\Gamma(M_R))) = ann_f(\Gamma(M_{R/I}))$. □

4. Graphs associated with divisible and free modules over R

We start this section with the following observation on the action of R on M.

Lemma 4.1. *Let M be an R-module. Then the following hold.*

(i) *If the action of R on M is faithful, then*

$$ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R)).$$

(ii) *If the action of R on M is not faithful, then*

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)).$$

Proof. (i) Since the action of R on M is faithful, so the annihilator ideal is a nonzero ideal. That is, $Ann(M) \neq (0)$. Let $x \in \widehat{A_t(M)}$. Then, $[x : M_R] \neq 0$ and there exists $0 \neq y \in M$ such that $Ann(M) \subset [x : M_R]$, $[x : M_R][y : M_R]M = 0$ with $Ann(M) \subset [y : M_R] \subset R$. It follows that $x \in \widehat{A_s(M)}$ and the vertices x and y are adjacent in $ann_t(\Gamma(M_R))$ if and only if they are adjacent in $ann_s(\Gamma(M_R))$. Therefore, $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R))$.

(ii) Similar to part (i). □

In the following result, we consider the graphs associated with free modules over an integral domain R. We show that the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty if and only if $R \cong M$. Moreover, we show graphs $ann_t(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ are empty and the graph $ann_t(\Gamma(M_R))$ is complete if and only if $M \ncong R$.

Proposition 4.2. *Let M be a free R-module, where R is an integral domain. Then the following hold.*

(i) *$ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty graphs if and only if $R \cong M$.*

(ii) *$ann_t(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are empty graphs and the graph $ann_f(\Gamma(M_R))$ is complete if and only if $M \ncong R$.*

Proof. (i) Suppose the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ and $ann_t(\Gamma(M_R))$ are empty. Then

$$ann_f(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = ann_t(\Gamma(M_R)).$$

Therefore, by Theorem 3.12, M is a prime multiplication-like module. Further, by Theorem 3.16, M is an indecomposable module and so $M \cong R$.

Conversely, if M and R are isomorphic, it is clear that all the annihilating graphs are empty.

(ii) Suppose that $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = \phi$ and $ann_f(\Gamma(M_R))$ is a complete graph. Then $\widehat{A_f(M)} \neq \phi$. Therefore, $M \ncong R$.

For the converse, let $M = \oplus_{\lambda\in\Omega} R$, where Ω is an index set with $|\Omega| \geq 2$. Let $0 \neq x = (x_\lambda)_{\lambda\in\Omega} \in M$, where $x_\lambda \in R$, for each $\lambda \in \Omega$. Then, $x_\mu \neq 0$, for some $\lambda \neq \mu \in \Omega$ and also $[x : M_R]M = \oplus_{\lambda\in\Omega}[x : M_R] \subseteq Rx = R(x_\lambda)_{\lambda\in\Omega}$. If $[x : M_R] \neq 0$ and $0 \neq z \in [x : M_R]$, then we put $y_\mu = 0$ and $y_\lambda = z$, for each $\mu \neq \lambda$. Therefore, $(y_\lambda)_{\lambda\in\Omega} \oplus_{\lambda\in\Omega} [x : M_R]$, and so there exist $l \in R$ such that $(y_\lambda)_{\lambda\in\Omega} = l(x_\lambda)_{\lambda\in\Omega}$. It follows that $0 = y_\mu = tx_\mu$ and $z = tx_\lambda$, for each $\mu \neq \lambda$. Since R is an integral domain and $x_\mu \neq 0$, $t = 0$ which implies that $z = 0$, a contradiction. Thus, $[x : M_R] = 0$ for each $x \in M$. Hence the graph $ann_f(\Gamma(M_R))$ is complete and since M is faithful R-module, by Lemma 4.1, $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R)) = \phi$. □

Let M be an R-module. Then we say M is *divisible* if $rM = M$ for all $0 \neq r \in R$. If R is a principal integral domain, then M is injective if and only if it is divisible. Over R, the divisible modules are exactly the injective modules. However, over other domains divisible modules need not to be injective. Further, we say that M is a *virtually divisible* module if $Ann(M/N) = Ann(M)$ for each proper submodule N of M. Also, M is a *weakly virtually divisible module* if $Ann(M/Rn) = Ann(M)$ for each proper cyclic submodule Rn of M (that is, $[x : M_R] = Ann(M)$ for each $0 \neq x \in M$ with $Rx \neq M$).

In the following result, we give the nature of all the annihilating graphs associated with weakly virtually divisible R-modules.

Theorem 4.3. *Let M be weakly virtually divisible R-module such that M is not cyclic. Then the following hold.*

(i) *$ann_t(\Gamma(M_R))$ is an empty graph and $ann_f(\Gamma(M_R))$ is a complete graph.*
(ii) *If the action of R on M is faithful, then $ann_s(\Gamma(M_R))$ is an empty graph.*
(iii) *If the action of R on M is not faithful, then $ann_s(\Gamma(M_R))$ is a complete graph.*

Proof. (i) Since M is not cyclic and M is weakly virtually divisible module, $[x : M_R] = Ann(M)$, which implies that the graph $ann_t(\Gamma(M_R))$ is empty and the graph $ann_f(\Gamma(M_R)))$ is complete.

(ii) If R acts on M such that the action on M is faithful, then by Lemma 4.1, $ann_t(\Gamma(M_R)) = ann_s(\Gamma(M_R))$. So $ann_s(\Gamma(M_R))$ is an empty graph.

(iii) If the action of R on M is not faithful, then by Lemma 4.1, $ann_f(\Gamma(M_R))$ $= ann_s(\Gamma(M_R))$. By (i), $ann_f(\Gamma(M_R))$ is a complete graph. Thus, it follows that the graph $ann_s(\Gamma(M_R))$ is also complete. □

An R-module M is called *simple* if $M \neq (0)$ and it has no submodules except (0) and M. An R-module M is a *semi-simple* module if it is a direct sum of simple modules. Also, an R-module M is called a homogenous *semi-simple* R-module if it is a direct sum of isomorphic simple R-modules, that is, $Ann(M)$ is a maximal ideal of R.

Remark 4.4. Let R be a field and M a homogeneous semi-simple R-module. If M is simple, then all the annihilating graphs of M over R are empty. If M is not simple, then $ann_t(\Gamma(M_R))$ is an empty graph and the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are complete.

Now, we have the following observation regarding divisible and virtually divisible modules over an integral domain.

Lemma 4.5. *A module M over R is virtually divisible if and only if $P = Ann(M)$ is a prime ideal and M is a divisible R/P-module.*

Proof. Suppose $P = Ann(M)$ is a prime ideal of R and M is a divisible R/P-module. Then, clearly M is virtually divisible.

Conversely, suppose M is virtually divisible. Let $ab \in P$, where $a, b \in R$. Let $aM \neq 0$. Then, clearly aM is a nonzero submodule of M. If $aM \neq M$, then $Ann(M/aM) = Ann(M) = P$ (because M is virtually divisible module). So $a \in Ann(M/aM) = Ann(M)$, a contradiction. Thus, $aM = M$ and so $bM = baM = 0$. It follows that $b \in Ann(M) = P$. Therefore P is a prime ideal.

Further, let $0 \neq r \in R/P$. Then $rM \neq 0$. If $rM \neq M$, then by the same reasoning as above we have a contradiction. Thus $rM = M$ (that is, $(r + P)M = M$) and so M is a divisible R/P-module. □

In the next result, we show that if M is virtually divisible R-module and simple, then all the annihilating graphs associated with M are empty. Further, we show that if M is a non simple virtually divisible R-module, then the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ are complete, where as $ann_t(\Gamma(M_R))$ is an empty graph.

Theorem 4.6. *Let R be an integral domain and let M be an R-module. If M is virtually divisible R-module, then the following hold.*

(i) *If M is simple, then all the annihilating graphs associated with M over R are empty.*
(ii) *If M is not simple, then $ann_t(\Gamma(M_R))$ is empty and the graphs $ann_f(\Gamma(M_R))$, $ann_s(\Gamma(M_R))$ are complete.*

Proof. Let M be a virtually divisible R-module. By Lemma 4.5, $P = Ann(M)$ is a prime ideal and M is a divisible R/P-module. If $P = 0$, then M is a divisible R-module. If $P \neq 0$, then P is a maximal ideal and so M is a homogeneous semi-simple module. Now the result follows from Remark 4.4. □

Remark 4.7. Let R be an integral domain and let M be an R-module. If M is a divisible R-module and simple, then all the annihilating graphs of M over R are empty. However, if M is a divisible R-module but not simple, then the graph $ann_t(\Gamma(M_R))$ is empty, where as the graphs $ann_f(\Gamma(M_R))$ and $ann_s(\Gamma(M_R))$ are complete.

