LANDAU'S THEOREM ON SCORE SEQUENCES IN TOURNAMENTS

Dissertation submitted to the Department of Mathematics in partial fulfilment of the requirements for the award of

MASTER'S DEGREE IN MATHEMATICS

by

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CERTIFICATE

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Abstract

The work of this dissertation consists of two chapters. Chapter 1 gives a brief introduction of the problem along with introduction and basic definitions.

Chapter 2 deals with the Landau's theorem on score sequences in tournaments and its various proofs of the sufficiency. We discuss the bipartite and multipartite analogue of Landau's theorem. Also, we present the combinatorial characterization for score sequences in oriented graphs. Finally we discuss the necessary conditions for football sequences.

NOTATIONS

$\emptyset$	empty set
$\in, a \in A$	membership
$\notin, a \notin A$	nonmembership
$A - B$	set theoretic difference of sets
$A \subseteq B$	set inclusion
$A \subsetneq B$	proper set inclusion
$A \cup B$	union of two sets A and B
$A \cap B$	intersection of two sets A and B
A^c	complement of a set A
$f : X \to Y$	function (or mapping) with domain X and range in Y
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
SDR	system of distinct representatives
$a \equiv b(mod\ m)$	congruence modulo m of integers
G	simple graph
$G_1 \cong G_2$	G_1 is isomorphic to G_2
$d(u,v)$	distance between two vertices u and v
P_n	path on n vertices
C_n	cycle on n vertices
K_n	complete graph on n vertices
$K_{m,n}$	complete bipartite graph
$deg(v)$	degree of a vertex
$v_i \to v_j$	v_i dominates v_j

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Chapter 1

Introduction

Section 1.1

Background

The origin of Graph Theory can be traced back to Euler's work on the Konigsberg bridges problem which subsequently lead to the concept of an Eulerian graph. In the recent years Graph Theory has established itself as an important mathematical tool in a wide variety of subjects ranging from operational research and chemistry to genetics and linguistics and from computer sciences and geography to sociology and architecture. At the same time it has also emerged as a worthwhile mathematical discipline in its own. Graph Theory is a delightful playground for the exploration of proof techniques in discrete mathematics and as a result has applications in many areas of computing, social and natural sciences.

One of the important classes of graphs is digraphs (or directed graphs). The concept of digraphs is one of the richest theories in Graph Theory, mainly because of their applications to physical problems. For example, flow networks with valves in the pipes and electrical networks are represented by digraphs. They are applied in abstract representations of computer programs and are an invaluable tools in the study of sequential machines. They are also used for systems analysis in control theory.

In a world of choices and alternatives, rankings are becoming an increasingly important tool to help individuals and institutions make decisions. One of the popular ranking methods is the pairwise comparison of the objects. Many authors describe different applications for instance biological (Landau), chemical (Hakimi), network modelling (Kim et al. and Newman et al.), economical (Bazoki, Fulop, Keri, Poesz, Ronyai et al.) and human relation modeling (Liljeros et al.). The theory of tournaments is one such direction to deal with the problems of rankings. In the present times, tournament theory is applied from round-robin sports matches to the stock market to the design of computer chip pathways.

Section 1.2

Definitions

Definition 1.2.1. A simple graph $G(V,E)$ (or briefly, a graph G) consists of a set of objects $V = \{v_1, v_2, \dots, v_n\}$, $n \in \mathbb{N}$, called the vertex set and the set $E = \{e_1, e_2, \dots, e_m\}$, $m \in \mathbb{N}$, called the edge set, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The cardinality of the vertex set of a graph G is called the order of G and the cardinality of the edge set is called the size of G.

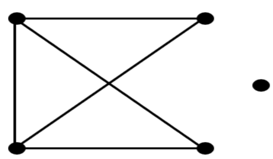

Figure 1.1: Simple graph

Definition 1.2.2. A multigraph G is a pair (V,E), where V is a nonempty set of objects called vertices and E is the multiset of edges. The number of times an edge $e = uv$ occurs in E is called the multiplicity of e and edges with multiplicity greater than one are called the multiple edges.

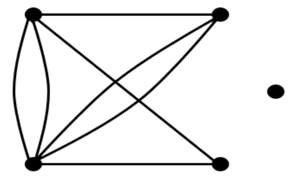

Figure 1.2: Multigraph

Definition 1.2.3. We denote by uv an edge from a vertex u to a vertex v. An edge of the form uu is called a loop at u and edges which are not loops are called proper edges. A loop with multiplicity greater than one is called a multiple loop. A general graph G is a pair (V,E), where V is a nonempty set of objects called vertices and E is a multiset of edges which may contain the loop also.

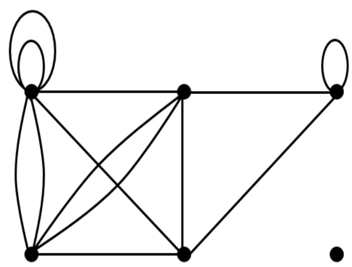

Figure 1.3: General graph

Definition 1.2.4. A simple graph $G(V,E)$ is said to be a complete graph if every two distinct vertices are adjacent in G. A complete graph on n vertices is denoted by K_n.

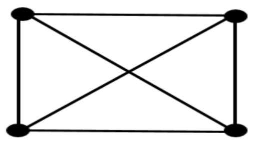

Figure 1.4: Complete graph K_4

Definition 1.2.5. A graph $G(V,E)$ is said to be r-partite (where r is a positive integer), if its vertex set can be partitioned into disjoint sets $V_1, V_2, \ldots, V_r$ with $V = V_1 \cup V_2 \cup \cdots \cup V_r$ such that uv is an edge of G if u is in some V_i and v is in some V_j, $i \neq j$, and is denoted by $G(V_1, V_2, \ldots, V_r, E)$.

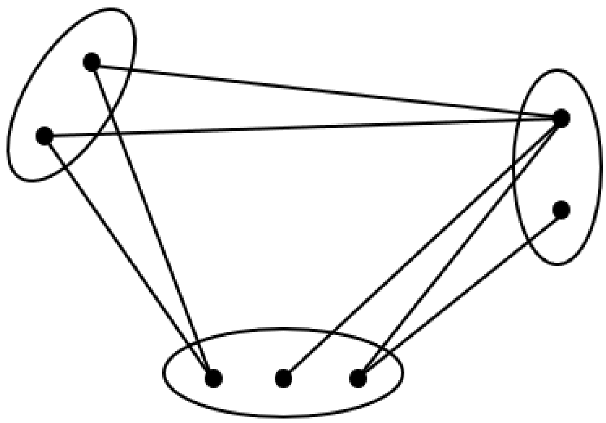

Figure 1.5: 3-partite graph

Definition 1.2.6. A graph $G(V,E)$ is said to be bipartite, or 2-partite, if its vertex set can be partitioned into two different sets V_1 and V_2 with $V = V_1 \cup V_2$, such that $uv \in E$ if $u \in V_1$ and $v \in V_2$. The bipartite graph is said to be complete if $uv \in E$ for every $u \in V_1$ and every $v \in V_2$. When $|V_1| = n_1$, $|V_2| = n_2$, we denote the complete bipartite graph by K_{n_1,n_2}.

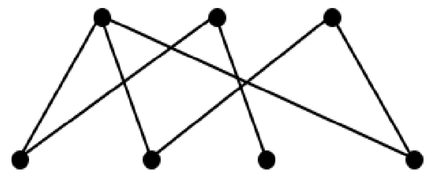

Figure 1.6: Bipartite graph

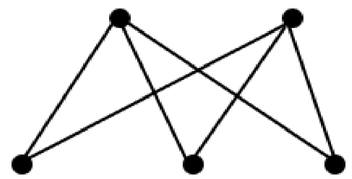

Figure 1.7: Complete bipartite graph $K_{2,3}$

Definition 1.2.7. A graph G is said to be connected if there is a path between every pair of vertices of G. A graph which is not connected is said to be disconnected.

Definition 1.2.8. The degree of a vertex v in a graph G, denoted by $deg_G(v)$, or simply $d_G(v)$ is the number of edges incident on v. We use $deg(v)$ or $d(v)$ to denote the degree of v in G when the graph G is understood. A vertex whose degree is equal to one is called a pendent vertex.

Definition 1.2.9. An alternating sequence of vertices and edges, beginning and ending with vertices such that no edge is traversed or covered more than once is called a walk. A vertex may appear more than once in a walk. The walk is said to be *open* if initial and terminal vertices of the walk are distinct, and *closed* if initial and termianl vertices are same. The number of edges in a walk is the *length* of the walk.
A path is an open walk in which no vertex (and therefore no edge) is repeated. A closed walk in which no vertex (and edge) is repeated is called a cycle. A path of lenght n is

called an n-path and is denoted by P_n. A cycle of lenght n is called an n-cycle and is denoted by C_n.

Definition 1.2.10. A directed graph (or digraph) D is a pair (V,A), where V is a finite non-empty set whose elements are called vertices and A is the subset of ordered pairs of distinct vertices called arcs. If the ordered pair (u,v) is an arc a, we say that a is directed from u to v. If each arc of a digraph is replaced by an edge, the resulting structure is a graph known as the underlying graph of the digraph.

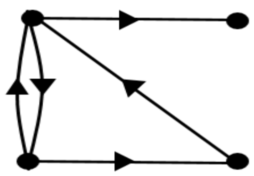

Figure 1.8: Digraph

Definition 1.2.11. In a digraph, the number of arcs directed away from the vertex v is called the outdegree of the vertex v, denoted by $od(v)$ or $d^+(v)$ and the number of arcs directed to the vertex v is called the indegree of the vertex v, denoted by $id(v)$ or $d^-(v)$. The degree (or total degree) of a vertex v in a digraph is defined by $d(v) = d^+(v) + d^-(v)$ and the ordered pair $(d^+(v), d^-(v))$ is called the degree pair of v.
A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolate*. A vertex v is called a *transmitter* or a *receiver* according as $d^+(v) > 0,\ d^-(v) = 0$ or $d^+(v) = 0,\ d^-(v) > 0$. A vertex v is called a *carrier* if $d^+(v) = d^-(v) = 1$.

Definition 1.2.12. Two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \to V_2$ such that f preserves adjacency of vertices, that is there is an edge between $f(u)$ and $f(v)$ in G_2 if and only if there is an edge between u and v in G_1, and is denoted by $G_1 \cong G_2$.

