Basic Mathematical Model of Cardiovascular System

A project dissertation submitted to the

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CERTIFICATE

This is to certify that the project dissertation entitled, **“Basic Mathematical Model of Cardiovascular System"** *being submitted by the students with the enrollments* ***21068120017, 21068120032*** *to the* ***Department of Mathematics, University of Kashmir, Srinagar,*** *for the award of Master’s degree in Mathematics, is an original project work carried out by them under my guidance and supervision.*

The project dissertation meets the standard of fulfilling the requirements of regulations related to the award of the Master’s degree in Mathematics. The material embodied in the project dissertation has not been submitted to any other institute, or to this university for the award of Master’s degree in Mathematics or any other degree.

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Contents:

Chapter 1

1.1 Introduction of Mathematical Biology

Mathematical biology aims at the mathematical representation and modeling of biological processes, using techniques and tools of applied mathematics. It can be useful in both theoretical and practical research. Describing systems in a quantitative manner means their behavior can be better simulated, and hence properties can be predicted that might not be evident to the experimenter. This requires precise mathematical models [1].

Mathematical biology is an interdisciplinary field that applies mathematical techniques, modeling, and computational methods to study and understand biological phenomena. It involves using mathematical models to describe and analyze various aspects of living organisms, from the cellular level to ecosystems. Mathematical biologists aim to gain insights into complex biological processes, predict outcomes, and test hypotheses.

This field encompasses a wide range of topics, including population dynamics, epidemiology, genetics, ecology, neurobiology, and more. Mathematical models can help researchers to simulate biological systems, make predictions about their behavior, and guide experimental design. Overall, mathematical biology plays a crucial role in advancing our understanding of biology and solving real-world problems related to healthcare, conservation, and more [2].

Mathematical biology is expanding and developing rapidly as scientists in biological sciences turn from descriptive experiments to more quantitative experiments. The diversity and complexity of living organisms means there are vastly more challenges for mathematicians to explain and predict biological systems through modeling.

The concept of mathematical Biology is not a new one. The Chinese, the ancient Egyptians, Indians, Babylonians and Greeks indulge in understanding and predicting the natural phenomena through their knowledge of mathematics. Mathematical Biology consists of simplifying real world problems and representing them as mathematical problems, solving the model and interpreting these solutions in the language of real world [3].

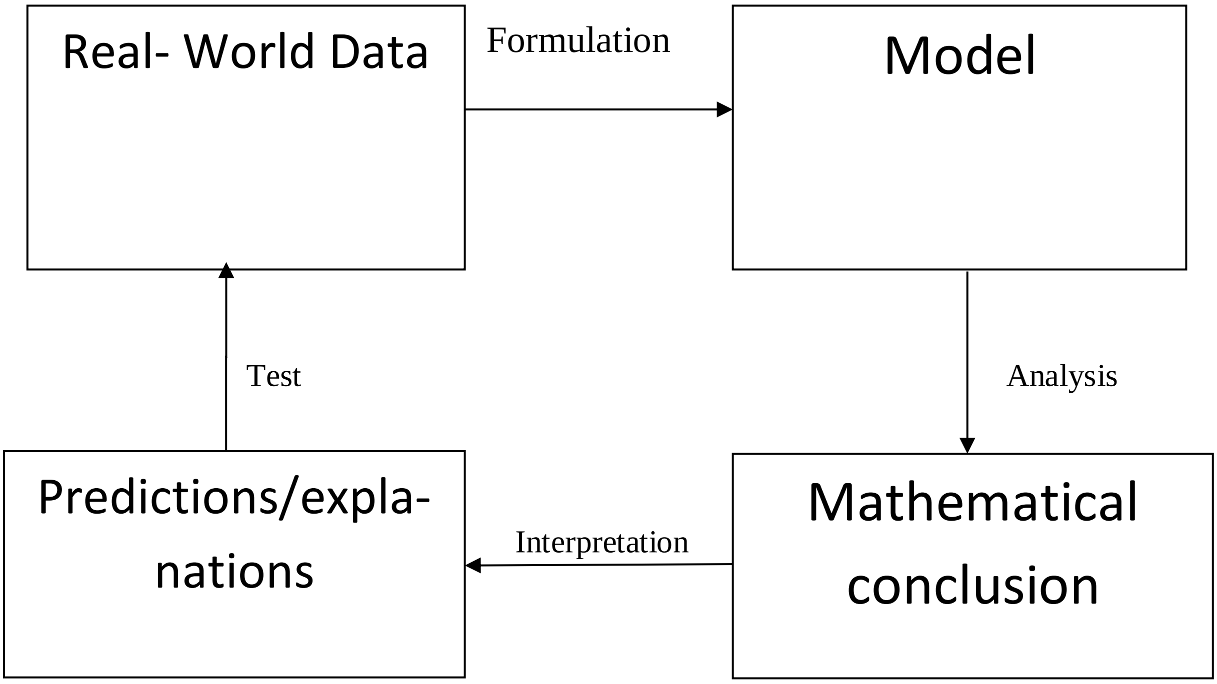

Fig. 1.1: Process of mathematical modeling [3].

1.2 Introduction To Human Physiology

Human physiology is the branch of biology that focuses on the study of how the various systems and organs in the human body function and interact to maintain life and health. It encompasses the study of processes such as digestion, respiration, circulation, nervous system function, and more, to understand how the body's internal mechanisms work to maintain homeostasis and support the overall well-being of an individual [4].

1. Homeostasis: Understanding how the body regulates and maintains a stable internal environment despite external changes.
2. Circulatory System: Discussions about the heart, blood vessels, and blood circulation, including topics like blood pressure, heart rate, and cardiovascular diseases.
3. Respiratory System: Exploring how the lungs and respiratory tract function, including the exchange of gases like oxygen and carbon dioxide.
4. Nervous System: Discussions on brain function, nerve signaling, and sensory perception, as well as neurological disorders.
5. Endocrine System: Examining the role of hormones in regulating various bodily functions and the impact of hormonal imbalances.
6. Digestive System: Understanding how the body processes food, absorbs nutrients, and eliminates waste, as well as digestive disorders.

7. Muscular and Skeletal Systems: Topics related to muscle function, bone structure, and movement, as well as conditions like osteoporosis and muscle disorders.
8. Reproductive System: Discussions on human reproduction, fertility, and reproductive health.
9. Immune System: Exploring how the body defends against pathogens and the development of vaccines and immunotherapy.
10. Metabolism: Discussions on energy metabolism, nutrition, and metabolic disorders like diabetes.

These discussions are essential for advancing our understanding of the human body and for developing treatments and interventions to maintain and improve human health. Researchers and healthcare professionals continually explore these topics to enhance our knowledge of human physiology and its impact on overall well-being [5].

1.3 Physiology, Anatomy and Function of Heart

1.3.1 Physiology of heart

Heart as a Pump: The heart is a muscular organ that functions as a pump, maintaining the circulation of blood to supply oxygen, nutrients, and remove waste products from body tissues.

The heart has four chambers: two atria (upper chambers) and two ventricles (lower chambers). The right atrium receives deoxygenated blood from the body, the left atrium receives oxygenated blood from the lungs, and the ventricles pump blood out of the heart.