The *socle* of a module M over ring R is denoted by $Soc(M)$ and is defined by $Soc(M) = \sum\{N : N \textit{ is a simple submodule of } M\}$. The following result characterizes all non-simple weakly virtually divisible modules.

Theorem 4.8. *The graph $ann_f(\Gamma(M_R))$ is complete if and only if M is a non-simple weakly virtually divisible module for which $Ann(M)$ is a prime ideal and $Soc(M) = 0$.*

Proof. Suppose M is a non-simple weakly virtually divisible R-module. Then, $Ann(M) = [x : M_R]$ for each $x \in M$. It follows that the graph $ann_f(\Gamma(M_R))$ is complete.

Conversely, suppose the graph $ann_f(\Gamma(M_R))$ is complete. This implies $ann_s(\Gamma(M_R))$ is a complete graph with vertices $\widehat{M}$. Therefore, for $x \neq y \in \widehat{M}$,

$[x : M_R]$ and $[y : M_R]$ are the two ideals of R such that $[x : M_R][y : M_R]M = 0$. Since, $Ann(M)$ is a prime ideal, either $[x : M_R]M = 0$ or $[y : M_R]M = 0$, that is, for each $0 \neq x, y \in M$, either $[x : M_R] = Ann(M)$ or $[y : M_R] = Ann(M)$. If $Ann(M) \subset [x_0 : M_R]$ for some $0 \neq x_0 \in M$, we show that $Rx_0 = \{0, x_0\}$. Let $rx_0 \neq x_0$, where $r \in R$. Since $[x_0 : M_R]M \subseteq Rx_0$ and $r[x_0 : M_R]M \subseteq Rrx_0$, we have $r[x_0 : M_R] \subseteq [rx_0 : M_R] = Ann(M)$. Therefore, $rM = 0$ and so $rx_0 = 0$. Thus, $Rx_0 = \{0, x_0\}$ which is a simple submodule of M and $Soc(M) \neq 0$, a contradiction. Therefore, $[x : M_R] = Ann(M)$ for each $x \in M$, that is, M is a weakly virtually divisible module. □

We conclude this section with the following open problems.

1. Let M be an R-module. Then $ann_t(\Gamma(M_R))$ is a connected graph and $diam(ann_t(\Gamma(M_R))) \leq 3$.

2. Let M be an R-module. Then $\dim(ann_t(\Gamma(M_R)))$ is a finite number if and only if the graph $ann_t(\Gamma(M_R))$ is finite.

3. The number $\dim(ann_t(\Gamma(M_R)))$ is finite if and only if M is finite over R.

Acknowledgements. The authors are grateful to the anonymous referees for their valuable suggestions and comments. The first author thanks UGC, New Delhi for the support under research project no. MRP-MAJOR-MATH-2013-8034.

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Shariefuddin Pirzada
Department of Mathematics
University of Kashmir
Srinagar 190006, India
E-mail address: pirzadasd@kashmiruniversity.ac.in

Rameez Raja
Department of Mathematics
University of Kashmir
Srinagar 190006, India
E-mail address: rameeznaqash@gmail.com

Acta Univ. Sapientiae, Mathematica, **10**, 2 (2018) xx–yy

Computing metric dimension of compressed zero divisor graphs associated to rings

S. Pirzada
Department of Mathematics, University of Kashmir, Srinagar, India
email: pirzadasd@kashmiruniversity.ac.in

M. Imran Bhat
Department of Mathematics, University of Kashmir, Srinagar, India
email: imran_bhat@yahoo.com

Abstract. For a commutative ring R with $1 \neq 0$, a compressed zero-divisor graph of a ring R is the undirected graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0],[1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : ann(x) = ann(y)\}$ and the two distinct vertices $[x]$ and $[y]$ of $Z(R_E)$ are adjacent if and only if $[x][y] = [xy] = [0]$, that is, if and only if $xy = 0$. In this paper, we study the metric dimension of the compressed zero divisor graph $\Gamma_E(R)$, the relationship of metric dimension between $\Gamma_E(R)$ and $\Gamma(R)$, classify the rings with same or different metric dimension and obtain the bounds for the metric dimension of $\Gamma_E(R)$. We provide a formula for the number of vertices of the family of graphs given by $\Gamma_E(R \times \mathbb{F})$. Further, we discuss the relationship between metric dimension, girth and diameter of $\Gamma_E(R)$.

1 Introduction

Beck [7] first introduced the notion of a zero divisor graph of a ring R and his interest was mainly in coloring of zero divisor graphs. Anderson and Livingston [3] studied zero divisor graph of non-zero zero divisors of a commutative ring R. For a commutative ring R with $1 \neq 0$, let $Z^*(R) = Z(R) \setminus \{0\}$ be the set

2010 Mathematics Subject Classification: 13A99, 05C78, 05C12
Key words and phrases: Metric dimension, zero-divisor graph, compressed zero-divisor graph, equivalence classes

of non-zero zero divisors of R. A *zero divisor graph* $\Gamma(R)$ is the undirected graph with vertex set $Z^*(R)$ and the two vertices x and y are adjacent if and only if $xy = 0$. This zero divisor graph has been studied extensively and even more the idea has been extended to the ideal based zero divisor graphs in [15, 23] and modules in [20]. Inspired by ideas from Mulay [16], we study the zero divisor graph of equivalence classes of zero divisors of a ring R. Anderson and LaGrange [4] studied this under the term compressed zero divisor graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0], [1]\}$, constructed by taking the vertices to be equivalence classes $[x] = \{y \in R \mid ann(x) = ann(y)\}$, for every $x \in R \setminus ([0] \cup [1])$ and each pair of distinct classes $[x]$ and $[y]$ is joined by an edge if and only if $[x][y] = 0$, that is, if and only if $xy = 0$. If x and y are distinct adjacent vertices in $\Gamma(R)$, we note that $[x]$ and $[y]$ are adjacent in $\Gamma_E(R)$ if and only if $[x] \neq [y]$. It is clear that $[0] = \{0\}$ and $[1] = R \setminus Z(R)$ and that $[x] \subseteq Z(R) \setminus \{0\}$, for each $x \in R \setminus ([0] \cup [1])$. Some results on the compressed zero divisor graph can be seen in [5].

For example, consider $R = \mathbb{Z}_{12}$. Here, $Z^*(R) = \{2, 3, 4, 6, 8, 9, 10\}$ is the vertex set of $\Gamma(R)$, see Fig 1(a). For the vertex set of $\Gamma_E(R)$, we have $ann(2) = \{6\}$, $ann(3) = \{4, 8\}$, $ann(4) = \{3, 6, 9\}$, $ann(6) = \{2, 4, 6, 8, 10\}$, $ann(8) = \{3, 6, 9\}$, $ann(9) = \{4, 8\}$, $ann(10) = \{6\}$.

So, $Z(R_E) = \{[2], [3], [4], [6]\}$ is the vertex set of $\Gamma_E(R)$, see Fig 1(b).

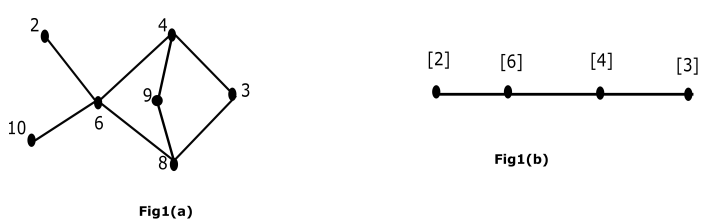

Figure 1: $\Gamma(\mathbb{Z}_{12})$ and $\Gamma_E(\mathbb{Z}_{12})$

We note that the vertices of the graph $\Gamma_E(R)$ correspond to *annihilator* ideals in the ring and hence *prime* ideals if R is a Noetherian ring in which case $Z(R_E)$ is called as the *spectrum* of a ring. Clearly $\Gamma_E(R)$ is connected and $diam(\Gamma_E(R)) \leq 3$. Also $diam(\Gamma_E(R)) \leq diam(\Gamma(R))$. Anderson and LaGrange [5] showed that $gr(\Gamma_E(R)) \leq 3$ if $\Gamma_E(R)$ contains a cycle and determined the structure of $\Gamma_E(R)$ when it is acyclic and the monoids R_E when $\Gamma_E(R)$ is a star graph. In [4], they also show that $\Gamma_E(R) \cong \Gamma_E(S)$ for a Noetherian or finite commutative ring S.