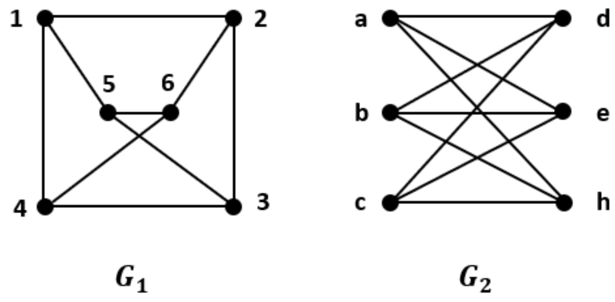

Figure 1.9: Two isomorphic graphs

Figure (1.9) shows the isomorphic graphs G_1 and G_2 under the mapping f, where $f(1) = a,\ f(2) = d,\ f(3) = c,\ f(4) = h,\ f(5) = e,\ f(6) = b.$

Definition 1.2.13. Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \longrightarrow V_2$ such that $(u, v) \in A_1$ if and only if $(f(u), f(v)) \in A_2$ and f is called an isomorphism from D_1 to D_2.

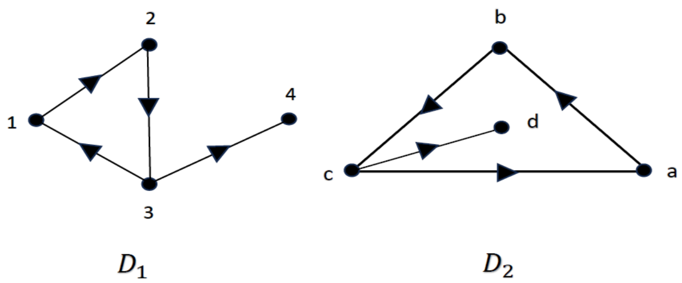

Figure 1.10: Two isomorphic digraphs

Figure (1.10) shows the isomorphic digraphs D_1 and D_2 under the mapping f, where $f(1) = a,\ f(2) = b,\ f(3) = c,\ f(4) = d.$

Definition 1.2.14. A digraph $D = (V,A)$ is said to be complete if for every pair of distinct points u and v in V both (u,v) and (v,u) are in A. Thus a complete digraph with n vertices has $n(n-1)$ arcs.

Definition 1.2.15. A digraph containing no symmetric pair of arcs is called an oriented graph.

Definition 1.2.16. A complete asymmetric digraph or a complete oriented graph is called a tournament, that is a tournament is an orientation of the complete graph K_n. Therefore in a tournament each pair of distinct vertices v_i and v_j is joined by one and only one of the oriented arcs (v_i,v_j) or (v_j,v_i). If the arc (v_i,v_j) is in T, then we say v_i dominates v_j and is denoted by $v_i \to v_j$. The number of arcs in a tournament on n vertices is $\frac{n(n-1)}{2}$.

Definition 1.2.17. A tournament is *strongly connected* or strong if for every two vertices u and v there is a path from u to v and a path from v to u. A strong component of a tournament is a maximal strong sub-tournament. A tournament is *regular* if every vertex has the same indegree and outdegree, which is possible only when the number of vertices is odd. A *nearly regular* tournament is a tournament that has an even number of vertices and each vertex has score either $\frac{n}{2}$ or $\frac{n}{2}-1$, where n is the number of vertices.

Definition 1.2.18. A triple in a tournament T is the subdigraph induced by any three vertices. A triple (u,v,w) in T is said to be *transitive* if whenever $(u,v) \in A(T)$ and $(v,w) \in A(T)$, then $(u,w) \in A(T)$. That is, whenever $u \to v$ and $v \to w$, then $u \to w$.

Definition 1.2.19. In a tournament T, the score $s(v_i)$ or simply s_i of a vertex v_i is the number of arcs directed away from v_i and the score sequence $S(T)$ is formed by listing the vertex scores in non-decreasing order. Clearly $0 \leq s_i \leq n-1$. Further no two scores can be zero and no two scores can be $n-1$. Tournament score sequences are also called score structures, score vectors and score lists.

A strong score sequence is a score sequence that corresponds to a strong tournament. A regular score sequence is a score sequence that corresponds to a regular tournament and a nearly regular score sequence is a score sequence that corresponds to a nearly regular tournament.

Definition 1.2.20. The number of times that a particular score occurs in a score sequence of a tournament is called the frequency of that score.

Definition 1.2.21. A score sequence is simple (uniquely realisable) if it belongs to exactly one tournament. Every score sequence of tournaments with fewer than five vertices is simple.

Definition 1.2.22. If $A = \{A_i : i \in \mathbb{N}\}$ is a family of sets, then a system of distinct representatives (SDR) of A is a set of elements $\{a_i : i \in \mathbb{N}\}$ such that $a_i \in A_i$ for every $i \in \mathbb{N}$ and $a_i \neq a_j$ whenever $i \neq j$. For example, if $A_1 = \{2,8\}, A_2 = \{8\}, A_3 = \{5,7\}, A_4 = \{2,4,8\}$ and $A_5 = \{2,4\}$, then the family $\{A_1, A_2, A_3, A_4\}$ has an SDR $\{2,8,7,4\}$, but the family $\{A_1, A_2, A_4, A_5\}$ has no SDR.

Chapter 2

Landau's theorem on tournament scores

Section 2.1

Introduction

Tournaments form a large class of directed graphs of order n. It is known that there are $\frac{2^{\binom{n}{2}}}{n!}$ (asymptotically) nonisomorphic tournaments of order n, which is same as for graphs of order n. This makes tournaments to provide a rich source for combinatorial investigations and various models for applied problems. A king in a tournament is a vertex x so that for every vertex y either x dominates y or there is a vertex z such that x dominates z and z dominates y. That is x is a king if x can reach every other vertex via either a 1-path or a 2-path. We know a transmitter, if one exists, is a king. In fact, every vertex of maximum score is a king, so every tournament contains at least one king.

We know that every sequence of non-negative integers need not be a score sequence. For example, the sequence $[3,3,3,3,3,3,3]$ is a score sequence realising the tournament given below

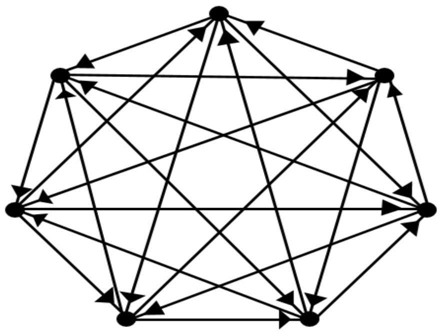

Figure 2.1: The Paley Tournament

However, the sequence $[0,1,1,4,4]$ satisfies all the necessary conditions, but is not a score sequence, since no tournament realises it. So, the need arises to characterise the sequences of non-negative integers which are score sequences. In this direction, we have the constructive and recursive criterion as follows. This is analogous of the Havel [22]-Hakimi [19] theorem for graphical sequences.

***Theorem* 2.1.1.** *A non-decreasing sequence $[s_i]_{i=1}^{n}$ of non-negative integers $n \geq 2$, is the score sequence of an n-tournament if and only if the new sequence*

$$[s_1, s_2, \ldots, s_m, s_{m+1} - 1, \ldots, s_{n-1} - 1]$$

where $m = s_n$, when arranged in non-decreasing order, is the score sequence of some $(n-1)$-tournament.

The main aim of this dissertation is to study the combinatorial characterisation of score sequences given by H. G. Landau [25]. This theorem is the analog of the Erdos-Gallai theorem [14] for graphical sequences. This theorem motivates to study the analogous versions in other types of tournaments, like bipartite and multipartite tournaments. Further this theorem has been used for combinatorial characterizations for the out-degrees in various types of digraphs including oriented graphs. It finds applications in the investigations of football sequences also. There are now several proofs of this fundamental result in tournament theory, clever arguments involving gymnastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, a constructive argument utilizing network flows, another one involving systems of distinct representatives. Landau's original proof appeared in 1953 [25], Matrix considerations by Fulkerson [15] (1960) led to a proof, discussed by Brauldi and Ryser [10] in (1991). Berge [7] in (1960) gave a network flow proof and Alway [3] in (1962) gave another proof. A constructive proof via matrices by Fulkerson [16] (1965), proof of Ryser (1964) appears in the monograph of Moon (1968). An inductive proof was given by Brauer, Gentry and Shaw [8] (1968). The proof of Mahmoodian [27] given in (1978) appears in the textbook by Behzad, Chartrand and Lesnik-Foster [6](1979). A proof by contradiction was given by Thomassen [38] (1981) and was adopted by Chartrand and Lesniak [13] in subsequent revisions of their 1979 textbook, starting with their 1986 revision. A nice proof was given by Bang and Sharp [5](1979) using systems of distinct representatives. Three years later in 1982, Achutan, Rao and Ramachandra-Rao [1] obtained a proof as a result of some slightly more general work. Bryant [12] (1987) gave a proof via a slightly different use of distinct representatives. Partially ordered sets were employed in a proof by Aigner [2] in 1984 and described by Li [26] in 1986 (his version appeared in 1989). Two proofs of sufficiency appeared in a paper by Griggs and Reid [18] (1996) one a direct proof and the second is self contained. Again two proofs appeared in 2009 one by Brauldi and Kiernan [11] using Rado's theorem from Matroid theory, and another inductive proof by Holshouser and Reiter [23] (2009). More recently Santana and Reid [37] (2012) have given a new proof in the vein of the two proofs by Griggs and Reid (1996).

The following is the statement of the Landau's theorem.

***Theorem* 2.1.2.** *A sequence of non-negative integers $[s_i]_{i=1}^{n}$ in non-decreasing order is a* *score sequence of a tournament if and only if for each subset $I \subseteq [n] = [1,2,\dots,n]$,*

$$\sum_{i\in I} s_i \geq \binom{|I|}{k}, \tag{1.1}$$

with equality when $|I| = n$.

Because of the monotonicity assumption $s_1 \leq s_2 \leq \cdots \leq s_n$, the inequalities (1.1) known as the Landau's inequalities are equivalent to

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}$$

for $1 \leq k \leq n$, with equality for $k = n$.

Hence Landau's theorem can be restated as follows.

***Theorem* 2.1.3.** *A non-decreasing sequence $[s_i]_{i=1}^{n}$ of non-negative integers is a score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2}, \tag{1.2}$$

for $1 \leq k \leq n$, with equality for $k = n$.