Blood flows through a series of valves and chambers in a one-way circuit. Deoxygenated blood returns to the right atrium, is pumped into the right ventricle, sent to the lungs for oxygenation, returns to the left atrium, and is then pumped into the left ventricle, which sends oxygenated blood to the rest of the body. Heart has valves (Muscular Flaps) which prevents back flow of blood. The atrioventricular (AV) valves (tricuspid and mitral/bicuspid) separate the atria from the ventricles, while the semilunar valves (pulmonary and aortic) separate the ventricles from the arteries.

Heart has its own Conductive system

The sinoatrial (SA) node generates electrical impulses, initiating each heartbeat. These impulses travel through the atria, causing them to contract, and then pass through the atrioventricular (AV) node, Delays signals from SA node to prevent

Over Stimulation of heart and helps it to maintain Its rhythm (72beats /min) called as Gatekeeper of Heart.

The cardiac cycle consists of systole (contraction) and diastole (relaxation) phases. During systole, the heart contracts, pushing blood out of Heart into Blood vessels and then to Body parts. During diasystole Heart relaxes and receives Deoxygenated blood into it and cycle goes on. The heart itself requires a constant supply of oxygenated blood. Coronary arteries deliver oxygen and nutrients to the heart muscle, ensuring it functions properly [5].

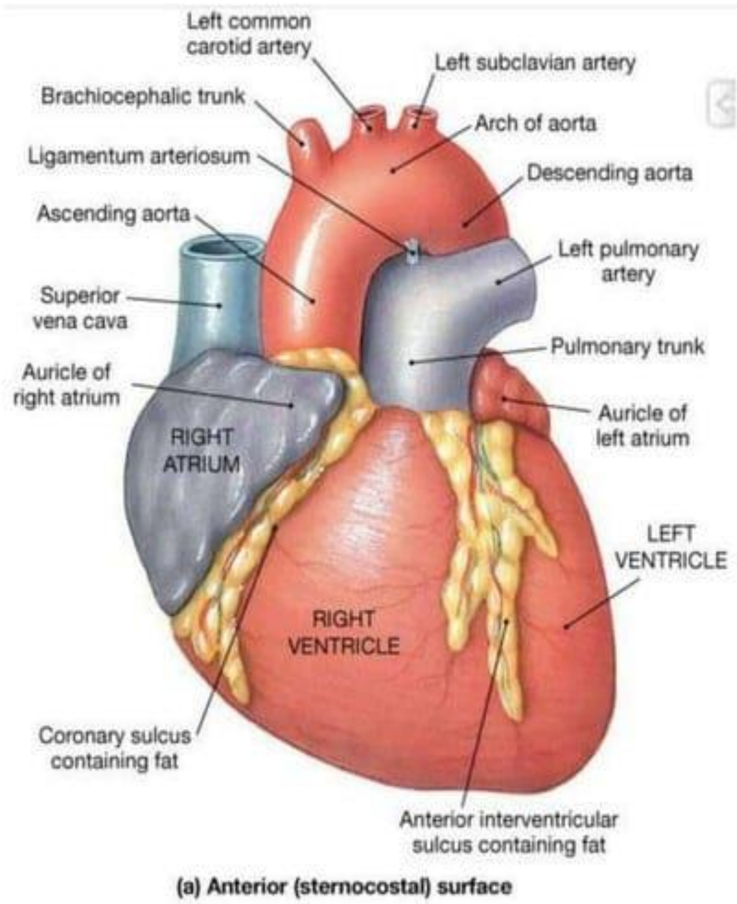

Fig 1.2: Human Heart [4].

1.3.2 Anatomy of Heart

Heart Is a Hollow Muscular Organ Situated In Thoracic Cavity (Chest of Human Body)

Enclosed In Pericardium (Covering Of Heart)

Shape Pyramidal In Shaped

Measurement

Length 12cm

Width 9cm

Weight 300g in Males

And 250g in Females

Placed - Obliquely In Chest Cavity

So That 1/3 OF Heart Is To Right of Median Plan

-2/3 of it is to Left of Median Plan

Chambers-4 Chambers

-Right Atrium & Right Ventricle

- Left Atrium and Left Ventricle

-On Surface,

Atria Separated From Ventricle by Atrioventricular Groove

Ventricle from Each Other by Interventricular Grooves.

External Features -

Apex

Base

- 4 Surface]

4 Border-Right, Left, Upper and Inferior

Apex-Conical

-Formed By Left Ventricle

Directed Downward and Forward and to the Left

Base-Formed by 2 Atria, Mainly by Left Atrium

→ Surface-

1) Sternocostal-Formed by Right Atrium and Right Ventricle

2) Diaphragmatic Surface-Formed by Left and Right Ventricle

3) Left Surface - Formed by Left Ventricle and Partly Left Atrium

Directed Upward, Backward and to the Left

4) Right Surface Formed by Atrium

Borders

1) Right Border

2) Left Border

3) Inferior Border

4) Upper Border [6].

Borders

Separating the surfaces of the heart are its borders. There are four main borders of the heart:

1- Right border – Right atrium

2- Inferior border – Left ventricle and right ventricle

3- Left border – Left ventricle (and some of the left atrium)

4- Superior border – Right and left atrium and the great vessels

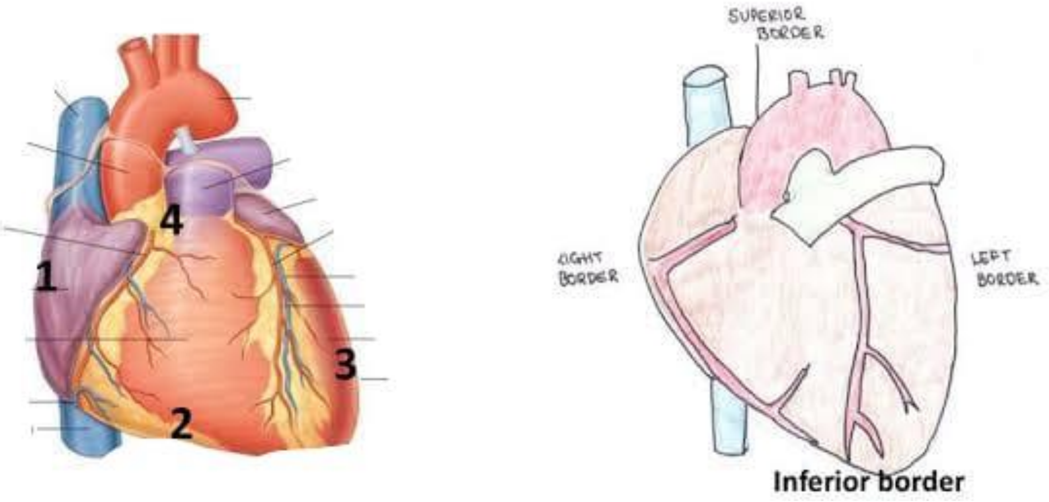

Fig.1.3: Borders of Heart [4].

1.3.3 Functions of Heart

Some important Terminologies

Blood Vessels of Heart

1) Arteries Blood Vessels which carry Blood away from Heart

2) Veins which carry Blood towards Heart

With no exception

3) Capillaries form the connection between the vessels that carry blood away from the heart (arteries) and the vessels that return blood to the heart (veins).