The compressed zero-divisor graph has some advantages over the earlier

studied zero divisor graph $\Gamma(R)$ as seen in [1, 2, 3] or subsequent zero divisor graph determined by ideal of R as seen in [15, 23]. For example, Spiroff and Wickham [[27], Proposition 1.10] showed that there are no finite regular graphs $\Gamma_E(R)$ for any ring R with more than two vertices. Further, they showed that R is a local ring (a ring R is said to be a local ring if it has a unique maximal ideal) if $\Gamma_E(R)$ is a star graph with at least four vertices.

Another important aspect of studying graphs of equivalence classes is the connection to associated primes of the ring. In general, all the associated primes of a ring R correspond to distinct vertices in $\Gamma_E(R)$. Through out, R will denote a commutative ring with unity, $U(R)$ its set of units. We will denote a finite field on q elements by $\mathbb{F}_q$, ring of integers modulo n by $\mathbb{Z}_n$ and all graphs are simple graphs in the sense that there are no loops. For basic definitions from graph theory we refer to [11, 17], and for commutative ring theory we refer to [6, 13].

A graph G is *connected* if there exists a path between every pair of vertices in G. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of the shortest $u - v$ path in G. If such a path does not exist, we define $d(u, v)$ to be infinite. The *diameter* of a graph is the maximum distance between any two vertices of G. The diameter is 0 if the graph consists of a single vertex. Also, the *girth* of a graph G, denoted by $gr(G)$, is the length of a smallest cycle in G. Slater [25] introduced the concept of a *resolving set* for a connected graph G under the term locating set. He referred to a minimum resolving set as a reference set for G and called the cardinality of a minimum resolving set (reference set) the location number of G. Independently, Harary and Melter [12] discovered these concepts as well but used the term metric dimension, rather than location number. The concept of metric dimension has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [8, 9], robot navigation [14], combinatorial optimization [24], sonar and coast guard Loran [26]. We adopt the terminology of Harary and Melter.

In this paper, we study the notion of metric dimension of $\Gamma_E(R)$. We explore the relationship between metric dimension of $\Gamma_E(R)$ and $\Gamma_E(R)$. We obtain the metric dimension of $\Gamma_E(R)$ whenever it exists. We also classify the rings having the same or different metric dimension and obtain bounds for the metric dimension of $\Gamma_E(R)$. We also provide relationship between the metric dimension, girth and diameter of $\Gamma_E(R)$.

2 Metric dimension of some graphs $\Gamma_E(R)$

Let G be a connected graph with $n \geq 2$ vertices. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the k-vector as the metric representation (locating code) of v with respect to W as

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

The set W is a resolving set of G if distinct vertices have distinct metric representations (codes) and a resolving set containing the minimum number of vertices is called a *metric basis* for G and the *metric dimension*, denoted by $dim(G)$, of G is the cardinality of a metric basis. If W is a finite metric basis, we say that $r(v|W)$ are the metric coordinates of vertex v with respect to W. The only vertex of G whose metric coordinate with respect to W has 0 in its ith coordinate of $r(v|W)$ is $\{w_i\}$. So the vertices of W necessarily have distinct metric representations. Since only those vertices of G that are not in W have coordinates all of which are positive, it is only these vertices that need to be examined to determine if their representations are distinct. This implies that the metric dimension of G is at most $n-1$. In fact for every connected graph G of order $n \geq 2$, we have $1 \leq dim(G) \leq n-1$.

For example, consider the graph G given in Figure 2. Take $W_1 = \{v_1, v_3\}$. So, $r(v_1|W_1) = (0,1)$, $r(v_2|W_1) = (1,1)$, $r(v_3|W_1) = (1,0)$, $r(v_4|W_1) = (1,1)$, $r(v_5|W_1) = (2,1)$. Notice, $r(v_2|W_1) = (1,1) = r(v_4|W_1)$, therefore W_1 is not a resolving set. However, if we take $W_2 = \{v_1, v_2\}$, then $r(v_1|W_2) = (0,1)$, $r(v_2|W_2) = (1,0)$, $r(v_3|W_2) = (1,1)$, $r(v_4|W_2) = (1,2)$, $r(v_5|W_2) = (2,1)$. Since distinct vertices have distinct metric representations, W_2 is a minimum resolving set and thus this graph has metric dimension 2.

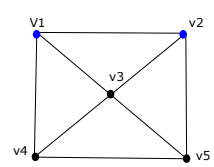

Figure 2: $dim(G) = 2$

Now, we have the following observation.

Lemma 1 *A connected graph G of order n has metric dimension 1 if and only if $G \cong P_n$, where P_n denotes a path on n vertices of length $n-1$.*

Proof. Suppose $G \cong P_n$. Let $x_1 - x_2 - \cdots - x_n$ be a path on n vertices of G. Since $d(x_i, x_1) = i - 1$ for $1 \leq i \leq n$, it follows $\{x_1\}$ is a minimum resolving set and therefore metric basis for $\Gamma_E(R)$. So $dim(P_n) = 1$.

Conversely, let G be not a path. Then either G is a cycle or it contains a vertex v whose degree is at least 3. But, G can not be a cycle as $dim(G) = 2$, see [[18], Lemma 2.3]. Let u_1, u_2, ..., u_k be the vertices adjacent to v. Since $dim(G) = 1$ and if $W = \{w\}$ is a metric basis for G, then the metric representation of every vertex has a single coordinate. If d is the length of the shortest path from v to w, the coordinates of each u_i with respect to W is one of $\{d-1, d, d+1\}$, but $d(u_i, w) = d$ can not occur for all i $(1 \leq i \leq k)$. Therefore, it follows that at least two adjacent vertices of v have the same metric coordinates, which is a contradiction. Hence G is a path. □

A graph $G(V, E)$ in which each pair of distinct vertices is joined by an edge is called a complete graph. A *complete* graph of n vertices is denoted by K_n. A graph G is said to be *bipartite* if its vertex set V can be partitioned into two sets V_1 and V_2 such that every edge of G has one end in V_1 and another in V_2. A bipartite graph is complete if each vertex of one partite set is joined to every vertex of the other partite set. We denote the complete bipartite graph with partite sets of order m and n by $K_{m,n}$. More generally, a graph is complete r-partite if the vertices can be partitioned into r distinct subsets, but no two elements of the same subset are adjacent. Based on the above definitions, we have the following observations.

Proposition 1 *The metric dimension of the compressed zero divisor graph $\Gamma_E(R)$ is 0 if and only if the zero divisor graph $\Gamma(R)$ of R ($R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$) is a complete graph.*

Proof. If $\Gamma(R) \cong K_n$, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z^*(R)$. Let $v_1, v_2, \ldots, v_n$ be the zero divisors of $\Gamma(R)$, then $[v_1] = [v_2], \cdots = [v_n]$ implies that all the vertices of $\Gamma(R)$ would collapse to a single vertex in $\Gamma_E(R)$ and we know the metric dimension of a single vertex graph is 0.

Conversely, assume that $\Gamma(R)$ is not isomorphic to K_n. Then $\Gamma(R)$ contains at least one vertex not adjacent to all the other vertices. Thus $|\Gamma_E(R)| \geq 2$, so that $dim(\Gamma_E(R)) \geq 1$. □

We can also obtain the converse part by letting $dim(\Gamma_E(R)) = 0$. Then $\Gamma_E(R) = \{[a]\}$ for some $a \in Z^*(R)$, that is, $\Gamma_E(R)$ is a graph on a single vertex, which then implies $\Gamma(R)$ is either isomorphic to a single vertex or a

complete graph K_n, for all $n \geq 1$. If G is a connected graph of order $n \geq 2$, we say two distinct vertices u and v are distance similar, if $d(u,a) = d(v,a)$ for all $a \in V(G) - \{u,v\}$. It can be seen that the distance similar relation ($\sim$) is an equivalence relation on $V(G)$ and two distinct vertices are distance similar if either $uv \notin E(G)$ and $N(u) = N(v)$, or $uv \in E(G)$ and $N[u] = N[v]$. Further we can find several results on metric dimension for zero divisor graphs of rings in [18, 19, 21].