Proof **Necessity.** If a sequence of non-negative integers $[s_i]_{i=1}^{n}$, in the non -decreasing order is the score sequence of an n-tournament T, then the sum of the first k scores in the sequence counts exactly one each arc in the sub-tournament W induced by $[v_1, v_2, \dots, v_k]$ plus each arc from W to $T - W$. Therefore, the sum is at least $\frac{k(k-1)}{2}$, the number of arcs in W. Also, since the sum of the scores of the vertices counts each arc of the tournament exactly once, the sum of the scores is the total number of arcs, that is $\frac{n(n-1)}{2}$. $\square$

Brauldi and Shen [9] obtained stronger inequalities for scores in tournaments. These inequalities are individually stronger than Landau's inequalities, although collectively the two sets of inequalities are equivalent.

***Theorem* 2.1.4.** [9] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \dots, n\}$,*

$$\sum_{i \in I} s_i \geq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{2} \binom{|I|}{2} \tag{1.3}$$

with equality when $|I| = n$.

It can be seen that equality can often occur in (1.3), for example, equality holds for regular tournaments of odd order n whenever $|I| = k$ and $I = \{n-k+1, \dots, n\}$. Further Theorem 2.1.4 is best possible in the sense that, for any real $\varepsilon > 0$, the inequality

$$\sum_{i \in I} s_i \geq (\frac{1}{2} + \varepsilon) \sum_{i \in I} (i-1) + (\frac{1}{2} - \varepsilon) \binom{|I|}{2}$$

fails for some I and some tournaments, for example, regular tournaments. Brauldi and Shen [9] further observed that while an equality appears in (1.3), there are implications concerning the strong connectedness and regularity of every tournament with the score sequence S. Brauldi and Shen also obtained the upper bounds for scores in tournaments.

***Theorem* 2.1.5.** [9] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \dots, n\}$,*

$$\sum_{i \in I} s_i \leq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{4} |I| (2n - |I| - 1),$$

with equality when $|I| = n$.

Section 2.2

Various proofs of Sufficiency

We now present the different proofs for the sufficiency of Landau's theorem.

Landau's proof. [25] This proof is by induction on n. The case $n = 2$ is obvious. Let $n \geq 2$ and asssume that the result holds for sequences of length n satisfying Landau's conditions. Suppose the sequence $[s_1, s_2, \ldots, s_n, s_{n+1}]$ is a sequence of integers satisfying Landau's conditions. Then

$$s_{n+1} \leq \binom{n+1}{2} - \binom{n}{2} = n.$$

Assume that this inequality is actually an equality. Then $[s_1, s_2, \ldots, s_n]$ satisfies Landau's conditons, so the induction hypothesis implies there is an n-tournament W with score sequence $[s_1, s_2, \ldots, s_n]$. Add a new vertex to W which dominates every vertex in W to obtain an $(n+1)$-tournament with score sequence $[s_1, s_2, \ldots, s_n, s_{n+1}]$, as required.

So, assume that $s_{n+1} < n$, and define $d = n - s_{n+1} > 0$. A rather subtle argument on the indices of the scores (not given here) implies that there are at least d values of i for which $s_i - (i-1) > 0$. The d largest of the scores with such indices are each reduced by 1, and all of the other $n+1-d$ scores are unchanged. Let the resulting sequence from 1 to n be denoted by $[w_1, w_2, \ldots, w_n]$. Another involved argument on the indices of the scores (not given here) implies that the sequence $[w_1, w_2, \ldots, w_n]$ satisfies Landau's conditions. So, by the induction hypothesis, there is a tournament W with score sequence $[w_1, w_2, \ldots, w_n]$. Add a new vertex x to W that is dominated by exactly the d vertices of W whose scores were one of the d numbers reduced by 1(the other $n-d$ vertices of W are to be dominated by x). This new $(n+1)$- tournament has score sequence $[s_1, s_2, \ldots, s_n, s_{n+1}]$ as required.

In any case $[s_1, s_2, \ldots, s_n, s_{n+1}]$ is the score sequence. By induction, the result follows. □

Fulkerson's proof. [16] Fulkerson used network flow theory to determine the conditions for the existence of a square matrix with zero trace and bounded integral entries, in which row sums are bounded above and column sums are bounded below. This result is discussed in the 1991 monograph by Brualdi and Ryser [10]. Using the i^{th} row sum bound to be s_i, where $0 \le s_1 \le s_2 \le \cdots \le s_n$ satisfy Landau's conditions in (1.2) and the j^{th} column sum bound to be $n-1-s_j$, the existence of the adjacency matrix of a digraph D without loops is assured by Fulkerson's theorem [16]. The out degrees of the vertices of D are given by $s_1, s_2, \ldots, s_n$ and D contains $\frac{n(n-1)}{2}$ arcs. A bit of straight forward translation is necessary to see this implication. However, there is no assurance that D is a tournament, for D may contain (directed) 2 cycles. However, by suitably adding a new arc between some two non-adjacent vertices in D, deleting an arc is some 2-cycle in D, and reversing the orientations of the arcs of a suitable path in D, a digraph D^* can be obtained with out-degrees $s_1, s_2, \ldots, s_n$ but with fewer 2-cycles than in D. By repeated application of this process, D can be transformed into a tournament with score sequence $[s_1, s_2, \ldots, s_n]$. So, Landau's theorem [25] follows from Fulkerson's theorem [16] if it is specialized as indicated to yield D and if 2-cycles that might be present in D are removed.

To see how D^* can be obtained from D, suppose that D contains at least one 2-cycle. As D contains $\frac{n(n-1)}{2}$ arcs, D must contain at least one pair of non-adjacent vertices, say x and y. Define D_0 to be the sub-digraph of D induced by all the vertices of D reachable from either x or y by paths in D. By this definition, D contains no arcs of the form $w \to z$, where w is in D_0 and z is not in D_0. Consequently, the out degree in D_0 of each vertex in D_0 is equal to its out-degrees in D. Let $|V(D_0)| = k$. Since the sum of any k out-degrees of vertices in D is at least as large as the sum of the k smallest out-degrees in D,

$$\begin{aligned}
|A(D_0)| &= \sum \{d^+(w) : w \in V(D_0)\} \ge \sum_{i=1}^{k} s_i \\
&\ge \binom{k}{2} = \binom{|V(D_0)|}{2}.
\end{aligned}$$

As D_0 is lacking an arc between x and y, this inequality implies that there must be some 2-cycle in D_0. By definition of D_0, both vertices on any such 2-cycle must be reachable from the set $\{x, y\}$. Among all such 2-cycles choose one, say given by $u \to v \to u$, so that the length of the path from the set $\{x, y\}$ to the set $\{u, v\}$ is as small as possible. Let P

denote such a shortest path and without loss of generality, assume that u is reachable from x via P (so neither y nor v is on P). Note that, by the choice of P, no arc on P is contained in a 2-cycle. Alter D by adding new arc $x \to y$, by deleting arc $v \to u$, by reversing the orientation of every arc of D on P and by reversing the arc $u \to v$ to obtain D^*. No multiple arcs are created as no arc of P was on a 2-cycle, so D^* is a digraph and D^* has fewer 2-cycles than D, as claimed. This completes the proof. □

Ryser's proof. [36] We induct on n. The cases $n = 1$ or 2 are clear. Let $n > 2$. Suppose that the result is true for all sequences of length less than n which satisfy Landau's conditions and suppose that $[s_1, s_2, \ldots, s_n]$ satisfy Landau's conditions. First note that $1 \leq s_n \leq n-1$, so s_n is an index on one of the $s_i's$. Denote s_n by m. Let j be the least index such that $s_j = s_m$, and let k be the largest index such the $s_k = s_m$ and $k \leq n-1$. Then in the sequence $[s_1, s_2, \ldots, s_n]$ note that $s_{j-1} < s_j = \cdots = s_m = \cdots = s_k < s_{k+1}$. Reduce exactly $n-1-m$ of integers $s_1, s_2, \ldots, s_{n-1}$ by one to form a new sequence $t_1, t_2, \ldots, t_{n-1}$ of length $n-1$ as follows: Define $t_i = s_i - 1$, if $j \leq i \leq (k-(m-j)-1)$, or if $k+1 \leq i \leq n-1$, and define $t_i = s_i$, if $1 \leq i \leq j-1$, or if $(k-(m-j) \leq i \leq k)$. The sequence $t_1, t_2, \ldots, t_{n-1}$ satisfies Landau's conditions, as is shown below, so by induction hypothesis there is an $(n-1)$-tournament W with score sequence $[t_1, t_2, \ldots, t_{n-1}]$. Add a new vertex x to W, add arcs from x to each of the m vertices of W with scores $t_i = s_i$ whenever $1 \leq i \leq j-1$ or $(k-(m-j) \leq i \leq k$, and add arcs from each of the remaining $n-1-m$ vertices of W to x. This results in an n-tournament with score sequence $[s_1, s_2, \ldots, s_n]$ as required.
To see why Landau's conditions are satisfied by $[t_1, t_2, \ldots, t_{n-1}]$, note that $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$, because of the choice of j and the definitions of the $t_i's$. Also,

$$\begin{aligned}\sum_{i=1}^{n-1} t_i &= (\sum_{i=1}^{n} s_i) - m - (n-1-m) \\ &= \binom{n}{2} - (n-1) = \binom{n-1}{2}.\end{aligned}$$

It remains to verify the inequalities in Landau's conditions. Suppose that some inequality is not true. Let h be the least value such that $1 \leq h \leq n-2$ and

$$\sum_{i=1}^{h} t_i < \binom{h}{2}.$$

If $h = 1$, then $t_1 < 0$, so $s_1 = 1, j = 1$(this is the j defined above), and $s_m = s_j = s_1 = 1$. This means that $m = 1,2$ or 3 and that the original $s_i's$ are given by [1,1,1] or [1,1,2,2] or [1,1,1,3]. Each of these sequences is the score sequence of a tournament with respective $t_i's$ given by [0,1], [0,1,2], [1,1,1], a contradiction. So, $1 < h \leq n-2$. Also, $j \leq h$ because $t_i = s_i$ whenever,$1 \leq i \leq j-1$.