Arteries mainly carry oxygenated Blood except Pulmonary Arteries

Veins Carry Mainly Deoxygenated Blood except Pulmonary Vein

2. Major veins

Superior Vena cava Carrier Blood from upper half of Body to Heart (in Right Atrium) enters from above and Inferior Vena cava Carrier Blood from Lower half of Body to Heart (in Right Atrium) enters from below

3. Major Artery.

Aorta form arch of Aorta

Gives Right Brachiocephalic Artery

And Left Common Carotid and Left Sub clavian [4].

Working of Heart

The works of heart usually follow two pathways:

Systematic Circulation through which heart pumps blood in blood vessels and Body organs receives oxygenated blood.

Pulmonary circulation Deoxygenated is taken back to Heart then to lungs for oxygenation then back to Heart and the again to body cycle goes on.

During this Blood flows twice into heart which we called double Circulation which is the characteristic of Mammals (e.g. Humans, Monkey etc.)

Below is the sequence of events Described:

Pulmonary Circulation

Deoxygenated blood returns to the right atrium from the body via two large veins, the superior and inferior vena cava.

The right atrium contracts, pushing blood through the tricuspid valve into the right ventricle. When the right ventricle contracts, it pushes this deoxygenated blood through the pulmonary valve into the pulmonary artery. The pulmonary artery carries this blood to the lungs, where it receives oxygen and releases carbon dioxide during gas exchange. Oxygenated blood returns to the heart via the pulmonary veins, entering the left atrium.

Systemic Circulation: Oxygenated blood in the left atrium is pumped into the left ventricle.

When the left ventricle contracts, it forces this oxygen-rich blood through the aortic valve into the aorta, the body's largest artery. The aorta branches into smaller arteries, which carry oxygenated blood to all parts of the body, including organs and tissues. In capillaries, oxygen and nutrients are exchanged for carbon dioxide and waste products.

Deoxygenated blood is collected in veins and eventually returns to the right atrium, starting the process again in pulmonary circulation.

In summary, pulmonary circulation moves deoxygenated blood to the lungs for oxygenation, while systemic circulation distributes oxygen-rich blood to the body's tissues. This continuous cycle ensures that every cell in the body receives the oxygen and nutrients it needs while eliminating waste products.

This cycle last for about 0.8 seconds and is called as Cardiac Cycle

Cardiac cycle have

Atrial Systole= 0.1s

Atrial Diasystole= 0.7 s

Ventricular Systolic= 0.3s

Ventricular Diasystole= 0.5s

During this Blood distribution (Working of Heart) we hear usually 2 sounds of Heart

S_1 (Lub)

When AV Valve Close (Auruculo Ventricular Valves i.e. Bi-cuspid and Tricuspid Valve Closes)

S_2 (Dub)

SL valve closes (Semilunar Valves i.e. aortic and pulmonary)

There are some other sounds S_3 and S4 [7].

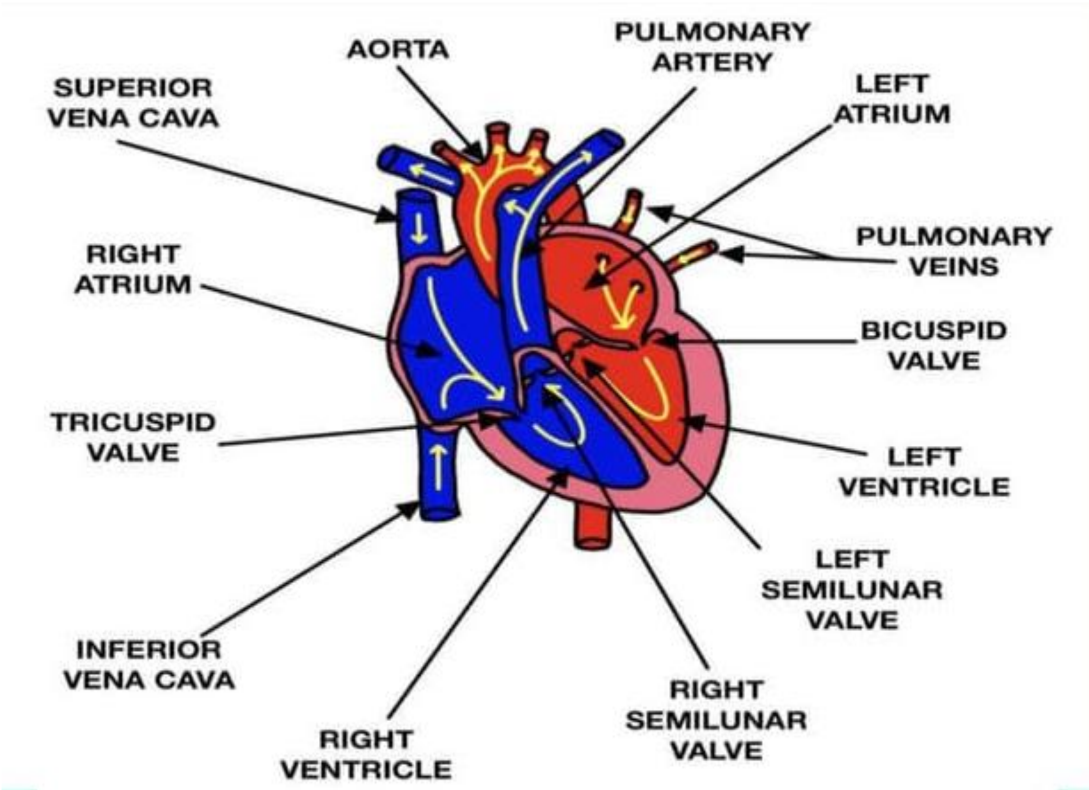

Fig1.4: Structure of Heart [4].

1.4 Some Mathematical Techniques in Modeling Human Heart

The study and understanding of the human heart involve various techniques and technologies. Some of the key techniques used in studying the human heart include:

Electrocardiography (ECG or EKG): ECG records the electrical activity of the heart, providing information about heart rate, rhythm, and any abnormal electrical patterns. It is commonly used in diagnosing arrhythmias and other cardiac issues.

Echocardiography: This uses ultrasound waves to create images of the heart's structure and function. It can assess heart valves, chamber size, and blood flow, making it valuable in diagnosing conditions like heart valve diseases and heart failure.

Cardiac Catheterization: Invasive procedure involving the insertion of a catheter into blood vessels leading to the heart. It helps measure blood pressure, obtain coronary angiograms, and perform interventions like angioplasty or stent placement for coronary artery disease.

Cardiac MRI (Magnetic Resonance Imaging): Provides detailed images of the heart's structure and function, including the assessment of cardiac muscle and tissue. It is especially useful for diagnosing heart diseases and assessing heart function.

Cardiac CT (Computed Tomography): Uses X-rays to create cross-sectional images of the heart and blood vessels. It is valuable for diagnosing coronary artery disease and evaluating heart anatomy.

Stress Testing: This includes exercise stress tests or pharmacological stress tests to assess how the heart responds to increased workload. It helps diagnose coronary artery disease and evaluate heart function.