Proposition 2 *The metric dimension of $\Gamma_E(R)$ is 1 if $\Gamma(R)$ is isomorphic to a complete bipartite graph $K_{m,n}$, with m or $n \geq 2$.*

Proof. Let $\Gamma(R)$ be isomorphic to a complete bipartite graph $K_{m,n}$ with two distance similar classes V_1 and V_2. Let $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$ such that $u_i v_j = 0$ for all $i \neq j$. Clearly, each of V_1 and V_2 is an independent set. We see that $[u_1] = [u_2] = \cdots = [u_m]$ and $[v_1] = [v_2] = \cdots = [v_n]$, so that V_1 and V_2 each represents a single vertex in $\Gamma_E(R)$. Since the graph is connected, $\Gamma_E(R)$ is isomorphic to $K_{1,1}$, a path on two vertices. Therefore by Lemma 1, we have $dim(\Gamma_E(R)) = 1$. $\square$

Remark 1 *Note that the converse of this result need not be true, the graph illustrated in Fig.1 being a counter example. However, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma_E(R) \cong K_{1,1}$ with metric dimension 1 and $\Gamma(R) \cong \Gamma_E(R)$.*

One of the important differences between $\Gamma(R)$ and $\Gamma_E(R)$ is that the later can not be complete with at least three vertices, as seen in [[27], Proposition 1.5]. However, if $\Gamma_E(R)$ is complete r-partite, then $r = 2$ and $\Gamma_E(R) \cong K_{n,1}$, for some $n \geq 1$, see [[27], Proposition 1.7]. A second look at the above result allows us to deduce some facts about star graphs. A complete bipartite graph of the form $K_{n,1}, n \in \mathbb{N} \cup \{\infty\}$ is called a *star graph*. If $n = \infty$, we say the graph is an infinite star graph.

Corollary 1 *If R is a ring such that $\Gamma_E(R)$ is a star graph $K_{n,1}$ with $n \geq 2$, then $dim(\Gamma_E(R)) = n - 1$.*

Proof. First we identify a centre vertex of $K_{n,1}$ adjacent to n vertices. Then partition the vertex set V of order $n+1$ into two distance similar classes, with centre vertex in one class V_1 and the remaining n vertices in another class V_2 which is clearly an independent set. Choose a subset of vertices W of V and

$u \sim v$. Then $r(u|W) = r(v|W)$ whenever both $u, v \notin W$. Hence the metric basis contains all except at most two vertices one from each class V_i, $1 \leq i \leq 2$. Therefore, $dim(K_{n,1}) = |V(\Gamma_E(R))| - 2 = n + 1 - 2 = n - 1$. □

For example the metric dimension of $K_{1,3}$ is 2, see Figure 3.

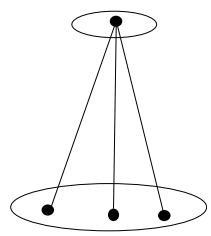

Figure 3: $dim(K_{1,3} = 2)$

Corollary 2 *If R is a commutative ring such that $\Gamma_E(R)$ has at least $n \geq 3$ vertices, then $dim(\Gamma_E(R)) \neq n - 1$.*

Proof. Suppose $dim(\Gamma_E(R)) = n - 1$, $(n \geq 3)$. Then, by [[18], Lemma 2.2], $\Gamma_E(R)$ is a complete graph on n vertices which is a contradiction to the argument prior to Corollary 1. Therefore $dim(\Gamma_E(R)) \neq n - 1$. □

Remark 2 *It is not known whether for each positive integer n, the star graph $K_{n,1}$ can be realized as $\Gamma_E(R)$ for some ring R. However, there is a ring $R = \mathbb{Z}_2[x, y, z]/(x^2, y^2)$ whose $\Gamma_E(R)$ is a star graph with infinitely many ends, that is, $\Gamma_E(R)$ is an infinite star graph. This ring also shows that the Noetherian condition is not enough to force $\Gamma_E(R)$ to be finite, see [27]. For $n = 3$, if the local ring R is isomorphic to $\mathbb{Z}_4[x]/(x^2)$ or $\mathbb{Z}_2[x, y]/(x^2, y^2)$ or $\mathbb{Z}_4[x, y]/(x^2, y^2, xy-2, 2x, 2y)$, then $\Gamma_E(R) \cong K_{1,3}$ and therefore $dim(\Gamma_E(R)) = 2$. For $n = 4$, if the local ring R is isomorphic to $\mathbb{Z}_8[x, y]/(x^2, y^2, 4x, 4y, 2xy)$, then $\Gamma_E(R) \cong K_{1,4}$ and therefore $dim(\Gamma_E(R)) = 3$. For $n = 5$, if $R \cong \mathbb{Z}_2[x, y, z]/(x^2, y^2, z^2, xy)$, then $\Gamma_E(R) \cong K_{5,1}$ and therefore $dim(\Gamma_E(R)) = 4$. This star graph $K_{1,5}$ is the smallest star graph that can be realized as $\Gamma_E(R)$, but not as a zero divisor graph.*

By definition of the compressed zero divisor graph $\Gamma_E(R)$ of a ring R, it is clear that each vertex in $\Gamma_E(R)$ is a representative of a distinct class of zero

divisor activity in R. Thus, $dim(\Gamma_E(R)) \leq dim(\Gamma(R))$. However, the strict inequality holds if $\Gamma_E(R)$ has at least 3 vertices.

Example 1 *In the rings* $R = \frac{\mathbb{Z}_2[x,y]}{(x^2,xy,2x)}$, $R = \frac{\mathbb{Z}_4[x,]}{(x^2)}$, $R = \mathbb{Z}_{16}$, $R = \frac{\mathbb{Z}_8[x]}{(2x,x^2)}$, *it is easy to find that* $dim(\Gamma_E(R)) < dim(\Gamma(R))$.

It will be interesting to see the family of rings in the equality $dim(\Gamma_E(R) = dim(\Gamma(R)$ occurs.

A ring R is called a *Boolean ring* if $a^2 = a$ for every $a \in R$. Clearly a Boolean ring R is commutative with char(R) = 2, where char(R) denotes the characteristic of a ring R. More generally, a commutative ring is *von Neumann regular* ring if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$, or equivalently, R is a reduced zero dimensional ring, see [[13], Theorem 3.1]. A Boolean ring is clearly a von Neumann regular, but not conversely. For example, let $\{F_i\}_{i\in I}$ be a family of fields, then $\prod_{i\in I} F_i$ is always von Neumann regular, but it is Boolean if and only if $F_i \cong \mathbb{Z}_2$ for all $i \in I$. Also the set $B(R) = \{a \in \ R \mid a^2 = a\}$ of idempotents of a commutative ring R becomes a Boolean ring with multiplication defined in the same way as in R, and addition defined by the mapping $(a, b) \mapsto a + b - 2ab$. In [[13], Lemma 3.1], if $r, s \in \Gamma(R)$, the conditions $N(r) = N(s)$ and $[r] = [s]$ are equivalent if R is a reduced ring, and these are equivalent to the condition $rR = sR$ if R is a von Neumann regular ring. Furthermore, if R is a von Neumann regular ring and $B(R)$ is the set of idempotent elements of R, the mapping defined by $e \mapsto [e]$ is isomorphism from the subgraph of $\Gamma(R)$ induced by $B(R) \setminus \{0, 1\}$ onto $\Gamma_E(R)$ [[13], Proposition 4.5]. In particular, if R is a Boolean ring (i.e., R = B(R)), then $\Gamma_E(R) \cong \Gamma(R)$. From this discussion, we have the following characterization.