Define $f = \max(h,k)$, and let t denotes the number of values of i not exceeding h such that $t_i = s_i - 1$. Then $m \leq f - t$ (This can be checked by considering three cases for $h : j \leq h \leq (k-(m-j))-1)$ or $(k-(m-j)) \leq h \leq k$ or $k+1 \leq h \leq n-2$. Then

$$\begin{aligned}
\binom{n}{2} &= \left(\sum_{i=1}^{h} s_i\right) + \left(\sum_{i=h+1}^{f} s_i\right) + \left(\sum_{i=f+1}^{n-1} s_i\right) + m \\
&= \left(\sum_{i=1}^{h} t_i\right) + t + \left(\sum_{i=h+1}^{f} s_i\right) + \left(\sum_{i=f+1}^{n-1} s_i\right) + m \\
&< \binom{h}{2} + t + (f-h)s_h + \left(\sum_{i=f+1}^{n-1} s_i\right) + m \\
&\leq \binom{h}{2} + (f-h)s_h + \left(\sum_{i=f+1}^{n-1} s_i\right) + f \\
&\leq \binom{h}{2} + (f-h)h + \left(\sum_{i=f+1}^{n-1} s_i\right) + f \\
&\leq \binom{f}{2} + f(n-f) \leq \binom{n}{2}.
\end{aligned}$$

The fourth-to-last inequality follows from $m \leq f - t$. The third from last inequality follows from $s_h \leq h$ (recall that by the choice of h, $\sum_{i=1}^{h-1} t_i \geq \binom{h-1}{2}$; so, if $s_h > h$, then $t_h > h-1$ and

$$\sum_{i=1}^{h} t_i > \binom{h-1}{2} + (h-1) = \binom{h}{2},$$

a contradiction to the choice of h). The next-to-last inequality can be checked using the same three cases for h as mentioned above.The last inequality follows from the fact that the function of f on the interval $[1, n-1]$ given by $\left(\frac{f(f-1)}{2} + f(n-f)\right)$ has maximum value $\frac{n(n-1)}{2}$ *at* $f = n-1$. In summary, the above calculations lead to

$$\binom{n}{2} < \binom{n}{2},$$

a contradiction. So, there is no such h, and Landau's conditions hold for $(t_1, t_2, \ldots, t_{n-1})$. This completes the proof. $\square$

Mahmoodian's proof. [27] We induct on n. We note that the cases $n = 1, 2$ are obvious. Therefore, assume that $n > 2$ and that the result holds for values less than n.
Suppose that for some $k,\ 1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} s_i = \binom{k}{2}.$$

By the induction hypothesis , there is a k-tournament V with score sequence $[s_1, s_2, \ldots, s_k]$. Also,

$$\begin{aligned} s_{k+1} - k &= \sum_{i=1}^{k+1} s_i - \sum_{i=1}^{k} s_i - k \geq \binom{k+1}{2} - \binom{k}{2} - k \\ &= \binom{k}{2} + k - \binom{k}{2} - k = 0. \end{aligned}$$

So, $0 \leq s_{k+1} - k \leq s_{k+2} - k \leq \cdots \leq s_n - k$. Moreover, for $(k+1) \leq j \leq n$,

$$\begin{aligned} \sum_{i=k+1}^{j} (s_i - k) &= \left(\sum_{i=k+1}^{j} s_i\right) - k(j-k) = \left(\sum_{i=1}^{j} s_i\right) - \left(\sum_{i=1}^{k} s_i\right) - k(j-k) \\ &\geq \binom{j}{2} - \binom{k}{2} - k(j-k) = \binom{j-k}{2}, \end{aligned}$$

with equality for $j = n$.
Thus by the induction hypothesis, there is a $(n-k)$-tournament U with score sequence

$[s_{k+1}-k, s_{k+2}-k, \ldots, s_n-k]$. Now, form an n-tournament T from U and V (where, without loss of generality, the vertex sets of U and V are disjoint) by letting every vertex in U dominate every vertex in V. Tournament T has score sequence $[s_1, s_2, \ldots, s_n]$.
To complete the proof, suppose that for all $k,\ 1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} s_i > \binom{k}{2}.$$

Define m as follows;

$$m = \min\left\{ \sum_{i=1}^{k} s_i - \binom{k}{2}; 1 \leq k \leq n-1) \right\}$$

Then $0 < m \leq s_1$. Consider the non-decreasing sequence $[s_1 - m, s_2, \ldots, s_{n-1}, s_n + m]$. Then Landau's conditions are valid for this sequence of n integers, and in addition, for some index k between 1 and $n-1$,

$$(s_1 - m) + \sum_{i=2}^{k} s_i = \binom{k}{2}.$$

By the case treated above, there is an n-tournament W with score sequence $[s_1 - m, s_2, \ldots, s_{n-1}, s_n + m]$. Let vertex x have score $s_1 - m$ in W, and let vertex y have score $s_n + m$ in W. Since $(s_n + m) - (s_1 - m) \geq 2m$, there are at least $2m - 1 \geq m$ vertices in W that are dominated by y and dominate x. Let $v_1, v_2, \ldots, v_m$ be m such vertices, so that $y \to v_i \to x$ is a 2-path in W for each $i,\ 1 \leq i < m$. Reverse the orientation of the $2m$ arcs in these m (internally vertex - disjoint) 2-paths so as to obtain a tournament T with score sequence $[s_1, s_2, \ldots, s_n]$ as required. Hence the result. $\square$

Thomassen's proof. [39] The proof is by contradiction. Assume that all sequences of non-negative integers in non-decreasing order of length fewer than n, satisfying given conditions be the score sequences.

Let n be the smallest length and s_1 be the smallest possible such that the sequence $S = [s_i]_{i=1}^{n}$ is not a score sequence.

We consider two cases,

(a). Equality in (1.2) holds for some $k < n$.

(b). Each inequality in (1.2) is strict for all $k < n$.

Case (a). Assume that $k(k < n)$ is the smallest such that

$$\sum_{i=1}^{k} s_i = \binom{k}{2}.$$

Clearly, the sequence $[s_1, s_2, \ldots, s_k]$ satisfies (1.2) and is a sequence of length less than n. Therefore, by the given assumption $[s_1, s_2, \ldots, s_k]$ is a score sequence of some tournament, say T_1. Now

$$\begin{aligned}
\sum_{i=1}^{p}(s_{k+i} - k) &= \sum_{i=1}^{p+k} s_i - \sum_{i=1}^{k} s_i - pk \\
&\geq \binom{p+k}{2} - \binom{k}{2} - pk \\
&= \binom{p}{2},
\end{aligned}$$

for each p, $1 \leq p \leq n-k$, with equality when $p = n-k$. Since $p < n$, therefore by the minimality of n, the sequence $[s_{k+1} - k, s_{k+2} - k, \ldots, s_n - k]$ is the score sequence of some tournament T_2.

The tournament T of order n consisting of disjoint copies of T_1 and T_2, such that each vertex of T_2 dominates every vertex of T_1 has score sequence $S = [s_i]_{i=1}^{n}$, a contradiction.

Case (b). Assume that each inequality in (1.2) is strict for all $k < n$. Clearly, $s_1 > 0$.

Consider the sequence $S' = [s'_i]_{i=1}^{n}$, where

$$s'_i = \begin{cases} s_i - 1, & i = 1 \\ s_i + 1, & i = n \\ s_i, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{i=1}^{k} s'_i = \left(\sum_{i=1}^{k} s_i\right) - 1 > \binom{k}{2} - 1 \geq \binom{k}{2}$$

for all k, $1 \leq k < n$.

Also,

$$\sum_{i=1}^{n} s'_i = \left(\sum_{i=1}^{n} s_i\right) - 1 + 1 = \sum_{i=1}^{n} s_i = \binom{n}{2}.$$

Thus, the sequence $S' = [s'_i]_{i=1}^n$ satisfies conditions (1.2) and therefore by the minimality of s_1 is a score sequence of some tournament T. Let x be a vertex having score $s_n + 1$ and y be a vertex having score $s_1 - 1$.

Since $s_x > s_y$, therefore T has a path from x to y of length ≤ 2. By reversing the arcs of that path, we obtain a tournament with score sequence $[s_1, s_2, \ldots, s_n]$, again a contradiction. □

For the next proof of sufficiency, we shall require the following result.

Hall's SDR theorem [20]. A family of finite non-empty sets $A = \{A_i : 1 \leq i \leq r\}$ has an SDR if and only if for every k, $1 \leq k \leq r$, union of any k of these sets contains at least k elements.

Bang's and Sharp's proof. [5] Assume that the non-decreasing sequence of integers $[s_1, s_2, \ldots, s_n]$ satisfies Landau's conditions. Let $X_1, X_2, \ldots, X_n$ be pairwise disjoint sets with $|X_i| = s_i$, for $1 \leq i \leq n$. Consider these n sets as the vertices of a complete graph. The goal of the proof is to orient the edges of this complete graph so as to obtain a tournament with score sequence $[s_1, s_2, \ldots, s_n]$. The orientation of the edges between X_i and X_j will be determined as follows. Form the $\frac{n(n-1)}{2}$-set $F = \{X_i \cup X_j : 1 \leq i < j \leq n\}$.

For $1 \leq k \leq (n(n-1)/2)$, consider the union of any k members of F, and let A denote the set of distinct subscripts on the $X_i's$ that make up these k members of F. Note that

$$\binom{|A|}{2} \geq k.$$

Now, the cardinality of the union of these k members of F is

$$|\bigcup_{i \in A} X_i| = \sum_{i \in A} |X_i| = \sum_{i \in A} s_i \geq \sum_{i=1}^{|A|} s_i \geq \binom{|A|}{2} \geq k.$$

That is the union of any k members of F contains at least k elements. By Hall's theorem on systems of distinct representatives, F has a system of distinct representatives, i.e., $\frac{n(n-1)}{2}$ distinct elements, called representatives, from the union of members of F so that each set of F contains one of the representatives. For each $1 \leq i \leq j \leq n$, orient the edge between X_i and X_j from X_i to X_j if and only if the representative of the member $X_i \cup X_j$ of F is in X_i. As X_i and X_j are disjoint, no edge is given two orientations, i.e., no 2-cycles result. This construction yields n-tournament T. Moreover, each element of $X_1 \cup X_2 \cup \cdots \cup X_m$ appears exactly once as a representative in the above system of $\frac{n(n-1)}{2}$ distinct representatives, since

$$|\bigcup_{i=1}^{n} X_i| = \sum_{i=1}^{n} |X_i| = \sum_{i=1}^{n} s_i = \binom{n}{2}.$$

So, each element of X_i appears exactly once as a representative. That is, the score of X_i is $|X_i| = s_i,\ 1 \leq i \leq n$. Thus T has the score sequence $[s_1, s_2, \ldots, s_n]$. □

In the mid 1980's, two researchers placed the proof of Landau's theorem in the context of a special poset. We have the following definition.