Holter Monitor and Event Monitor: Portable devices used to record heart rhythms over an extended period. They are useful for diagnosing intermittent irregular heartbeat.

Blood Tests: Measure various biomarkers in the blood, such as cardiac enzymes and troponin (type of protein found in the muscles of your heart) to assess heart health and diagnose conditions like heart attacks.

Nuclear Cardiology: Involves the use of radioactive tracers to assess blood flow to the heart muscle and detect areas with reduced blood supply, often used in stress testing.

Cardiac Electrophysiology Studies (EPS): Invasive procedures that involve threading catheters into the heart to map its electrical system and diagnose and treat irregular heartbeat.

These techniques play crucial roles in diagnosing and managing various heart-related conditions, allowing healthcare professionals to provide appropriate treatment and care for patients with heart issues [8].

Chapter 2

2.1 Surrey Work and Paper review

Numerical simulation of physiological and pathological changes in the human cardiovascular system had become an active research area in the past decades. Various models had been proposed to study the dynamics of the cardiovascular system (Snyder and Rideout, 1969, Heldt et al., 2002, Liang et al., 2009, Wang et al., 2013)[12]. Among these studies, lumped parameter models were usually used to study the global responses of the whole circulation system (Snyder and Rideout, 1969, Melchior et al., 1992, Ursino, 1998, Pennati et al., 1997, Sun et al., 1997, Heldt et al., 2002, Ellwein et al., 2008)[13]. Mathematical models of the heart played critical roles in investigating the global responses of human cardiovascular system. Due to their important roles in pumping blood into the circulation system, ventricular models had been paid more attention than atrial models in most modeling studies (Suga et al., 1973, Melchior et al., 1992, Drzewiecki et al., 1996, Pennati et al., 1997, Heldt et al., 2002, Ottesen and Danielsen, 2003)[13]. Models of the whole heart were also developed and used in some studies, while only a few of them included arbitrary heart rate (Sun et al., 1997, Ursino, 1998, Vollkron et al., 2002, Liang et al., 2009, Muller and Toro, 2014)[12].

There were mainly three ways to include arbitrary heart rate. Heldt et al. (2002) used 'Bazett formula' to determine the systolic time interval, which linked the systolic time interval of the present beat to duration of cardiac cycle that preceded it, meanwhile the length of the present cardiac cycle was determined by means of an integral pulse frequency modulation model of the sinoatrial node. Ottesen and Danielsen (2003) presented a different paradigm for modeling ventricular contraction with heart rate changes through using a polynomial expression of the activity function, which contained more features and would generate results in good agreement with experimental data from Regen et al. (1993) and Mulier (1994). Liang and Liu (2006) used a constant time parameter to scale the value of systolic elastance when the cardiac cycle changes. Ottesen's paradigm was chosen in this study for building the whole heart model with consideration of contractility varying with heart rate changes.

Modeling atrial contraction with heart rate changes was also valuable for analyzing the global responses of human cardiovascular system with closed loop regulation, because atrial contraction played an increasingly significant role in ventricular filling as heart rate increases Mohrman and Heller (2010)[14].

1. The Lumped Parametric Model was proposed by Synder and Rideout in the year 1969. The purpose of this model is to develop a mathematical model of Human Cardiovascular system. This model was divided into three parts: Systemic Circulation, Pulmonary Circulation and the Heart. The main concept of this model is describing the system based on the vessel diameters and simulating mathematical equations with active electrical elements.

It is useful to understand the anatomy of human cardiovascular system and related syndromes. This model deals with vessels pressure and blood flow at certain time.

A lumped parameter model in the cardiovascular system is a simplest mathematical representation that considers the system as a collection of interconnected compartments or "lumps" rather than modeling every individual component in detail. In the context of the cardiovascular system, these compartments typically represent different parts of the circulatory system, such as the heart, arteries, veins, and organs. Here are some advantages and disadvantages of using a lumped parameter model.

Advantages

1. Simplicity: Lumped parameter models are relatively simple and require fewer equations and parameters compared to detailed, distributed models.

2. Computationally Efficient: Due to their simplicity, lumped parameter models are computationally efficient and can be solved quickly, making them suitable for real-time simulations and clinical applications.

3. Parameter Estimation: Lumped parameter models can be calibrated using clinical data, allowing researchers to estimate 44 model parameters and customize the model for individual patients or specific scenarios.

Disadvantages

1. Lack of Spatial Detail: Lumped parameter models lack spatial detail because they treat entire compartments as single entities.

2. Oversimplification: Due to their simplifications, these models may not capture all the intricate physiological processes and phenomena that occur in the cardiovascular system.

3. Limited usefulness for research: While lumped parameter models are useful for certain clinical applications and educational purposes, they may not be suitable for in depth research into highly detailed and specific cardiovascular phenomena [11].

2. A Mathematical model of human heart including the effects of heart contractility varying with heart rate changes, model was proposed by J. T. Ottesen and M. Densielsen in the year 2003. The pumping heart is described by a new mathematical approach which considers the heart as a pressure source depending on time, volume and flow. This new approach allows a separation between isovolumic and ejecting heart properties. The computed results cover most of the features of the human ventricle during normal and altered vascular conditions. It is shown that the time-varying elastance concept is disqualified as an independent description of the heart; it follows from isovolumic heart properties and an ejection

effect which consists of positive and negative effects of ventricular blood ejection [12].

3. The 0D Resistive-Compliant model was proposed by Ambrosi et al., in the year 2012[9]. The 0D Resistive-Compliant model is a simplified mathematical representation used in the field of cardiovascular modeling to describe the behavior of blood vessels and the flow of blood within them. It is often used in lumped parameter models of the cardiovascular system. In order to understand the parameters LCR (inductance, capacitance and resistance) in this model, it will be analyzed below discussed. This approach is usually called capacitance-resistance model (CRM) [9].

4. A global multiscale mathematical model for the human circulation with emphasis on the venous system model was proposed by Lucas O Muller and Toro in the year 2014. They present a global, closed loop, multiscale mathematical model for the Human circulation including the arterial system, the venous system, the Heart, the pulmonary circulation and the microcirculation. A distinctive feature of their model is the detailed description of the various systems particularly for intracranial and extracranial veins. Medium to large vessels are described by 1-Dimensional hyperbolic system while the rest of the components are described by 0-Dimensional model represented by differential-algebraic equations [10].

2.2 **Mathematical Modeling of the cardiovascular system**

2.2.1. The 0D Resistive-Compliant models:

Before get to the full explanation of the 0D model, some ideas must be discussed in advance. To make analogy with an electric system is not exclusive of cardiovascular models. That analogy is vastly used in many fields (FIRESTONE, 1933; HOLANDA, 2015) in order to simplify the system. This approach is usually called capacitance-resistance model (CRM).