Proposition 3 *Let R be a reduced commutative ring with unity. Then, metric dimension of the zero divisor graph $\Gamma(R)$ equals to metric dimension of its corresponding compressed zero divisor graph if R is a Boolean ring.*

Note that the converse of this result is not true in general. For example, the graphs in Figure 4 being a counter example, where $dim(\Gamma(\mathbb{Z}_6)) = dim(\Gamma_E(\mathbb{Z}_6))$, but R is not a Boolean ring.

Corollary 3 *Let R and S be commutative reduced rings with unity* 1. *If* $\Gamma(R) \cong \Gamma(S)$, *then* $dim(\Gamma_E(R)) = dim(\Gamma_E(S))$.

Figure 4: $dim(\Gamma(\mathbb{Z}_6)) = dim(\Gamma_E(\mathbb{Z}_6)) = 1$

Remark 3 *As seen in [[21], Theorem 2], for the graph* $\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)$ *of a finite Boolean ring*

$$dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) \leq n, \quad dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) \leq n-1$$

for $n = 2, 3, 4$ *and* $dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) = n$ *for* $n = 5$*. This is also true for* $\Gamma_E(R)$*, follows by Proposition 3. The case* $n > 5$ *is still open.*

3 Bounds for the metric dimension of $\Gamma_E(R)$

In this section, we investigate the role of metric dimension in the study of the structure of the graph $\Gamma_E(R)$. We also obtain metric dimension of some special type of rings that exhibit $\Gamma_E(R)$. Pirzada et al [18] characterized those graphs $\Gamma(R)$ for which the metric dimension is finite and for which the metric dimension is undefined [[18], Theorem 3.1]. The analogous of this result is as follows.

Theorem 1 *Let* R *be a commutative ring. Then*
(i) $dim(\Gamma_E(R))$ *is finite if and only if* R *is finite.*
(ii) $dim(\Gamma_E(R))$ *is undefined if and only if* R *is an integral domain.*

However, $dim(\Gamma_E(R))$ may be finite if R is infinite. For example, $R = \mathbb{Z}[x, y]/(x^3, xy)$ has $\Gamma_E(R)) \cong K_{1,3} + e$ (or paw graph), see Figure 5, and therefore has $dim = 2$.

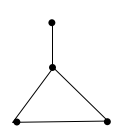

Figure 5:

The following lemma will be used to find the metric dimension of finite local rings.

Lemma 2 *If* R *is a finite local ring, then* $|R| = p^n$*, for some prime* p *and some positive integer* n*.*

Now, we have the following results.

Proposition 4 *If R is a local ring with $|R| = p^2$ and $p = 2, 3, 5$, then $dim(\Gamma_E(R))$ is either 0 or undefined.*

Proof. Consider all local rings of order p^2 with p a prime. According to [[10], p. 687] local rings of order p^2 are precisely $\mathbb{F}_{p^2}$, $\frac{\mathbb{F}_p[x]}{(x^2)}$, and $\mathbb{Z}_{p^2}$. If R is a field of order p^2, i.e., $R \cong \mathbb{F}_{p^2}$, then $\Gamma_E(R)$ is an empty graph, which implies $dim(\Gamma_E(R))$ is undefined. If R is not a field and $|R| = p^2$, i.e., $R \cong \frac{\mathbb{F}_p[x]}{(x^2)}$, or $\mathbb{Z}_{p^2}$ then we can find that $\Gamma_E(R)$ is a single vertex, when $p = 2,\ 3$ or 5 which then immediately gives that $dim(\Gamma_E(R)) = 0$. $\square$

From the above result, we also observe that $dim(\Gamma(R)) = dim(\Gamma_E(R))$, if

$$R \cong \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

Proposition 5 *If R is a local ring (not a field) of order*
(i) p^3 with $p = 2$ or 3, then $dim(\Gamma_E(R))$ is 0, and $dim(\Gamma_E(R)) = 1$ only if $R \cong \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_8$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 6)$ or $\mathbb{Z}_{27}$
(ii) p^4 with $p = 2$, then $dim(\Gamma_E(R))$ is 0, 1 or 2.

Proof. (i) The following is the list of all the local rings of order p^3.

$$\mathbb{F}_{p^3},\ \frac{\mathbb{F}_p[x,y]}{(x,y)^2},\ \frac{\mathbb{F}_p[x]}{(x^3)},\ \frac{{\mathbb{Z}_p}^2[x]}{(px, x^2)},\ \frac{{\mathbb{Z}_p}^2[x]}{(px, x^2 - p)}$$

Case(a). When $p = 2$, the equivalence classes of the zero divisors in the local rings $\mathbb{Z}_2[x,y]/(x,y)^2$ and $\mathbb{Z}_4[x]/(2x, x^2)$ are same and is given by $[a] = \{x, y, x+y\}$ for any zero divisor a of the first ring and $[b] = \{2, x, x+2\}$ for any zero divisor b of the second ring, that is, they get collapsed to a single vertex. Therefore $dim(\Gamma_E(R)) = 0$. However, $\Gamma_E(R)$ of the rings $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_8$ and $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ is isomorphic to the graph $K_{1,1}$, which then, by Lemma 1, gives $dim_E(R) = 1$.
Case(b). When $p = 3$ in the above list of local rings, we find that the compressed zero divisor graph structure of the rings $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2-3)$, $\mathbb{Z}_9[x]/(3x, x^2-6)$ and $\mathbb{Z}_{27}$ is same and is isomorphic to $K_{1,1}$. Then, by Lemma 1, we have $dim_E(R) = 1$. Also, in the rings $\frac{{\mathbb{Z}_3}^2[x]}{(3x, x^2)}$ and $\frac{\mathbb{Z}_3[x,y]}{(x,y)^2}$, the equivalence classes of all the zero divisors is same and is given by $[a] = \{3, 6, x, 2x, x+$

$3, x+6, 2x+3, 2x+6\}$ for any non-zero zero divisor a of the first ring and $[b] = \{x, 2x, y, 2y, x+y, 2x+y, x+2y, 2x+2y\}$ for any non-zero zero divisor b of the later ring. Thus, $\Gamma_E(R)$ for both rings is a graph on a single vertex and follows that $dim(\Gamma_E(R) = 0$.
(ii) Consider the local rings of order p^4, when $p = 2$. Corbas and Williams [10] conclude that there are 21 non-isomorphic commutative local rings with identity of order 16. The rings with $dim(\Gamma_E(R)) = 0$ are $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y, z]/(x, y, z)^2$ and $\mathbb{Z}_4[x]/(x^2 + x + 1)$. The rings with $dim(\Gamma_E(R)) = 1$ are $Z_2[x]/(x^4)$, $Z_2[x, y]/(x^3, xy, y^2)$, $Z_4[x]/(2x, x^3 - 2)$, $\mathbb{Z}_4[x]/(x^2 - 2)$, $\mathbb{Z}_8[x]/(2x, x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_4[x]/(x^2 - 2x - 2)$, $\mathbb{Z}_8[x]/(2x, x^2 - 2)$, $\mathbb{Z}_4[x]/(x^2 - 2x)$, $\mathbb{Z}_2[x]/(x^4)$ and $\mathbb{Z}_2[x]/(x^4)$. Further the rings with $dim(\Gamma_E(R)) = 2$ are $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y]/(x^2, y^2)$ and $\mathbb{Z}_2[x, y]/(x^2 - y^2, xy)$. □

Now, we find the metric dimension of $\Gamma_E(\mathbb{Z}_n)$.

Proposition 6 *Let p be a prime number.*
(i) If $n = 2p$ and $p > 2$, then $dim(\Gamma_E(\mathbb{Z}_n)) = 1$.
(ii) If $n = p^2$, then $dim(\Gamma_E(\mathbb{Z}_n)) = 0$.

Proof. (i) If $p = 2$, since $\Gamma_E(\mathbb{Z}_4)$ is a graph with single vertex. So, $dim(\Gamma_E(\mathbb{Z}_4) = 0$.

If $p > 2$, the zero divisor set of $\mathbb{Z}_n$ is $\{2, 2.2, 2.3, \ldots, 2.(p-1), p\}$. Since, $char(\mathbb{Z}_n) = 2p$, it follows that p is adjacent to all other vertices. Thus the equivalence classes of these zero divisors are given by
$[p] = \{2, 2.2, 2.3, \ldots, 2.(p-1)\}$, $[2] = [2.2] = \cdots = [2.(p-1)] = \{p\}$.