Definition. A vector $X = (x_1, x_2, \ldots, x_n)$ is *majorized* by a vector $Y = (y_1, y_2, \ldots, y_n)$ denoted by $X \prec Y$, if $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$, whenever $1 \leq k \leq n$, with equality for $k = n$.

The majorization relation was used by Ryser to present necessary and sufficient conditions for the existence of $(0,1)$ matrices with given row sums and given column sums. This result is now called the Gale-Ryser theorem as Gale [17] also gave a treatment. A 1979 monograph by Marshall and Olkin [28] discussed majorization of general vector.

For a vector S, let T(S) denote the set of n-tournaments with score sequence S. Let S_0 denote the specific vector $(0,1,2,\ldots,n-1)$.

These notations allow Landau's theorem to be simply stated as follows.

***Theorem* 2.2.1.** (Landau's theorem) *For every vector S, $T(S) \neq \phi$ if and only if $S \prec S_0$.*

We require the following lemma for the next proof of sufficiency given by Aigner [2] and Li [26].

***Lemma* 2.2.2.** *If S covers S' and $T(S) \neq \phi$, then $T(S') \neq \phi$.*

Proof. Assume that an n-tournament T has a score sequence S and let vector S' be given by $S' = [s_1, s_2, ..., s_i, \ldots, s_j, \ldots, s_n]$, where i and j are two indices for the two non-zero terms in $S' - S$, $1 \leq i < j \leq n$, then

$$S = [s_1, ..., s_{i-1}, s_i - 1, s_{i+1}, \ldots s_{j-1}, s_j + 1, s_{j+1}, ..., s_n].$$

In T, choose a vertex x of score $s_j + 1$ and choose a vertex y of score $s_i - 1$. As $s_j + 1 - (s_i - 1) \geq 2$, there is a vertex z, distinct from x and y, so that $x \longrightarrow z \longrightarrow y$ in T. Reverse the orientation of the 2-path $x \longrightarrow z \longrightarrow y$ in T to obtain a new n-tournament with score sequence S' as required. This proves the lemma. $\square$

Aignar and Li proof. [2, 26] First note that if $L(S_0) = \{S : S \text{ is a vector and } S \prec S_0\}$, then $(L(S_0), \prec)$ is a poset. In addition, $L(S_0)$ is lattice with unique maximum element S_0 and unique minimum element, which is as follows,

$$\begin{cases} \left(\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2}\right), & \text{if } n \text{ is odd.} \\ \left(\frac{n}{2}-1, \frac{n}{2}-1, \ldots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Also, S covers S' in $(L(S_0), \prec)$ iff $S' - S$ contains exactly two non-zero terms $a+1$ in some position i and $a-1$ in some different position j, where $i < j$.
If S and S' are distinct vectors in $L(S_0)$ and $S \prec S'$, then there exists in $(L(S_0), \prec)$ a chain

$$S = S^0 \prec S^1 \prec S^2 \prec, \ldots, \prec S^m = S',$$

so that $S^{(i)}$ covers $S^{(i-1)}$, $1 \leq i \leq m$.
Clearly, $T(S_0) \neq \phi$, as S_0 is the score sequence of the transitive n-tournament. Let $S \prec S_0$. Choose a "covering" chain in $(L(S_0), \prec)$, say, $S = S^0 \prec S^1 \prec S^2 \prec, \ldots, \prec S^m = S_0$, where $S^{(i)}$ covers $S^{(i-1)}$, $1 \leq i \leq m$. Therefore, by *Lemma* 2.2.2 and induction $T(S) \neq \phi$. □

Griggs and Reid proof (Majorization). [18] Let $S = [s_1, s_2, \ldots, s_n]$ be an integer sequence satisfying conditions (1.2). Starting with the transitive n-tournament, denoted TT_n, we successively reverse the orientation of the two arcs in selected 2-paths until we construct a tournament with score sequence S. Suppose that at some stage we have obtained n-tournament U with score sequence S_1 =[u$_1, u_2, \ldots, u_n$], such that, for $1 \leq k \leq n$, $\sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} u_i$ (with equality for $k = n$.) This holds initially, when $U = TT_n$, by our hypothesis concerning S, since TT_n has score sequence $S_n = [0, 1, \ldots, n-1]$. If $S_1 = S$, we are done (S is the score sequence of U), so suppose that $S_1 \neq S$. Let α denote the smallest index such that $u_\alpha < s_\alpha$. Let β denote the largest index such that $u_\beta = u_\alpha$. Since $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} u_i \left(= \binom{n}{2} \right)$, by (1.2) there exists a smallest index $\gamma > \beta$ such that $u_\gamma > s_\gamma$. By maximality of β, $u_{\beta+1} > u_\beta$, and by minimality of γ, $u_\gamma > u_{\gamma-1}$. We have $[u_1, u_2, \ldots, u_{\alpha-1}] = [s_1, s_2, \ldots, s_{\alpha-1}]$, $u_\alpha = \cdots = u_\beta < s_\alpha \leq \cdots \leq s_\beta \leq s_{\beta+1}$, $s_{\beta+1} \geq u_{\beta+1}, \ldots, s_{\gamma-1} \geq u_{\gamma-1}$, $s_\gamma > u_\gamma$, and, of course, $u_\gamma \leq \cdots \leq u_n$ and $s_\gamma \leq \cdots \leq s_n$. Then $u_\gamma > s_\gamma \geq s_\beta > u_\beta$, or $u_\gamma \geq u_\beta + 2$. So, if vertex v_i in U has score u_i, $1 \leq i \leq n$, there must be a vertex v_λ, $\lambda \neq \beta, \gamma$, such that $v_\gamma \to v_\lambda \to v_\beta$ in U. Reversing this 2- path yields an n-tournament U' with score sequence $S_1' = [u_1', u_2', \ldots, u_n']$, where

$$u_i' = \begin{cases} u_\gamma - 1, & i = \gamma \\ u_\beta + 1, & i = \beta \\ u_i, & \text{otherwise.} \end{cases}$$

By choice of indices, $u_1' \leq u_2' \leq \cdots \leq u_n'$. It is easy to check that for $1 \leq k \leq n$, $\sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} u_i'$.

We know that for n-tuples of real numbers $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$, the metric $d(A,B) = \sum_{i=1}^{n} |a_i - b_i|$. Then for the sequences S, S_1, S_1' above, $d(S_1', S) = d(S_1, S) - 2$. Now, under modulo 2, $d(S_1, S) = \sum_{i=1}^{n}(u_i - s_i) = \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} s_i = 0$. So, eventually, after $\frac{1}{2}\, d(S_n, S)$ such steps we arrive at $S_1 = S$ and U realizes S. □

Griggs and Reid (Basic proof). [18] The specific sequence $S_n = [0, 1, 2, \ldots, n-1]$ satisfies conditions (1.2) as it is the score sequence of the transitive n-toumament. If sequence $S \neq S_n$ satisfies (1.2), then $s_1 \geq 0$ and $s_n \leq n-1$, so S must contain a repeated term. The object of this proof is to produce a new sequence S' from S which also satisfies (1.2), is "closer" to S_n than is S, and is a score sequence if and only if S is a score sequence. We find the first repeated term of S, reduce its first occurrence in S by 1 and increase its last occurrence in S by 1 in order to form S'. The process is repeated until the sequence S_n is obtained. We now prove the validity of this procedure.

Let $S \neq S_n$ be a sequence satisfying (1.2). Define k to be the smallest index for which $s_k = s_{k+1}$, and define m to be the number of occurrences of the term s_k in S. Note that $k \geq 1$ and $m \geq 2$, and that either $k+m-1 = n$ or $s_k = s_{k+1} = \cdots = s_{k+m-1} < s_{k+m}$. Define S' as follows: for $1 \leq i \leq n$,

$$s_i' = \begin{cases} s_i - 1, & i = k \\ s_i + 1, & i = k+m-1 \\ s_i, & \text{otherwise.} \end{cases}$$

Then $s'_1 \leq s'_2 \leq \cdots \leq s'_n$.

If S' is the score sequence of some n-toumament T in which vertex v_i has score s'_i, $1 \leq i \leq n$, then, since $s'_{k+m-1} > s'_k + 1$, there is a vertex in T, say v_p, for which $v_{k+m-1} \to v_p$ and $v_p \to v_k$. Reversal of those two arcs in T yields an n-tournament with score sequence S. On the other hand, if S is the score sequence of some n-tournament W in which vertex v_i has score s_i, $1 \leq i \leq n$, then we may suppose that $v_k \to v_{k+m-1}$ in W, for otherwise, interchanging the labels on v_k and v_{k+m-1} does not change S. Reversal of the arc $v_k \to v_{k+m-1}$ in W yields an n-toumament with score sequence S'. That is, S' is a score sequence if and only if S is a score sequence.

Next, we show that $\sum_{i=1}^{j} s_i > \binom{j}{2}$, $k \leq j \leq k+m-2$. Suppose, on the contrary, that for some j, $k \leq j < k+m-2$, $\sum_{i=1}^{j} s_i \leq \binom{j}{2}$. Conditions (1.2) imply that $\sum_{i=1}^{j} s_i \geq \binom{j}{2}$, so equality holds. Then, again by (1.2),

$$s_{j+1} + \binom{j}{2} = s_{j+1} + \sum_{i=1}^{j} s_i = \sum_{i=1}^{j+1} s_i \geq \binom{j+1}{2} = \binom{j}{2} + j.$$

So, $s_{j+1} \geq j$. As $s_j = s_{j+1}$, $s_j \geq j$. Thus,

$$\sum_{i=1}^{j} s_i = \sum_{i=1}^{j-1} s_i + s_j \geq \binom{j-1}{2} + s_j \geq \binom{j-1}{2} + j = \binom{j}{2} + 1 > \binom{j}{2},$$

a contradiction to our supposition. So, $\sum_{i=1}^{j} s_i > \binom{j}{2}$, $k \leq j \leq k+m-2$.