In order to understand the parameters RLC (resistance, inductance and capacitance) of this model, it will be analyzed the circuit in Figure (2.1) that has a font, a resistance and a switch. When the switch is open, no current passes through the resistance, whereas when it is closed, there is a current in the system. Now, if this switch opens and closes the circuit regularly, that current would be on and off with time. But if a capacitor is added, like in Figure (2.2), when the switch is closed the capacitor charges and when the switch is open, the capacitor discharges maintaining the resistor under a current. Now, if once again the switch goes on and off regularly, the presence of the capacitor will ensure a non-null current until the next cycle (AMBROSI et al., 2012; BLANCO and FEIJÓO, 2011). In the cardiovascular model, when the switch is on, is the equivalent to systole; and when it is off, diastole. The electrical resistance is analogue to the vessel resistance that opposes to the flow. Finally, as the vessels are elastic, when the high pressure flow comes in, it dilates and stores potential elastic energy, releasing it back to the fluid as the pressure diminishes in order to maintain the fluid circulating. That is called compliant model, thus the name of this model based on this analogy: Resistive-Compliant model (AMBROSI et al., 2012; BLANCO and FEIJÓO, 2011)[9].

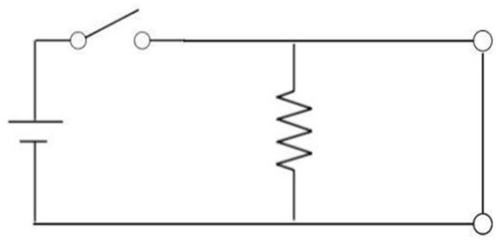

Fig.2.1: Resistive circuit with switch [9].

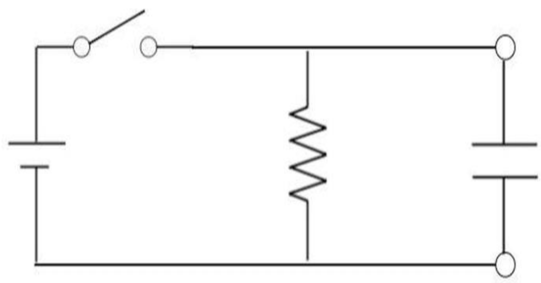

Fig.2.2: Capacitive-resistive circuit with switch [9].

	Capacitor Resistor circuit	Cardiovascular system
Driving force	Voltage difference, ΔE	Pressure difference, ΔP
Flow equation	Ohm's law, I = ΔE/R	Q = ΔP/R
Resistance, R	f_1(material property, Ac, L)	f_2(tissue property, Ac, L)
Capacitance, c	C = q/ΔE	C = ΔV/ΔP

Table 2.1: *CRM's underlying analogies*.

The study from WESTERHOF et al. (1969) is among the firsts to present the electric-circuit analogy applied to the cardiovascular system, and to use the CRM to an arterial tree. They considered Newton's law and continuity equations to model the flow in any length of the artery, as shown in Equations (2.1) and (2.2), respectively.

$$-\Delta P = L\frac{dQ}{dt} + RT \qquad (2.1)$$

$$\frac{dV}{dT} = C\frac{dP}{dT} \qquad (2.2)$$

where

Resistance, $R = 8\pi\mu/A^2$

Capacitance, $C = A/E\ (h/2R)$

Inductance, $L = \rho/A$

And:

μ is the blood viscosity;

ρ is the blood density;

E is the Young's modulus of the blood vessel;

A is a cross-sectional area of the segment of artery;

h is the wall thickness of the vessel.

WANG et al. (1989) also applied this analogy to their model to simulate disease conditions (the effect of stenosis in a few cases), using a coefficient α to correct the three inicial parameters (R, C and L) in order to represent a stenosed artery. The study results are representative, even in disease cases. It is important to point out that their study only approached the coronary arteries, and the RCL parameters where dependent of the length of the segment of artery (after integration) that difficult its estimation. Although further developments were made, the capacitance-resistance model is since the most used macroscopic model in the field of cardiovascular systems [9].

Here are some advantages and disadvantages of this model:

Advantages:

1. Simplicity: The 0D Resistive-Compliant model is relatively simple and computationally efficient. It represents blood vessels as simple resistive and compliant elements, making it easier to work with compared to more complex models.
2. Conceptual Understanding: This model provides a conceptual understanding of how blood flow and pressure interact within the cardiovascular system. It can help researchers and clinicians gain insights into the fundamental dynamics of blood circulation.

3. Parameter Tuning: The model allows for the tuning of resistive and compliant parameters to match observed clinical data. This flexibility makes it useful for customizing the model to specific patients or scenarios.
4. Real-Time Applications: Due to its simplicity and efficiency, the 0D Resistive-Compliant model is suitable for real-time simulations, making it valuable for clinical applications, such as patient monitoring and decision support.

Disadvantages:

1. Lack of Spatial Detail: Like other lumped parameter a model, the 0D Resistive-Compliant model lacks spatial detail. It treats entire vascular segments as single entities, which may not capture complex spatial variations accurately.
2. Limited Precision: The model's simplifications may lead to limited precision in certain applications, especially when high spatial and temporal resolution is required.
3. Assumptions Required: To use this model, certain assumptions about the relationships between resistance, compliance, and blood flow need to be made. These assumptions may not always accurately reflect the real physiological processes.
4. Limited Applicability: The 0D Resistive-Compliant model may not be suitable for in-depth research into highly detailed and specific cardiovascular phenomena, as it simplifies the complex interactions occurring in the vascular system [9].

2.2.2 Modeling Blood Flow In Human Heart:

We know that blood is carried from heart to various parts of the body and eventually returned to heart. In fact, blood is carried through system of elastic tubes-the arteries, capillaries and veins. The blood returns to the heart without actually leaving the system. This process is known as circulation of blood or flow of blood as discussed above.

We also know that proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the body in human beings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and are damages ranging from minor discomfort to death, in worst case. Therefore a better understanding of the physiology of the system is essential. Mathematical modeling of the system is aimed at this:

As a first step in modeling, we shall first identify the essential characteristics of blood flow.

We list them below:

I) Blood is a non-homogeneous fluid

II) Blood vessels are elastic, they branch repeatedly

III) Blood flow is unsteady or pulsatile

IV) Blood flow is generally laminar except for flow near heart

(a) Viscosity

Suppose a force is applied to a portion of a mass of a fluid it will begin to flow but if the force is removed the movement will be brought to rest. On the other hand, if a similar portion of a fluid is kept in moving, the movement will be transferred to the rest of the fluid. This property is analogous to that of friction between solid bodies.

Now we shall explain the concept of Viscosity of a fluid based on the following simple experiment. Consider the motion of a fluid between two long parallel plates one of which is rest and the other one is moving with a constant velocity U parallel to itself as shown in Figure-2.3

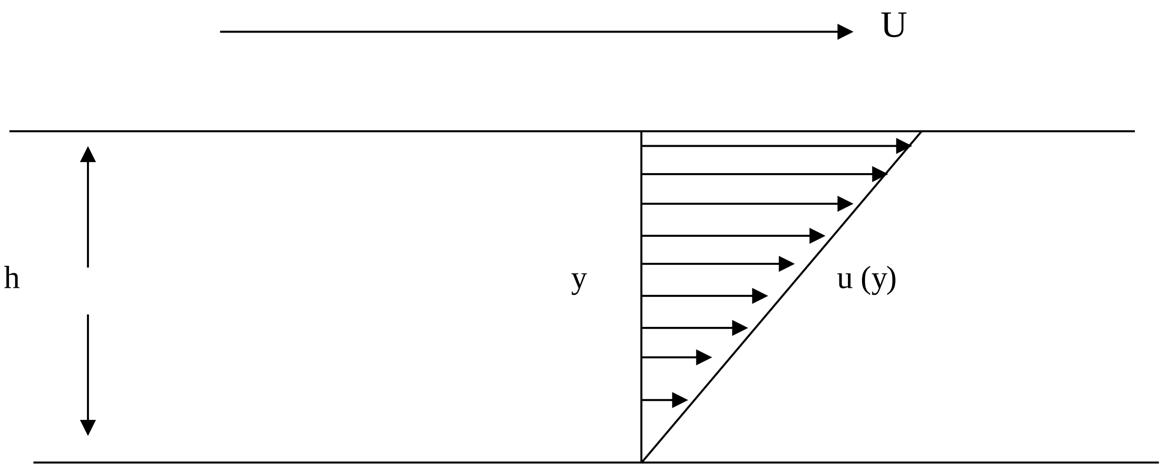

Fig.2.3: Motion of fluid between parallel plates [3].