So, the vertex set of $\Gamma_E(\mathbb{Z}_n)$ is $Z(R_E) = \{[p], [2x]\}$ for any positive integer $x = 1, 2, \ldots, p-1$. Thus $\Gamma_E(\mathbb{Z}_n)$ is a path P_2 which then, by Lemma 1, gives $dim(\Gamma_E(\mathbb{Z}_n) = 1$.
(ii) If $n = p^2$ and $p > 2$, the zero divisor set of $\mathbb{Z}_n$ is $\{p, p.2, p.3, \ldots, p(p-1)\}$. Since $char(\mathbb{Z}_n) = p^2$, it follows that the equivalence class of all these zero divisors is same and is $\{p, p.2, p.3, \ldots, p.(p-1)\}$. Thus, $\Gamma_E(R)$ in this case is a graph on a single vertex and therefore $dim(\Gamma_E(R) = 0$. □

From the above result, we have the following observations.

Corollary 4 *Let p be a prime number*
(i) If $n = 2p$ and $p > 2$, then $|\Gamma_E(\mathbb{Z}_n)| = 2$.
(ii) If $n = p^2$, then $|\Gamma_E(\mathbb{Z}_n)| = 1$.
(iii) If $n = p^k$, $k > 3$ and $p > 2$, then $|\Gamma_E(\mathbb{Z}_n)| = k - 1$.

Proof. (i) and (ii) follow from Proposition 6.
(iii) When $n = p^k$, $k > 3$ and $p \geq 2$, the zero divisors of $\mathbb{Z}_n$ are $Z(\mathbb{Z}_n) = \{up^i | u \in U(\mathbb{Z}_n)\}$, for $i = 1, 2, \ldots, k-1$. Now the equivalence classes of zero divisors are $[up] = \{up^{k-1}\}$, $[up^2] = \{up^{k-1}, up^{k-2}\}$, $\ldots$, $[up^{k-1}] = \{up^{k-1}, up^{k-2}, \ldots, up^2, up\}$.
In this way, we get $k-1$ distinct equivalence classes. Thus, $|\Gamma_E(\mathbb{Z}_n)| = k-1$. □

Corollary 5 *$dim(\Gamma_E(\mathbb{Z}_n)) \leq 2k-2$, where $n = p^k$, for any prime $p > 2$ and $k > 3$.*

Proof. By [[18], Theorem 2.1], If G is a connected graph with G partitioned into m distance similar classes that consist of a single vertex, then $dim(G) \leq |V(G)| + m$.

Using part (iii) of Corollary 4, the result follows. □

The following important lemma, which is used later in the proof of several results, provides a combinatorial formula for the number of vertices of the compressed zero divisor graph $\Gamma_E(R \times \mathbb{F}_q)$.

Lemma 3 *Let R be a finite commutative local ring with unity 1 and $|R| = p^k$ and let $\mathbb{F}_q$ be a finite prime field. Then $|Z^*((R \times F_q)_E)| = 2k$ or $2(1 + |Z^*(R_E)|$.*

Proof. Let R be a finite commutative local ring with unity and $|R| = p^k$, $k \geq 1$. We consider the following three cases.
Case 1. $R \cong \mathbb{F}_p$, for some prime p. Then the zero divisor set of $Z^*(\mathbb{F}_p \times \mathbb{F}_q) = \{\{(a, 0)\}, \{(0, x)\}\}$, for every $a \in U(R)$ and $0 \neq x \in \mathbb{F}_q$. Now, to find the equivalence classes of these zero divisors, we can find that the set $\{(a, 0)\}$ and $\{(0, x)\}$ respectively correspond to vertices $[(a, 0)]$ and $[(0, x)]$ in $\Gamma_E(R \times \mathbb{F}_q)$, for any $a \in U(R)$ and for any $x \in \mathbb{F}_q$. Therefore, $|Z^*(R \times \mathbb{F}_q)_E| = 2k$, where $k = 1$.
Case 2. $R \cong \mathbb{Z}_p^k$, $(k \geq 2)$. The equivalence class of each element $(a, 0)$, for every $a \in U(R)$ is same, since $[(a, 0)] = \{(0, x)\}$, for all $x \in \mathbb{F}_q$. In this way, we get one vertex of $\Gamma_E(R \times \mathbb{F}_q)$. Also, the equivalence classes of each element $(0, x)$, for every $0 \neq x \in \mathbb{F}_q$ is same, since $[(0, x)] = \{(a, 0)\}$. So, this gives another vertex of $\Gamma_E(R \times \mathbb{F}_q)$. Moreover, for any unit u in R, we get two zero divisor sets of equivalence classes given by

$$Z_1 = \{[(up, 0)], [(up^2, 0)], \ldots, [(up^{k-1}, 0)]\}$$
$$Z_2 = \{[(up, 1)], [(up^2, 1)], \ldots, [(up^{k-1}, 1)]\}.$$

We note that there is no other possible equivalence class. Claim $[(up^{k-1}, 1)] = [(up^{k-1}, x_i]$, for all $1 \leq i \leq q-2$. If $[(up^{k-1}, 1)] \neq [(up^{k-1}, x_i]$, there exists some zero divisor in $R \times \mathbb{F}_q$, say $(a_1, 0)$ adjacent to $(up^{k-1}, 1)$ but not adjacent to (up^{k-1}, x_i), which is a contradiction.

The total number of zero divisors is $|Z^*((R \times \mathbb{F})_E)| = 2 + |Z_1| + |Z_2| = 2 + k - 1 + k - 1 = 2k$ or $2 + 2|Z^*(R_E)| = 2(1 + |Z^*(R_E)|)$.

Case 3. R is a local ring other than $\mathbb{F}_p$ and $\mathbb{Z}_{p^k}$. So, we consider all local rings R with $|R| = p^k$, especially $k = 2$, 3 or 5 and the rings of order p^2, p^3 or p^4 are mentioned in proof of Proposition 4 and 5. Then the set of zero divisors of equivalence classes include

$$[(a, 0)],\ a \in U(R)$$
$$[(0, x_i)], \text{ for any } i,\ 1 \leq i \leq q-2$$
$$Z_1 = \{[(a_1, 0)], [(a_2, 0)], \ldots, [(a_r, 0)]\}$$
$$Z_2 = \{[(a_1, 1)], [(a_2, 1)], \ldots, [(a_r, 1)]\}.$$

where $a_1, a_2, \ldots, a_r$ are the non-zero zero divisors of the set $Z(R_E)$.

There is no other possible equivalence class as a zero divisor. Claim $[(a_i, 1)] = [(a_i, x_j)]$, $1 \leq i \leq r$ and $1 \leq j \leq q-2$. For if, $[(a_i, 1)] \neq [(a_i, x_j)]$, there exists some zero divisor $(a_k, 0)$ adjacent to one of $[(a_i, 1)]$ or $[(a_i, x_j)]$, but not to the other, which is a contradiction.

Thus, $|Z^*((R \times \mathbb{F}_q)_E)| = 2 + 2|Z^*(R_E)| = 2(1 + |Z^*(R_E)|$. $\square$

Example 2 *Consider the ring $\mathbb{Z}_8 \times \mathbb{Z}_3$, Here, $R = \mathbb{Z}_{2^3}$, $k = 3$, and $U(R) = \{1, 3, 5, 7\}$. For the zero divisors of equivalence classes, we have*
$[(1, 0)] = \{(0, 1), (0, 2)\}$, $[(3, 0)] = \{(0, 1), (0, 2)\}$, $[(5, 0)] = \{(0, 1), (0, 2)\}$,
$[(7, 0)] = \{(0, 1), (0, 2)\}$.
Also, $[(2, 0)] = \{(0, 1), (0, 2), (4, 0), (4, 1), (4, 2)\}$,
$[(4, 0)] = \{(0, 1), (0, 2), (2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$,
$[(6, 0)] = \{(0, 1), (0, 2), (4, 0), (4, 1), (4, 2)\}$.
Moreover, $[(0, 1)] = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0)\}$,
$[(0, 2)] = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0)\}$, $[(2, 1)] = \{(4, 0)\}$,
$[(4, 2)] = \{(2, 0), (4, 0), (6, 0)\}$, $[(4, 1)] = \{(2, 0), (4, 0), (6, 0)\}$, $[(2, 2)] = \{(4, 0)\}$,
$[(6, 1)] = \{(4, 0)\}$, $[(6, 2)] = \{(4, 0)\}$
Thus, $|\Gamma_E(\mathbb{Z}_8 \times \mathbb{Z}_3)| = \{[(0, 1)], [(1, 0)], [(2, 0)], [(4, 0)], [(2, 1)], [(4, 1)]\}$.
Using Lemma 3, we can directly have, $|\Gamma_E(\mathbb{Z}_8 \times \mathbb{Z}_3)| = 2 \times 3 = 6$.