Now, we can show that S satisfies (1.2) if and only if S' satisfies (1.2). If S satisfies (1.2), then

$$\sum_{i=1}^{j} s'_i = \begin{cases} \sum_{i=1}^{j} s_i, & \text{if } j \leq k-1 \\ \sum_{i=1}^{k-1} s_i + (s_k - 1) + \sum_{i=k+1}^{j} s_i, & \text{if } k \leq j \leq k+m-2 \\ \sum_{i=1}^{k-1} s_i + (s_k - 1) + \sum_{i=k+1}^{k+m-2} s_i + (s_{k+m-1} + 1) + \sum_{i=k+m}^{j} s_i, & \text{if } j \geq k+m-1. \end{cases}$$

In cases $j \leq k-1$ and $j \geq k+m-1$, we see that $\sum_{i=1}^{j} s_i' = \sum_{i=1}^{j} s_i \geq \binom{j}{2}$. In cases $k \leq j \leq k+m-2$, the strict inequality established above implies that $\sum_{i=1}^{j} s_i' = \left(\sum_{i=1}^{j} s_i\right) - 1 > \binom{j}{2} - 1$. So, S' satisfies (1.2). On the other hand if S' satisfies (1.2), then it is clear that S satisfies (1.2).

Define a total order on integer sequences that satisfy (1.2) as follows: $A = [a_1, a_2, \ldots, a_n] \leq B = [b_1, b_2, \ldots, b_n]$ if either $A = B$, or $a_n < b_n$, or for some i, $1 \leq i < n$, $a_n = b_n$, $a_{n-1} = b_{n-1}, \ldots,$ $a_{i+1} = b_{i+1}$, $a_i < b_i$. Clearly, $\leq$ is reflexive, antisymmetric, transitive, and satisfies comparability. Write $A < B$ if $A \leq B$, but $A \neq B$. Note that, for any sequence $S \neq S_n$ satisfying (1.2), $S < S_n$, where S_n is the fixed sequence $[0, 1, 2, \ldots, n-1]$, the score sequence for the transitive n-tournament. We have shown above that for every sequence $S \neq S_2$ satisfying (1.2) we can produce another sequence S' satisfying (1.2) such that $S < S'$. Moreover, S is a score sequence if and only if S' is a score sequence. So, by repeated application of this transformation starting from the original sequence satisfying (1.2) we must eventually reach S_n. Thus, S is a score sequence, as required. $\square$

A *matroid* M consists of a non-empty finite set X and an integer-valued function ρ defined on the set of subset of X, satisfying the following.

(i). $0 \leq \rho(A) \leq |A|$, for each subset A of X,

(ii). If $A \subseteq B \subseteq X$, then $\rho(A) \leq \rho(B)$,

(iii). For any $A, B \subseteq X$, $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$.

Let M be a *matroid* on X with rank function denoted by $\rho(\cdot)$. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of n subsets of X. A *transversal* of $\mathscr{A}$ is a set S of n elements of X which can be ordered as $x_1, x_2, \ldots, x_n$ so that $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The transversal S is an independent transversal of $\mathscr{A}$ provided that S is an independent set of the matroid M.

Using Rado's theorem [30, 33] for the existence of an independent transversal of family of subsets of a set on which a matroid is defined, we give a proof of Landau's theorem for the existence of a tournament with a prescribed sequence.

Rado's theorem. [30, 33]. *The family $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ of subsets of the set X on which matroid M is defined has an independent transversal if and only if*

$$\rho(U_{i\in k} A_i) \geq |K|, (K \subseteq \{1, 2, \ldots, n\}).$$

Proof of Landau's theorem using Rado's theorem. Assume that

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, (k = 1, 2, \ldots, n) \tag{2.4}$$

with equality for $k = n$.

Note that (2.4) is equivalent to

$$\sum_{i\in K} s_i \geq \binom{|K|}{2} \quad (K \subseteq \{1, 2, \ldots, n\}).$$

Let $X = \{(i, j); 1 \leq i, j \leq n, i \neq j\}$. Consider the matriod M on X whose circuits are the $\binom{n}{2}$ disjoint sets $\{(i, j), (j, i)\}$ of two pairs in X with $i \neq j$. Thus , a subset E of X is independent if and only if it does not contain a symmetric pair $(i, j), (j, i)$ with $i \neq j$. We have $\rho(X) = \binom{n}{2}$. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be the family of subsets of X, where

$$A_i = \{(i, j) : 1 \leq j \leq n, j \neq i\}, (i = 1, 2, \ldots n). \tag{2.5}$$

Let $s_1, s_2, \ldots, s_n$ be the sequence of non-negative integers with $s_1 + s_2 + \cdots + s_n = \binom{n}{2}$. There exists a tournament with score sequence $s_1, s_2, \ldots, s_n$ if and only if there exists $P_1, P_2, \ldots, P_n$ with $P_i \subseteq A_i$ and $|P_i| = s_i$, $(1 \leq i \leq n)$ such that $P = P_1 \cup P_2 \cup \cdots \cup P_n$ is an independent set of M, equivalently, if and only if the family

$$\mathscr{A}' = \{\underbrace{A_1, \ldots, A_1}_{s_1}, \underbrace{A_2, \ldots, A_2}_{s_2}, \ldots, \underbrace{A_n, \ldots, A_n}_{s_n}\}$$

has an independent transversal. The desired tournament has vertices $1,2,\ldots,n$ and an arc from i to j if and only if (i,j) is in P_i. The independence of P then implies that there is no arc from j to i.

It follows from Rado's theorem that $\mathscr{A}'$ has an independent transversal provided that

$$\rho(\cup_{i\in K}A_i) \geq \sum_{i\in K} s_i \quad (K \subseteq \{1,2,\ldots,n\}). \tag{2.6}$$

From the definition of M, we see that

$$\rho(\cup_{i\in K}A_i) = \binom{k}{2} + k(n-k), \tag{2.7}$$

where $k = |K|$. By (2.7), the rank of $\cup_{i\in k}A_i$ depends only on $k = |K|$. By the monotonicity assumption on the s_i, $\sum\limits_{i\in K} s_i$ is largest when $K = \{n-k+1,\ldots,n\}$. Thus (2.6) is equivalent to

$$\binom{k}{2} + k(n-k) \geq \sum_{i=n-k+1}^{n} s_i. \tag{2.8}$$

Since

$$\sum_{i=1}^{n} s_i = \binom{n}{2},$$

therefore, (2.8) becomes

$$\sum_{i=1}^{n-k} s_i \geq \binom{n}{2} - \binom{k}{2} - k(n-k). \tag{2.9}$$

It follows that (2.6) is equivalent to

$$\begin{aligned}\sum_{i=1}^{p} s_i &\geq \binom{n}{2} - \binom{n-p}{2} - p(n-p), \quad (p = 1,2,\ldots.n)\\ &= \binom{p}{2},\end{aligned}$$

which proves Landau's theorem. $\square$

Santana and Reid proof. [37] Define $S_n = \{[s_1, s_2, \dots, s_n] : s_i \in \mathbb{Z},\ 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n,\ \sum_{i=1}^{k} s_i \geq \binom{k}{2},\ 1 \leq k \leq n, \text{ and } \sum_{i=1}^{n} s_i = \binom{n}{2}\}$. So, S_n is the set of non-decreasing integral n-tuples that satisfy Landau's conditions (1.2). Define the order $\preceq$ on sequences in S_n as follows: for $A_n = [a_1, a_2, \dots, a_n]$ and $B_n = [b_1, b_2, \dots, b_n]$ in S_n, $A_n \preceq B_n$ if and only if either $A_n = B_n$, or $a_n < b_n$, or for some $i, 1 \leq i < n, a_n = b_n, a_{n\text{-}1} = b_{n\text{-}1}, \dots, a_{i+1} = b_{i+1}, a_i < b_i$. Then $\preceq$ is a total order on S_n with maximum element $Tr_n = [0, 1, 2, \dots, n\text{-}1]$, the score sequence of the transitive n-tournament (i.e., the n-tournament with no directed cycles), and with minimum element R_n, the score sequence of either a regular n-tournament, if n is odd, or a nearly-regular score sequence, if n is even. We claim that the following algorithm will transform a given sequence in S_n into the sequence R_n via jumps down $(S_n, \preceq)$ in such a way that each sequence in the process is a score sequence if and only if the sequence reached by a single jump is a score sequence. These jumps usually do not involve two sequences such that one covers the other in $(S_n, \preceq)$. And, we will show that the first new sequence obtained is a strong score sequence.

Algorithm.

(i) Begin with $S_0 = [s_1, s_2, \dots, s_n] \neq R_n$, where $S_0 \in S_n$.

(ii) For $S_l = [s_1, s_2, \dots, s_n],\ l \geq 0$, find indicies p and $q,\ p < q$, where $s_1 = s_2 = \cdots = s_p < s_{p+1}$ and $s_{q-1} < s_q = s_{q+1} = \cdots = s_n$, and replace s_p with $s_p + 1$ and s_q with $s_q - 1$. Note that p may be 1 and q may be n.

(iii) Set this new n-tuple as S_{l+1} and relabel the scores as $s_1, s_2, \dots, s_n$.

(iv) If $S_{l+1} = R_n$, the regular or nearly-regular score sequence, then stop. Otherwise, return to (ii).

If a non-decreasing integer sequence satisfies (1.2) and is neither the regular nor nearly-regular score sequence, then it is clear that such a p and $q,\ p < q$, as in part (ii) exist.

Consider the following example where the appropriate p^{th} and q^{th} positions are underlined, and $A \succcurlyeq B$ means $B \preceq A$.

$S_0 = [1,1,2,3,4,5,6,6] \succcurlyeq S_1 = [1,2,2,3,4,5,5,6] \succcurlyeq S_2 = [2,2,2,3,4,5,5,5] \succcurlyeq S_3 = [2,2,3,3,4,4,5,5] \succcurlyeq S_4 = [2,3,3,3,4,4,4,5] \succcurlyeq S_5 = [3,3,3,3,4,4,4,4]$. Note that the jumps are not necessarily between sequences so that one covers the other in $(S_8, \preceq)$. For example, $S_1 \succcurlyeq [2,2,2,2,4,5,5,6] \succcurlyeq S_2$, so S_1 does not cover S_2.

We note the following for above algorithm.