Let the distance between the plates be h and the fluid velocity be u. Assume that the fluid pressure is constant throughout the fluid. Due to cohesive nature of fluid it adheres to the plates. The fluid velocity at the lower plate is zero and that at upper plate is U. This is because the upper plate is moving and the lower plate is at rest. So we get

$u = 0$, when $y = 0$

$u = U$, when $y = h$

Experimentally, it is observed that the fluid velocity distribution is linear and as such it is given by

$$u(y) = \frac{U}{h}y \tag{2.3}$$

Where y is the direction at right angles to the flow. In order to support the motion it is necessary to apply a tangential force to the upper plate. Experimentally it is observed that this force, taken per unit area, is proportional to the velocity U of the upper plate and inversely to the distance h. If **τ** denotes the force, then **τ** is directly proportional to U/h.

This is denoted by

$$\tau \propto \frac{U}{h} \tag{2.4}$$

Many researchers have studied this property; the first theoretical consideration was made by Newton in which he considered the motion imparted to a large volume of fluid by the rotation of a long cylinder suspended in it. The hypothesis on which he based his derivation was that the resistance which arises from the defect of slipperiness of the parts of the liquid, other things being equal is proportional to the velocity with which the parts of the liquid are separated from one another. 'Defect of slipperiness' was the term used to describe what we now call viscosity. This hypothesis emphasizes immediately that in a fluid moving relative to a surface there are laminae (plate or layer) slipping on one another and so moving at different velocities. There is thus a velocity gradient i.e., du/dy in this case in a direction perpendicular to the surface. This gradient is usually called the rate of shear. In modern terms, the velocity gradient is written as du/dy, where y is the distance from the axis. The resistance or force is denoted as **τ**.

Then by Newton's hypothesis

$$\tau = \mu \frac{du}{dy} \qquad (2.5)$$

Where μ is a constant. Note that when we differentiate the expression given in equation (2.3) and substitute for du/dy in (2.1), we get the expression given in (2.4). **μ** in (2.5) is called the proportionality constant which gives the measure of the viscosity of the fluid; **μ** is also called the coefficient of viscosity.

(b) Poiseuille's Law:

Poiseuille's law is the relation between flow rate and pressure gradient-for fluid flow in a rigid cylindrical tube under a pressure gradients. (Note that the pressure gradient is the pressure drop per unit length$\frac{dp}{dz} = \lim_{\Delta z \to 0} \frac{\Delta p}{\Delta z}$)

In order that we can understand the flow properties of biological fluids such as blood which my exhibit non-Newtonian properties it is first necessary to discuss the behavior of simple or Newtonian fluids.

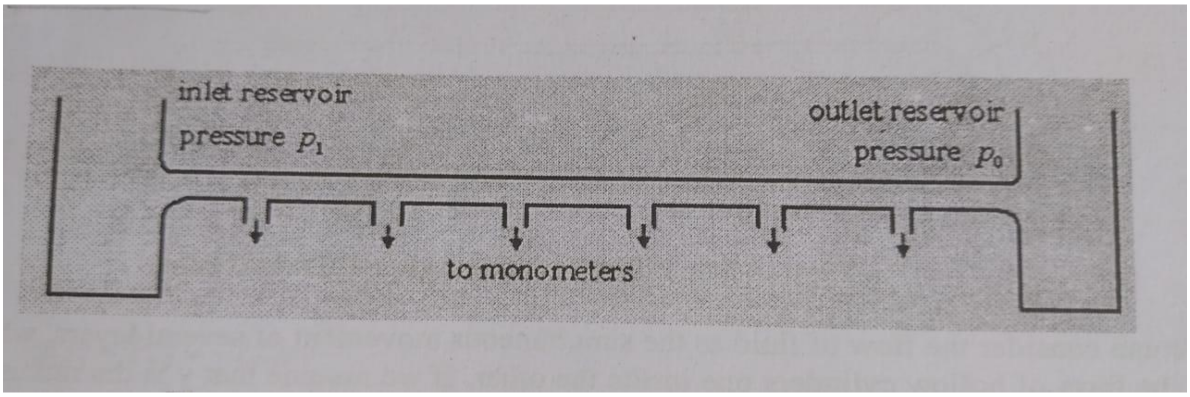

Fig. 2.4: Flow properties of simple fluid [3].

Let us look at the flow properties of a simple liquid like water in a very long horizontal pipe. Imagine that the pipe is circular in cross-section and d units in diameter as shown in Figure-2.4.

Its entrance and exit are connected to large reservoirs so that the pressure drops between the ends of the tube may be maintained constant and a steady flow of water through the pipe is achieved. Small side hole, or lateral, pressure tappings are made in the pipe at frequent intervals along its length and these tappings are connected to a series of manometers. It is thus possible to measure the pressure drop per unit length or pressure gradient along the pipe.

If the pressure at the inlet to the pipe is p_1 and that at the outlet P_0, then we shall observe that, as P_1-P_0 (or ΔP) is increased by raising the level in the upstream reservoir, so is the flow rate V through the pipe.

It was Poiseuille in 1840, who as a first step towards understanding the mechanics of the circulation, published a quantitative study of the flow properties in a pipe very remote from the entrance, and flow conditions in this region are now named after him. In addition to varying the flow rate and tube size, Poiseuille also studied the effect of viscosity on the flow conditions. Here we found that as viscosity was increased so was the pressure gradient necessary to maintain a given flow-rate.

Now to derive Poiseuille's formula we make use of Newton's second law of motion, which says that

$$Mass \times acceleration =$$

$$body\ force + pressure\ gradient\ force + viscous\ force \qquad (2.6)$$

Let us consider the fluid flow through a circular tube with length L and diameter D = 2R, which is small compared with the length. We assume that the rate of flow is constant i.e., flow is steady. We also assume that the fluid velocity everywhere inside the tube is laminar/stream lined. As we know for a laminar flow, the velocity is purely in the direction parallel to the axis of the cylinder. The fluid velocity at the inner surface is zero and it reaches maximum value on the axis (here axis means axis of the cylinder.)