Remark 4 *Lemma 3 holds if we replace $\mathbb{F}_q$ by any finite field $\mathbb{F}$. More generally, let R be any finite commutative ring with unity 1. We know $R \cong R_1 \times R_2$,*

where each R_i, $1 \leq i \leq 2$, is a local ring. If either R_1 or R_2 is a field, the number of vertices is always given by the formula $2(1 + |Z^*(R_{1E})|$ or $2(1 + |Z^*(R_{2E})|$, since the equivalence classes of zero divisors of $\Gamma_E(R_1 \times R_2)$ are always of the form $\{[(0,1)], [(1,0)], [(a,0)], [(0,b)], [(a,1)], [(1,b)], [(a,b)]$, where a and b are the non-zero zero divisors and $Z^*(R_{1E})$, $Z^*(R_{2E})$ denote the number of zero divisor equivalence classes of R_1 and R_2 respectively. The result holds trivially if both R_1 and R_2 are fields.

Theorem 2 *Let R be a finite commutative local ring with unity 1 and finite field $\mathbb{F}_q$. Then, $dim(\Gamma_E(R \times \mathbb{F}_q)) = 1$ or at most $4k$ or $4t$ where $k \geq 2$ and t are integers, $t = 1 + |Z^*(R_E)|$.*

Proof. Let R be a finite commutative local ring with unity 1. We consider the following three cases.
Case 1. R is a field. Then, by Case 1 of Lemma 3, $\Gamma_E(R \times \mathbb{F}_q)$ is a path on two vertices. Therefore, by Lemma 2.1, $dim(\Gamma_E(R \times \mathbb{F}_q)) = 1$.
Case 2. $R \cong \mathbb{Z}_{p^k}$, $k \geq 2$. In this case, we partition the vertices into distance similar classes in $\Gamma_E(R)$ given by

$$V_1 = \{[(a,0)]\}, \text{ for any } a \in U(R)$$
$$V_2 = \{[(0,x)]\}, \text{ for any } x \in \mathbb{F}_q$$
$$Z_1 = \{[(up,0)]\},\ Z_2 = \{[(up^2,0)]\},\ \ldots,\ Z_{k-1} = \{[(up^{k-1},0)]\}$$
$$W_1 = \{[(up,1)]\},\ W_2 = \{[(up^2,1)]\},\ \ldots,\ W_{k-1} = \{[(up^{k-1},1)]\}$$

Then, $dim(\Gamma_E(R \times \mathbb{F}_q) \leq |Z^*((R \times \mathbb{F}_q)_E) + m$ where m is the number of distance similar classes that consist of a single vertex. Hence by case 2 of Lemma 3, we have

$$dim(\Gamma_E(R \times \mathbb{F}_q) \leq 2k + 2(k-1) + 2 = 4k.$$

Case 3. R is a local ring other than $\mathbb{Z}_p^k$ and $\mathbb{F}_p^k (k \geq 1)$. Then, by Case 3 of Lemma 3, $dim(\Gamma_E(R \times \mathbb{F}_q)) \leq 2(1 + |Z^*(R_E)| + 2|Z^*(R_E)| + 2 = 4(1 + |Z^*(R_E)|) = 4t$ where t is any integer given by $t = 1 + |Z^*(R_E)|$. □

We say that a graph G has a bounded degree if there exists a positive integer M such that the degree of every vertex is at most M. In the next theorems, we obtain an upper bound for the number of zero divisors in a finite commutative ring R with unity 1 with finite metric dimension. The analogous of these results holds in case of $\Gamma_E(R)$.

Proposition 7 *If $\Gamma(R)$ is a zero divisor graph with finite metric dimension k, then $|Z^*(R)| \leq 3^k + k$.*

Proof. Let $\Gamma(R)$ be a zero divisor graph with metric dimension k. We choose two vertices, say w_1 and w_2, from the metric basis W. Since the diameter of $\Gamma(R)$ is at most 3, each coordinate of metric representation is an integer between 0 and 3 and only the vertices of a metric basis have one coordinate 0. The remaining vertices must get a unique code from one of the 3^k possibilities. Therefore, $|Z^*(R)| \leq 3^k + k$. $\square$

Proposition 8 *Let R be a commutative ring and $\Gamma_E(R)$ be a corresponding compressed zero divisor graph with $|Z^*(R)| \geq 2$. Then $dim(\Gamma_E(R)) \leq |Z^*(R_E)| - d$, where d is the diameter of $\Gamma_E(R)$.*

Proof. By [[21], Theorem 5.2], if R is a commutative ring and $\Gamma(R)$ is the corresponding zero divisor graph of R such that $|Z^*(R)| \geq 2$, then $dim(\Gamma(R)) \leq |Z^*(R)| - d'$ where d' is the diameter of $\Gamma(R)$. Since

$$dim(\Gamma_E(R)) \leq dim(\Gamma(R)) \text{ and } |Z^*(R_E)| \leq |Z^*(R)|,$$

therefore

$$dim(\Gamma_E(R)) \leq |Z^*(R_E)| - d,$$

where d is the diameter of $\Gamma_E(R)$. $\square$

Proposition 9 *If $\Gamma(R)$ is a finite graph with metric dimension k, then every vertex of this graph has degree at most $3^k - 1$.*

Proof. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a metric basis of $\Gamma(R)$ with cardinality k. Consider a vertex v with metric representation

$$(d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).$$

If u is adjacent to v, then $r(v|W) \neq r(u|W)$ and $|d(v, w_i) - d(u, w_i)| \leq 1$ for all $w_i \in W$, $1 \leq i \leq k$. If d is distance from v to w_i, then the distance of u from w_i is one of the numbers $\{d, d-1, d+1\}$. Thus, there are three possible numbers for each of the k coordinates of $r(u|W)$, but $d(u, w_i) \neq d(v, w_i)$ for all $1 \leq i \leq k$. This implies that there are at most $3^k - 1$ different possibilities for $r(u|W)$. Since all vertices must have distinct metric coordinates, the degree of v is at most $3^k - 1$. $\square$

A graph G is *realizable* as $\Gamma_E(R)$ if $G \cong \Gamma_E(R)$ for some ring R. There are many results which imply that most graphs are not realizable as $\Gamma_E(R)$, like $\Gamma_E(R)$ is not a cycle graph, nor a complete graph with at least three vertices.

Proposition 10 *The metric dimension of realizable graphs* $\Gamma_E(R)$ *with* 3 *vertices is* 1.

Proof. Spiroff et al. proved that the only one realizable graph $\Gamma_E(R)$ with exactly three vertices as a graph of equivalence classes of zero divisors for some ring R is P_3, see Figure 6. Clearly, its metric dimension is 1.□

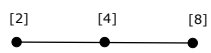

Figure 6: $\mathbb{Z}_{16}$

Proposition 11 *The metric dimension of realizable graphs* $\Gamma_E(R)$ *with* 4 *vertices is either* 1 *or* 2.

Proof. All the realizable graphs $\Gamma_E(R)$ on 4 vertices are shown in Figure 7. It is easy to see their metric dimension is either 1 or 2. □

Figure 7: $(\mathbb{Z}_4 \times \mathbb{F}_4)$ $\mathbb{Z}_4[x]/(x^2)$ $\frac{\mathbb{Z}[x,y]}{(x^3,xy)}$

Proposition 12 *The metric dimension of realizable graphs* $\Gamma_E(R)$ *with* 5 *vertices is either* 2 *or* 3.