***Lemma* 2.2.3.** *In the algorithm above, for all $l \geq 0$, if $S_l \neq R_n$ is in S_n, then S_{l+1} is in S_n. Thus, every sequence obtained by the algorithm is in S_n.*

***Lemma* 2.2.4.** *The algorithm produces a sequence of integral n-tuples beginning at S_0 and ending at R_n.*

***Lemma* 2.2.5.** *If, in the algorithm, S_{l+1} is the score sequence of some strong n-tournament for some $l \geq 0$, then S_l is the score sequence of some n tournament (not necessarily strong).*

A non-strong score sequence that might result in Lemma 2.2.5 occurs only in a special case as described next.

***Lemma* 2.2.6.** *If, in the algorithm, S_{l+1} is the score sequence of some strong n-tournament and S_l is the score sequence of some n-tournament which is not strong, then $l = 0$ and $S_l = S_0$.*

***Theorem* 2.2.7.** *If $S = [s_1, s_2, \ldots, s_n]$ is in S_n, then S is the score sequence of some n-tournament.*

Santana and Reid Proof. The theorem is clearly true if $S = R_n$. So, suppose $S \neq R_n$. By Lemma 2.2.4 the algorithm produces a sequence of n-tuples $S = S_0, S_1, \ldots, S_M$, terminating in $S_M = R_n$ for some integer $M \geq 1$ (Actually $M = \frac{1}{2}d(R_n, S)$). By Lemma 2.2.3, S_l

satisfies (1.2) with $s_1 \leq s_2 \leq \cdots \leq s_n$ for all l, $0 \leq l \leq M$. We now show, by induction on j, that S_{M-j} is the score sequence of some strong n-tournament, for all j, $0 \leq j < M$. If $j = 0$, then $S_{M-0} = S_M = R_n$, the regular or nearly-regular score sequence, which is strong. Now, suppose S_{M-j} is the score sequence of some strong n-tournament for some j, $0 \leq j < M-1$. Since S_{M-j} is the score sequence of some strong n-tournament, by Lemma 2.2.5, S_{M-j-1} is the score sequence of some n-tournament T. If T is not strong, then Lemma 2.2.6 implies that $S_{M-j-1} = S_0$. That is, $M-j-1 = 0$, a contradiction. Thus, T must be strong. So, by induction, S_l is the score sequence of some strong n-tournament, for all l, $1 \leq l \leq M$. In particular, S_1 is the score sequence of some strong n-tournament. By Lemma 2.2.5, $S_0 = S$ is the score sequence of some n-tournament, as desired. □

Section 2.3

Bipartite analogue of Landau's theorem

A *bipartite tournament* T is an orientation of a complete bipartite graph. The vertex set of T is the union of two disjoint nonempty sets X and Y, and arc set of T comprises exactly one of the pairs (x,y) or (y,x) for each $x \in X$ and each $y \in Y$. If the orders of X and Y are m and n respectively, T is said to be an $m \times n$ bipartite tournament.

A bipartite tournament may be used to represent competition between two teams and each player competes against everyone on the opposing team. The score s_v of the vertex v is the number of vertices it dominates and for a bipartite tournament there is a pair of score sequences, one sequence for each set. For example, the bipartite tournament shown below has score sequences $[4,3,2,0]$ and $[2,2,2,1]$.

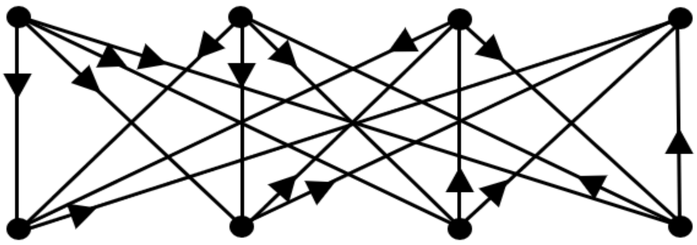

Figure 2.2: A bipartite tournament

Definition. A bipartite tournament is *reducible* if there is a nonempty proper subset of its vertex set to which there are no arcs from the other vertices, otherwise irreducible.

***Lemma* 2.3.1.** *If v and v' are vertices in the same partite set of a bipartite tournament T, if $s_v \leq s'_v$, and if there is a vertex w which is dominated by v and which dominates v', then there is another vertex w' which is dominated by v' and which dominates v, that is $v \to w \to v' \to w' \to v$ is a $4-$ cycle.*

The following recursive result is due to Gale [17]

***Theorem* 2.3.2.** *If $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ are sequences of non-negative integers in non-decreasing order, then A and B are the score sequences of some bipartite tournament if and only if the sequences $A' = [a_1, a_2, \ldots, a_{m-1}]$ and $B' = [b_1, b_2, \ldots, b_{a_m}, b_{a_m+1} - 1, \ldots, b_n - 1]$ are score sequences of some bipartite tournament.*

Proof. First suppose that A' and B' are the score sequences of a bipartite tournament T'. To the first partite set of T', add a new vertex v with arcs directed from it to vertices (in the second set) with scores $b_1, b_2, \ldots, b_{a_m}$, and to it from the others. This results in a bipartite tournament with score sequences A and B.

For the converse, it is sufficient to show that if A and B are the score sequences of a bipartite tournament, then in one realisation, a vertex (in the first set) of score a_m dominates vertices of scores $b_1, b_2, \ldots, b_{a_m}$. Among the bipartite tournament realisations of A and B, let T be the one in which a vertex x of of score a_m is such that the sum S of the scores of the vertices it dominates is as small as possible. Let $S > \sum_{j=1}^{a_m} b_j$. Then there exist vertices y and y' such that $x \to y', y \to x$ and $s_y < s'_y$. By Lemma 2.3.1, T has a $4-$cycle $x \to y' \to x' \to y \to x$, and if its arcs are reversed, the result is a bipartite tournament with the same sequences, but in which score sum of vertices dominated by x is less than before. Since the sum was assumed to be minimised, the result follows. $\square$

The following result by Moon [31] is the combinatorial characterization of score sequences in bipartite tournaments.

***Theorem* 2.3.3.** *A pair of sequences $A = [a_i]_1^m$ and $B = [b_j]_1^n$ of non-negative integers in non-decreasing order are the score sequences of some bipartite tournament if and only if*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \geq kl,$$

for $1 \leq k \leq m$ and $1 \leq l \leq n$, with equality when $k = m$ and $l = n$. Further more, the bipartite tournament is irreducible if and only if the inequality is strict except when $k = m$ and $l = n$.

Proof. **Necessity** In any bipartite tournament T, the combined scores of any collection of k vertices from the first set and l from the second must be at least kl, so that the inequalities certainly hold. Further, if T is irreducible, the inequality is strict unless $k = m$ and $l = n$.

Sufficiency If A and B satisfy the inequalities, we show that A' and B' satisfy the inequalities recorded as in construction of Theorem 2.3.2. It is easily seen that A' and B' are then in non-decreasing order, and further, their combined sum is

$$\sum_{1}^{m-1} a'_i + \sum_{1}^{h} b'_j = mn - a_m - (n - a_m) = (m-1)n.$$

For a fixed value of $k (1 \leq k \leq m-1)$, assume there is a value of l for which the inequality does not hold and let h denote the least such that

$$\sum_{1}^{k} a'_i + \sum_{1}^{h} b'_j < kh.$$

It follows from the minimality of h that $b'_h < k$, whence $b_h \leq k$. Now, let p and q be the least and greatest values of j for which $b_j = b_{a_m}$ and set $r = max(h,q)$. Since the first $p-1$ values of b_j were unchanged, we have $h \geq p$ and thus $b_h = \cdots = b_r$. Finally, let s denote the number of $j \leq h$ such that $b'_j = b_{j-1}$. If $h \leq q$, then $s \leq q - a_m$, and if $h > q$, then $s = (h-q) + (q-a_m) = h - a_m$. In either case, $a_m + s \leq r$. Therefore,

$$\begin{aligned} \sum_{1}^{k+1} a_i + \sum_{1}^{r} b_j &= \sum_{1}^{k} a'_i + \sum_{1}^{h} b'_j + \sum_{h+1}^{r} b_j + a_{k+1} + s \\ &< kh + (r-h)b_h + a_m + s \\ &\leq kh + (r-h)k + r < (k+1)r, \end{aligned}$$

which is a contradiction. Therefore A' and B' satisfy the inequalities, as required. It is easily seen that if the strict inequalities hold for A and B, no realisation can be reducible , completing the proof. □

Scores in multipartite tournaments

A *k-partite tournament* is a digraph obtained by orienting the edges of the complete k-partite graph. That is, its vertex set consists of k-vertex disjoint sets (or parts), and between every pair of distinct vertices from different parts there is exactly one arc. A *bipartite tournament* is a 2-partite tournament and a *multipartite tournament* is a k-partite tournament for some $k \geq 2$. The out-degree of a vertex x is called the score of x. Note that if all the parts in a multipartite tournament are singletons, then it is a tournament. Also, there may be many transmitters in a multipartite tournament since there is no arc between two vertices in the same part.

Figure (2.3) shows example of 3-partite tournament with the score sequences $[2,3,3],[1,3,3]$ and $[3,3]$.

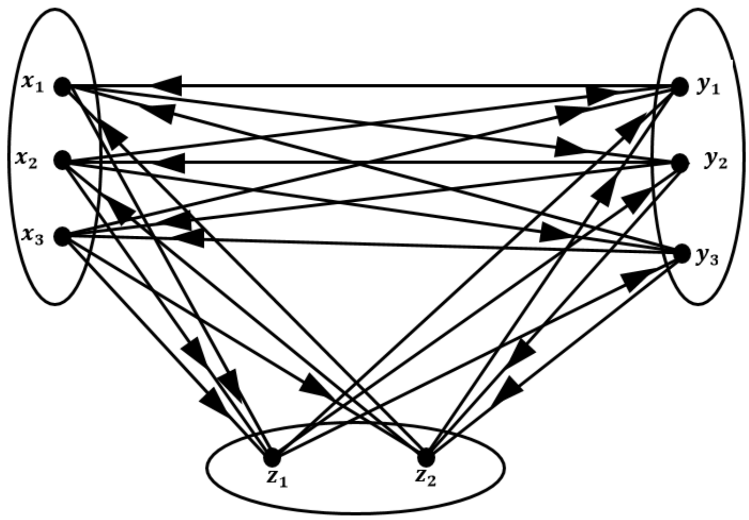

Figure 2.3: 3-partite tournament

A multipartite version of Landau's theorem was obtained by Moon [31] in 1962. It is discussed in his 1968 monograph [32]

Theorem 2.3.4. *Let $n_1, n_2, \dots, n_k$ be k positive integers. The k non-decreasing sequences of integers $S_i = [s^i_1, s^i_2, \dots, s^i_{n_i}]$, $1 \leq i \leq k$, form the score sequences of some k-partite tournament of order $n = n_1 + n_2 + \cdots + n_k$ if and only if*

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} s^i_j \geq \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j,$$

for all sets of k integers m_i satisfying $0 \leq m_i \leq n_i$, with equality holding when $m_i = n_i$ for all i, $1 \leq i \leq k$.

Section 2.4

Scores in oriented graphs

An oriented graph is a digraph with no symmetric pairs of directed arcs and without self loops. If D is an oriented graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and if $d^+(v)$ and $d^-(v)$ are respectively, the outdegree and indegree of a vertex v, then $a_v = n-1+d^+(v)-d^-(v)$ is called the score of v. Clearly, $0 \leq a_v \leq 2n-2$. The score sequence $A(D)$ of D is formed by listing the scores in non-decreasing order. One of the interpretations of an oriented graph is a competition between n teams in which each team competes with every other exactly once, with ties allowed. A team receives two points for each win and one point for each tie. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.
(i). An arc directed from u to v, denoted by $u(1-0)v$, (ii). An arc directed from v to u, denoted by $u(0-1)v$, (iii). There is no arc from u to v and there is no arc from v to u, and is denoted by $u(0-0)v$.

If $d^*(v)$ is the number of those vertices u in D which have $v(0-0)u$, then $d^+(v)+d^-(v)+d^*(v) = n-1$. Therefore, $a_v = 2\ d^+(v)+d^*(v)$. This implies that each vertex u with $v(1-0)u$ contributes two to the score of v. Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0-0)u$ contributes two(one each at u and v) to scores, therefore the sum total of all the scores is $2\binom{n}{2}$. With this scoring system, player v receives a total of a_v points.

Avery [4] obtained the following combinatorial characterization of score sequences in oriented graphs.

***Theorem* 2.4.1.** [4] *A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for each $I \subseteq [n] = \{1,2,\ldots,n\}$,*

$$\sum_{i \in I} a_i \geq 2\binom{|I|}{2}, \tag{4.10}$$

with equality when $|I| = n$.

Since $a_1 \leq a_2 \leq \cdots \leq a_n$, the inequality (4.10) are equivalent to

$$\sum_{i=1}^{k} a_i \geq 2\binom{k}{2}, \text{ for } k = 1,2,\ldots,n-1$$

with equality for $k = n$.

Another proof of Avery's theorem can be seen in Pirzada et al. [34]. A recursive characterization of score sequences in oriented graphs also appears in Avery [4].

***Theorem* 2.4.2.** [4] *Let A be a sequence of integers between 0 and $2n-2$ inclusive and let A' be obtained from A by deleting the greatest entry $2n-2-r$ say, and reducing each of the greatest r remaining entries in A by one. Then A is a score sequence if and only if A' is a score sequence.*

Theorem 2.4.2 provides an algorithm for determining whether a given non-decreasing sequence A of non-negative integers is a score sequence of an oriented graph and for constructing a corresponding oriented graph.

Section 2.5

Football sequences

If D is an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and if $d^+(v_i)$ and $d^-(v_i)$ are respectively the outdegree and indegree of a vertex v_i, define f_{v_i} (or briefly f_i) as

$$f_i = n - 1 + 2d^+(v_i) - d^-(v_i)$$

and call f_i as the football score(or briefly f-score) of v_i. Clearly

$$0 \leq f_{v_i} \leq 3(n-1).$$

The f-score sequence $F(D)$ (or briefly F) of D is formed by listing the f-scores in non-decreasing or non-increasing order. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.
(i). An arc directed from u to v, denoted by $u \to v$ and we write this as $u(1--0)v$.
(ii). An arc directed from v to u, denoted by $u \leftarrow v$ and we write this as $u(0--1)v$.
(iii). There is no arc directed from u to v and there is no arc directed from v to u, denoted by $u \sim v$ and we write this as $u(0--0)v$.
If $d^*(v)$ is the number of those vertices u in D for which we have $v(0--0)u$, then

$$d^+(v) + d^-(v) + d^*(v) = n - 1.$$

Therefore,

$$f_v = d^+(v) + d^-(v) + d^*(v) + 2d^+(v) - d^-(v) = 3d^+(v) + d^*(v).$$

This implies that each vertex u with $v(1--0)u$ contributes three to the f-score of v, and each vertex u with $v(0--0)u$ contributes one to the f-score of v.
Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0--0)u$ contributes two (one each at u and v) to f-scores, therefore

$$2\binom{n}{2} \leq \sum_{i=1}^{n} f_i \leq 3\binom{n}{2}.$$

We interpret an oriented graph as the result of a football tournament with teams represented by vertices in which the teams play each other once, with an arc from team u to team v if and only if u defeats v. A team receives three points for each win and one point for each draw (tie). With this f-scoring system, team v receives a total of f_v points.

We call the sequence $F = [f_1, f_2, \ldots, f_n]$ as the football sequence, if f_i is the f-score of some vertex v_i. Thus a sequence $F = [f_1, f_2, \ldots, f_n]$ of non-negative integers in non-decreasing order is a football sequence if it realizes some oriented graph. Several results on football sequences can be found in Ivanyi [24].

In an oriented graph the vertex of indegree zero is called a transmitter. This means that the transmitter represents that team in the game which does not lose any match.

***Theorem* 2.5.1.** *If the sequence $F = [f_1, f_2, \ldots, f_n]$ of non-negative integers in non-decreasing order is a football sequence then for $1 \le k \le n-1$ and $2\binom{k}{2} \le x_k \le 3\binom{k}{2}$,*

$$\sum_{i=1}^{k} f_i \ge x_k,$$

and for $2\binom{n}{2} \le x_n \le 3\binom{n}{2}$

$$\sum_{i=1}^{n} f_i = x_n.$$

***Lemma* 2.5.2.** *There is no oriented graph with n vertices whose f-score of some vertex is* $3n-4$.

Proof. Let D be an oriented graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let v_i be the vertex with f-score f_i. In case $v_i(1-0)v$ to all $v \in V - \{v_i\}$, then f-score of v_i is $3(n-1)$. If $v_i(1-0)v$ for all $v \in V - \{v_i, v_j\}$, for some $v_j \in V$ and $i \ne j$, then f-score of v_i is $3(n-2)+1 = 3n-5$. We note that the possible f-score can be $3(n-1)$ or $3(n-2)+1$. Thus the f-score f_i is either $3(n-1)$ or $f_i \le 3(n-2)+1 = 3n-5$. These imply that the f-score cannot be $3n-4$. □

***Lemma* 2.5.3.** *In an oriented graph with n vertices if the f-score f_i and n are of the same parity, then the vertex v_i with f-score f_i is not the transmitter.*

Proof. Let $D(V,A)$ be an oriented graph with $V = \{v_1, v_2, \ldots, v_n\}$ so that $f_{v_i} = f_i$. Let n and f_i be of same parity, that is either (a) n and f_i both are even or (b) n and f_i both are odd.

In D, let $v_i(1-0)u$, $v_i(0-0)w$ and $v_i(0-1)z$ with $u \in U$, $w \in W$, $z \in Z$ and $V = U \cup W \cup Z \cup \{v_i\}$. Further let $|U| = x$, $|W| = y$ and $|Z| = t$. Clearly

$$x + y + t = n - 1. \tag{5.11}$$

Case (a). $n-1$ is odd and f_i is even. We have $f_i = 3x + y$. Since f_i is even, $3x + y$ is even. Thus either (i) x is odd and y is odd, or (ii) x is even and y is even. In both cases, it follows from (5.11) that t is odd.
Case (b). $n-1$ is even and f_i is odd. So $3x + y$ is odd. This is possible if (iii) x is even and y is odd, or (ii) x is odd and y is even. In both cases, again it follows from (5.11) that t is odd.

Thus in all cases we have $|Z| = t = \ odd$, which implies that $|Z| \neq \phi$ so that there is at least one vertex z such that $z(1-0)v_n$. Hence v_i is not a transmitter. □

Lemma 2.5.3 shows that if the number of teams n and the f-score f_i are both odd or both even, then the team represented by v_i with f-score is not the transmitter, meaning it loses at least once in the competition.

***Theorem* 2.5.4.** *In an oriented graph with n vertices the vertex with f-score f_i is a transmitter if (i) n and f_i are of different parity and (ii) $f_i \equiv (n-1)(mod\ 2)$ and $f_i \equiv 3(n-1)(mod\ 2)$.*

Proof. Let $D(V,A)$ be the oriented graph with n vertices whose vertex set is $V = \{v_1, v_2, \cdots, v_n\}$. Let f-score of v_i be f_i and let v_i be the transmitter. Then in D, we have either $v_i(1-0)v_j$ or $v_i(0-0)v_j$ for all all $j \neq i$. Let U be the set of vertices for which $v_i(1-0)u$ and W be the set of vertices for which $v_i(1-0)w$ and let $|U| = x$ and

$|W| = y$.

Clearly

$$x + y = n - 1 \tag{5.12}$$

and

$$f_i = 3x + y. \tag{5.13}$$

Two cases can arise, (a) n is odd or (b) n is even.

Case (a). n is odd. Then $n-1$ is even so that $x+y$ is even. This is possible if either (i) x odd and y odd or (ii) x even and y even. In case of (i) $f_i = 3x + y = odd + odd = even$ and in case of (ii) $f_i = 3x + y = even + even = even$. Thus we see that n and f_i are of different parity.

Case (b). n is even, so that $n-1$ is odd and $x+y$ is odd. This is possible if either (iii) x odd and y even or (ii) x even and y odd. In both cases we observe that f_i is odd. Therefore again we obtain that n and f_i are of different parity.

Solving (5.12) and (5.13) together for x and y, we get

$$x = \frac{1}{2}[f_i - (n-1)] \tag{5.14}$$

$$y = \frac{1}{2}[3(n-1) - f_i]. \tag{5.15}$$

Clearly x and y are positive integers, thus the right hand sides of (5.14) and (5.15) are positive integers. This implies that $f_i - (n-1)$ and $3(n-1) - f_n$ are both divisible by 2. Hence $f_n \equiv (n-1)(mod\ 2)$ and $f_n \equiv 3(n-1)(mod\ 2)$. □

We note that the above conditions are only necessary but not sufficient. To find sufficient conditions for the football sequences is still an unsolved problem.

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