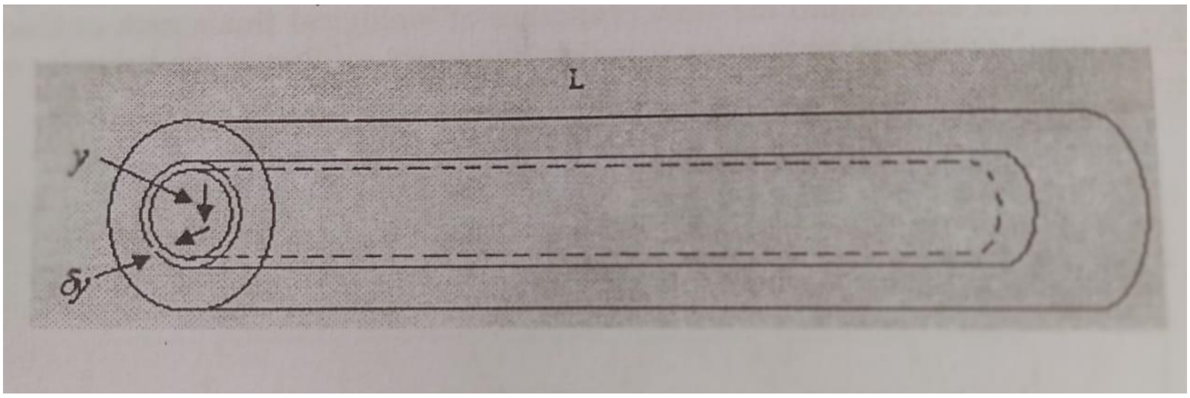

Fig.2.5: Fluid flow through cylindrical tube [3].

We can consider the flow of fluid as the simultaneous movement of several layers, which in the form of hollow cylinders one inside the other. If we assume that y is the radius of one of these cylinders, then y varies from 0 to R,

i.e., 0<y<R as shown in Figure-2.5

If we consider the fluid flow to be due to pressure differences at the ends of the tube from the higher to the lower one then the only force opposing this flow is viscous resistance. We know that this force is μ $\left(\frac{du}{dy}\right)$, and we find that the fluid particles are accelerated by the pressure difference and retarded by viscous resistance. If we look at equation (2.5), then we will find that the only forces present are pressure gradient force and viscous force. This is because, since the flow is a steady flow in a straight tube, the fluid is not subjected to any acceleration (i.e., when the flow is study, things do not change with respect to time). Therefore, LHS of equation (2.6) is zero also, since we are considering the flow in horizontal pipes, gravitational forces are not relevant and therefore the body force term also vanishes. Thus, equation (2.5) reduces to Pressure gradient force = - Viscous force. Now if F_{visc} denote the viscous force and $F(P)$ denote the pressure difference, then we have

$$F(P) = -F_{visc} \qquad (2.7)$$

(The negative sign indicates one force accelerates the motion, the other retards.)

Now we will calculate the LHS and RHS of equation (2.7), we first consider the RHS of on (2.7). Here note that each flow is in the form of cylindrical layer of length L and radius y, y varying from 0 to R. The viscous force acts on the surface and it is given by the slowing formula:

$$F_{visc} = Surface\ area\ of\ the\ cylinder \times Viscosity \times the\ velocity\ gradient \quad (2.8)$$

We have denoted the viscosity as **μ** and we know that the velocity gradient is given by du/dy. Therefore, we can write equation (2.8) as

$$F_{visc} = 2\pi y L \left(\mu \frac{du}{dy}\right) \quad (2.9)$$

Next we shall find the pressure difference.

Note that the force exerted by the pressure at an end of the cylinder is pressure at that end multiplied by the cross sectional area. Now if P_1 and P_2 respectively denote the pressures at her end of length L of the cylinder considered, then the required pressure difference is

$$F(P) = \pi y^2 (P_1 - P_2) \quad (2.10)$$

Substituting for $F(P)$ and F_{visc} in equation (2.7), we get

$$\pi y^2 (P_1 - P_2) = -2\pi y L \left(\mu \frac{du}{dy}\right)$$

$$y(P_1 - P_2) = -2L\mu \frac{du}{dy}$$

This gives the velocity gradient $\frac{du}{dy}$ as

$$\frac{du}{dy} = -y\left(\frac{P_1 - P_2}{2L\mu}\right) \quad (2.11)$$

the negative sign here implies u decreases when y increases. Also, note $P_1 > P_2$).

Now substituting this value of the velocity gradient in equation (2.5), we get the shear stress

$$\tau = \mu \frac{du}{dy}$$

$$= \mu \times (-y) \times \left(\frac{P_1 - P_2}{2L\mu}\right)$$

$$= -y\left(\frac{P_1 - P_2}{2L\mu}\right) \qquad (2.12)$$

Now if we consider the wall of the tube, then the radius y of the wall is R, therefore from equation (2.12) we get

$$Shear\ stress\ at\ the\ wall\ of\ the\ tube = -R\,\frac{(P_1 - P_2)}{2L} \qquad (2.13)$$

Thus, we have derived the equation describing the flow of fluid in a thin tube of length L, with pressures P_1 and P_2 at the ends.

Now we have to solve equation (2.9) to get the velocity u. Let us consider equation (2.9), since this equation is a first order linear ordinary differential equation. To find the solution we integrate on both sides of equation (2.9) and we velocity as

$$u(y) = -\frac{(P_1 - P_2)}{4\mu L} y^2 \pm C \qquad (2.14)$$

Where C is the constant of integration which is to be evaluated. To evaluate C, it is necessary to prescribe the boundary conditions. Here, we make use of the assumption made by New that the fluid in contact with the wall is at rest,

u = 0, when y = R

i.e., substituting the condition in equation (2.14) we get

$$C = -\frac{(P_1 - P_2)}{4\mu L} R^2$$

So that equation (1.14) reduces to

$$u(y) = -y^2 \frac{(P_1 - P_2)}{4\mu L} + R^2 \frac{(P_1 - P_2)}{4\mu L}$$

$$= (R^2 - y^2) \frac{(P_1 - P_2)}{4\mu L} \qquad (2.15)$$

Where u is the velocity component parallel to the axis, R is the radius of the cylinder, L is the length of the tube, μ is the viscosity of the fluid and P_1-P_2, the pressure drop.

Therefore, equation (2.15) describes the velocity of the fluid in a steady laminar flow. Now, let us see what equation (2.15) represents geometrically. Since equation (2.15) equation of a parabola where u = 0 when y = R and u is maximum when y = 0 i.e. at the axis of the tube as shown in Figure-2.6.

Our boundary conditions say that the velocity is zero at the wall. If the principle of conservation of mass is to hold well, the same amount of fluid should come out of every cross-section. The loss of velocity at the wall has to be compensated by maximum velocity at the centre.

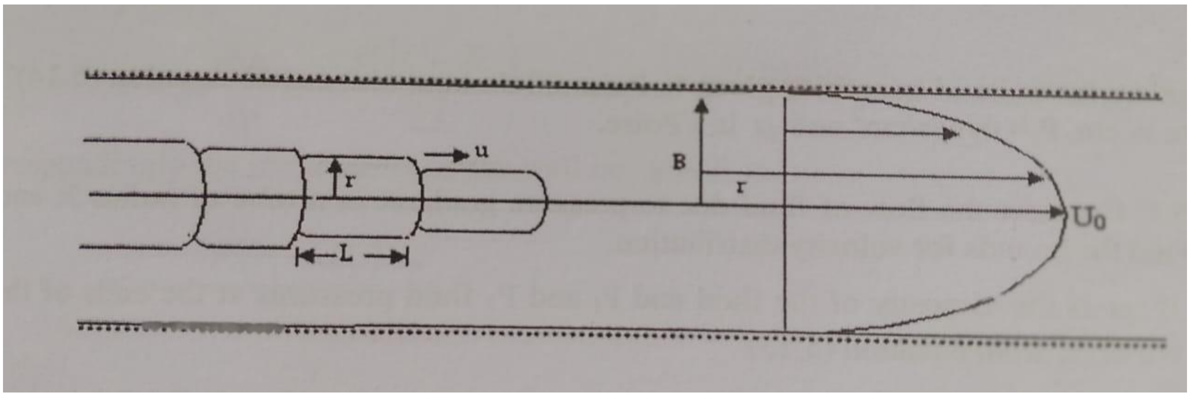

Fig. 1.10: The parabola shows the velocity profile in a steady laminar flow [3].

Thus, we find that the velocity distribution in a tube, with given pressure gradient is parabolic.

So, we have got an equation, which gives velocity distribution in a tube. Let us now find the volume of fluid, flowing through a section of the tube per unit time. Here we shall see how we will use equation (2.15) along its axis. That is, we have to determine the volume of the solid of revolution of parabola.

The required volume V of parabola of revolution

Then

$$V = \int_0^{2\pi} \int_0^{R} u(y)\, y dy d\theta$$

Now substituting for u in the above integral, we get

$$V = \frac{2\pi(P_1 - P_2)}{4L\mu} \int_0^{R} (R^2 - y^2)\, y dy$$

$$= \frac{(P_1 - P_2)\pi R^4}{8L\mu} \qquad (2.16)$$

Equation (2.16) is called Poiseuille's law and it says that the volume is proportional to the first power of the pressure drop per unit length, $(P_1 - P_2)/L$ and to the fourth power of the radius of the pipe R^4 and it is inversely proportional to the length of the tube as well as the viscosity of the fluid. This equation is a general solution for any problem of fluid flow trough cylindrical pipes, provided that the fluid flow satisfies all the assumptions, which are assumed in obtaining equation (2.16). The assumptions made are:

1) The fluid is homogeneous and its viscosity is the same at all rates of shear.

2) The fluid does not slip at the wall of the tube. This was the assumption that $u = 0$ when $y = R$ which was made in evaluating the constant of integration in equation (2.11).

3) The flow is laminar, i.e. the fluid is flowing parallel to the axis of inner surface wall of the tube.

4) The rate of flow is steady.

5) The tube is along with length much greater than the diameter of the tube.

Note: what are the units of quantities given in Poiseuille's equations (2.16)

R and L are in cm, $P = dynes/cm^2$ and μ is a Poise.

In the previous section we have used Poiseuille's law, which gives a relation between the rate of flow and the pressure difference existing while a fluid flows in a rigid tube of circular cross section.

Formulation: In above we have used Poiseuille's law, which gives a relation between the rate of flow and the pressure difference existing while a fluid flows in a rigid tube of circular cross section.

Let us now formulate a simple mathematical model for blood flow in arteries. Since the real situation is quite complex and including all the essential characteristics will make the model very complicated, let us make certain assumptions:

I) Blood is a homogeneous fluid
II) The flow is steady and laminar
III) The tube is rigid, long and straight

With these assumptions, the formulation leads to the steady flow of blood in a long rigid blood vessel for which Poiseuille's law is applicable. Thus, the velocity of blood in configuration corresponding to this simple model is

$$u(y)=\frac{(P_1-P_2)}{4\mu L}(R^2-y^2);\ 0 \leq y \leq R \qquad (2.17)$$

And the rate of flow is

$$V=\frac{\pi R^4(P_1-P_2)}{8\mu L} \qquad (2.18)$$

We can calculate the shear stress here by using the formula

$$\tau = \mu \frac{du}{dy} = -\frac{y(P_1\text{-}P_2)}{2L} \qquad (2.19)$$

Correspondingly the shear stress on the wall i.e. $y = R$ is

$$\tau = \frac{-R(P_1 - P_2)}{2L}$$

Case Study 1: For any given flow of fluid due to pressure gradient in a tube of radius R and length L,we have evaluated the bounds for velocity distribution by the formula

$$u(y) = \frac{(P_1 - P_2)}{4\mu L}(R^2 - y^2);\ 0 \le y \le R$$

Where μ is the viscosity of the fluid and $P = P_1 - P_2$ is fluid pressure at the ends of the tube

$$u(y) = \frac{P}{4\mu L}(R^2 - y^2), \qquad where\ P = P_1 - P_2$$

At $y = 0\ , u(0) = \frac{PR^2}{4\mu L}$ $and\ \ at\ y = R, u(R) = 0$

Therefore,

$$0 \le u(y) \le \frac{PR^2}{4\mu L}$$

Case study 2: For a blood vessel of constant radius R, length L and driving force $P = P_1 - P_2$. Show that the average velocity of flow is equal to half the maximum velocity and resistance

i.e., $\frac{(P_1 - P_2)}{V}$ is proportional to $\frac{L}{R^4}$.

Solution: We have,

$$\text{u(y)} = \frac{(P_1 - P_2)}{4\mu L}(R^2 - y^2)$$

$$= \frac{P}{4\mu L}(R^2 - y^2), 0 \leq \text{ y } \leq \text{ R}$$

$$where\ P = P_1 - P_2$$

Also we know that

$$\text{V} = \frac{\pi R^4}{8\mu L}(P_1 - P_2)$$

Average velocity of the blood in a vessel

$$= \frac{V}{\pi R^2} = \frac{R^2}{8\pi L}P = K\frac{R^2}{2} \qquad (2.20)$$

Also, maximum velocity

$$u_m = u|_{y=0} = \frac{P_1 - P_2}{4\mu L}R^2 = KR^2 \qquad (2.21)$$

From eq. (2.20) and (2.21), we get

Average velocity of the blood in a vessel $=\frac{1}{2}$ max. velocity

Resistance to the flow = $\frac{P_1 - P_2}{V}$

$\frac{8\mu L}{\pi R^4} = C\frac{L}{R^4},\ \ where\ C = \frac{8\mu}{\pi}$ is constant

This shows that the resistance to the flow is directly proportional to $\frac{L}{R^4}$.

2.3 Conclusion

The mathematical model of the human heart proposed in the dissertation can provide valuable insights into the complex dynamics of this vital organ. Through the simulations, we have been able to better understand the behavior of the heart's chambers, valves, and the flow of blood within them. The model has highlighted the importance of factors such as heart rate, contractility, and valve function in maintaining a healthy cardiac cycle. Furthermore, it will demonstrate the potential utility for predicting and analyzing various cardiac conditions and interventions, paving the way for future research and clinical applications (Pacemaker of Heart) in cardiology. Overall, the work underscores the significance of mathematical modeling in advancing the understanding of the human heart and its role in cardiovascular health.

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