Proof. The only realizable graphs of equivalence classes of zero divisors of a ring R with 5 vertices are shown in Figure 8. It is easy to see the metric dimension of the first three graphs is 2 and for the star graph is 3 (by Corollary 1). □

4 Relationship between metric dimension, girth and diameter of $\Gamma_E(R)$

In this section, we examine the relationship between girth, diameter and metric dimension of $\Gamma_E(R)$. Since $gr(\Gamma_E(R)) \in \{3, \infty\}$, it is worth to mention

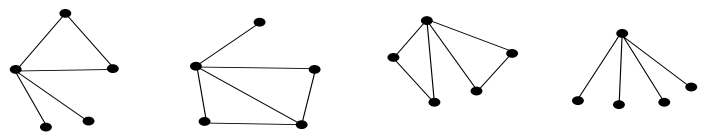

Figure 8: $(\mathbb{Z}_9[x]/(x^2)$, $\mathbb{Z}_{64}$, $Z_3[x,y]/(xy, x^3, y^3, x^2 - y^2)$, $\mathbb{Z}_8[x,y]/(x^2, y^2, 4x, 4y, 2xy)$

that, for a reduced commutative ring R with $1 \neq 0$, $gr(\Gamma_E(R)) = 3$ if and only if $gr(\Gamma(R)) = 3$ and that $gr(\Gamma_E(R)) = \infty$ if and only if $gr(\Gamma(R)) \in \{4, \infty\}$. However, if R is not reduced, then we may have $gr(\Gamma(R)) = 3$ and either $gr(\Gamma_E(R)) = 3$ or ∞. The following result gives the metric dimension of $\Gamma_E(R)$ in terms of the girth of $\Gamma_E(R)$ of a ring R.

Theorem 3 *Let R be a finite commutative ring with $gr(\Gamma_E(R)) = \infty$.*
(i) If R is a reduced ring, then $dim(\Gamma_E(R)) = 1$.
(ii) If $R \cong \mathbb{Z}_6, \mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, then $dim(\Gamma_E(R)) = |Z^*(R_E)| - 1$.
(iii) If $R \cong \mathbb{Z}_4$, $\mathbb{Z}_9$, $\mathbb{Z}_2[x]/(x^2)$, then $dim(\Gamma_E(R)) = 0$.
(iv) $\dim(\Gamma_E(R)) = 0$ or 1 if and only if $gr(\Gamma(R)) \in \{4, \infty\}$.

Proof. If R is a reduced ring and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we know $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Therefore, by Remark 4, R has two equivalence classes of zero divisors $[(0,1)]$ and $[(1,0)]$, adjacent to each other. Hence, $dim(\Gamma_E(R)) = 1$. Also, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then R being a Boolean ring, implies $\Gamma(R) \cong \Gamma_E(R)$. Therefore, by Case 1 of Lemma 3, the result follows. In part (ii), these rings are non reduced and $\Gamma_E(R)$ are isomorphic to $K_{1,1}$. Rings listed in part (iii) represents $\Gamma_E(R)$ on a single vertex, part (iv) follows from the above comments.□

We can also prove the Part (i) by using the fact that if R is reduced and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Thus $\Gamma(R)$ is a complete bipartite and the result follows from Proposition 2. Now, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R) \cong K_{1,1}$, whose metric dimension is 1. Since, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a Boolean ring, therefore by Proposition 3, we have $dim(\Gamma_E(R)) = 1$.

If R is a reduced ring with non-trivial zero divisor graph, then $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$ for some integer $k \geq 2$ and for finite fields $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k$. If R is not a reduced ring, then either R is local or $R \cong R_1 \times R_2 \times \cdots \times R_t$, for some integer $t \geq 2$ and local rings $R_1, R_2, \ldots, R_t$, where at least one R_i is not a

field. Now, we have the following observations for the finite commutative rings whose zero divisor graphs can be seen in [22].

Corollary 6 *If R is a finite commutative ring with unity 1 and $gr(\Gamma_E(R)) = \infty$, then the compressed zero divisor graph of the reduced rings $R \times \mathbb{F}$ where $\mathbb{F}$ is a finite field, is isomorphic to the compressed zero divisor graph of the following local rings with metric dimension 1, R being any local ring.*
$\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$, $\mathbb{Z}_8[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$, $\mathbb{Z}_9[x]/(3x, x^2-6)$, $\mathbb{Z}_9[x]/(3x, x^2-3)$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_{27}$.

Proof. The reduced rings $R \times \mathbb{F}$ with $gr(\Gamma_E(R)) = \infty$, all have compressed zero divisor graph isomorphic to $K_{1,1}$, by Case 2 of Lemma 3. Also, the local rings listed above have the same compressed zero divisor graph isomorphic to $K_{1,1}$. □

Proposition 13 *Let R be a finite commutative ring with 1 and $gr(\Gamma_E(R)) = \infty$. The following are the non reduced rings with $dim(\Gamma_E(R)) = 1$*
$\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_5 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_5 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2,x)^2$.

Proof. If R is not a local ring, we can write $R \cong R_1 \times R_2 \times \cdots \times R_k$, where $k \geq 2$ and each R_i is a local ring. In case of above rings $R \cong R_1 \times R_2$, where either R_1 or R_2 is a field. Therefore, using Remark 4, we have $|\Gamma_E(R)| = 4$ and it is easy to see that $\Gamma_E(R)$ isomorphic to a path on 3 vertices. Thus, $gr(\Gamma_E(R)) = \infty$ and $dim(\Gamma_E(R)) = 1$. □

If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, then it is easy to see that the three vertices $[(1,0,0)]$, $[(0,1,0)]$ and $[(0,0,1)]$ are adjacent with ends $[(0,1,1)]$, $[(1,0,1)]$, and $[(1,1,0)]$ respectively and thus $|\Gamma_E(R)| = 6$.

Proposition 14 *Let R be a reduced commutative ring and $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$. Then, $gr(\Gamma_E(R)) = 3$ and $dim(\Gamma_E(R)) = 2$.*

We now proceed to study the relationship between diameter and metric dimension of compressed zero divisor graphs. Since $diam(\Gamma_E(R)) \leq 3$, if $\Gamma_E(R)$ contains a cycle. We have the following results.

Theorem 4 *Let R be commutative ring and $\Gamma_E(R)$ be its corresponding compressed zero divisor graph.*
(i) $dim(\Gamma_E(R)) = 0$ if and only if $diam(\Gamma_E(R)) = 0$.
(ii) $dim(\Gamma_E(R)) = 0$ if and only if $diam(\Gamma(R)) = 0$ or 1, $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(iii) $dim(\Gamma_E(R)) = diam(\Gamma_E(R)) = 1$ if $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields.
(iv) $dim(\Gamma_E(R)) = 1$ and $diam(\Gamma_E(R)) = 3$, if R is non reduced ring isomorphic to the rings given in Proposition 13.
(v) $dim(\Gamma_E(R)) = 0$ if $Z(R)^2 = 0$ and $|Z(R)| \geq 2$.

Proof. (i) $dim(\Gamma_E(R)) = 0$ if and only if $\Gamma_E(R)$ is a single vertex graph if and only if $diam(\Gamma_E(R)) = 0$.
(ii) Let $dim(\Gamma_E(R)) = 0$. Then $\Gamma(R)$ is complete and thus $diam(\Gamma(R)) = 0$ or 1. Conversely, let $diam(\Gamma(R)) = 0$ or 1, then $\Gamma(R)$ is complete, thus $dim(\Gamma_E(R)) = 0$ unless $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(iii) Let $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, then by Case 1 of Lemma 3.8, $|\Gamma_E(R)| = 2$, since the only equivalence classes of zero divisors are $[(0,1)]$ and $[(1,0)]$. So, $\Gamma_E(R) \cong K_{1,1}$. Thus, $dim(\Gamma_E(R)) = diam(\Gamma_E(R)) = 1$.
(iv) Rings listed in this case correspond to a path of length 3.
(v) Let $|Z(R)| \geq 2$ and $(Z(R))^2 = 0$. Hence $ann(a) = ann(b)$, for each $a, b \in Z(R)^*$, which implies that $diam(\Gamma_E(R)) = 0$. Therefore, $dim(\Gamma_E(R)) = 0$. $\square$

Acknowledgements. This research is supported by the University Grants Commission, New Delhi with research project number MRP-MAJOR-MATH-2013-8034.

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Received: