RACSAM (2021) 115:124
https://doi.org/10.1007/s13398-021-01061-z

ORIGINAL PAPER

Bernstein and Turán-type inequalities for a polynomial with constraints on its zeros

Abdullah Mir[1] · **Daniel Breaz**[2]

Received: 8 December 2020 / Accepted: 8 May 2021

Abstract
This paper considers the well known Bernstein and Turán-type inequalities that relate the uniform norm of a polynomial to that of its derivative on the unit circle in the plane. Here, we establish some new inequalities that relate the uniform norm of the polynomial derivative and the polynomial, in case when the zeros are inside or outside some closed disk. The obtained results produce various inequalities that are sharper than the previous ones known in very rich literature on this subject.

Keywords Polynomial · Polar derivative · Rouché's theorem · Zeros

Mathematics Subject Classification Primary 30A10 · 30C10; Secondary 30D15

1 Introduction

Let $P(z) := \sum_{j=0}^{n} c_j z^j$ be an algebraic polynomial of degree n in the complex plane and $P'(z)$ is its derivative. The study of extremal problems of functions of complex variables and the results where some approaches to obtaining the polynomial inequalities are developed on using various methods of the geometric function theory is a fertile area in analysis for researchers. A classical result due to Bernstein [3] that relates an estimate of the size of the derivative and the polynomial for the uniform norm on the unit circle states that, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Inequality (1.1) was proved by Bernstein in 1912, and later, in 1930 (see [4]), Bernstein revisited this inequality and proved that, for two polynomial $P(z)$ and $Q(z)$ with degree of

✉ Daniel Breaz
dbreaz@uab.ro

Abdullah Mir
drabmir@yahoo.com

1 Department of Mathematics, University of Kashmir, Srinagar 190006, India

2 Department of Mathematics and Informatics, "1 Decembrie 1918" University of Alba Iulia Alba, G. Bethlen, Str. No. 5-9, Iulia Alba, Romania

Published online: 24 May 2021

$P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leq |Q(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leq |Q'(z)|$ on $|z| = 1$. In particular, for $Q(z) = z^n \max_{|z|=1} |P(z)|$, this classical result allows one to establish (1.1). It is important to mention that these inequalities of Bernstein have seen the starting point of a considerable literature in polynomial approximation, and over a period, these inequalities were generalized and extended in several directions, in different norms and for different classes of functions. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then (1.1) can be sharpened. This fact was observed by Erdös and later verified by Lax [13] by proving that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas, if the polynomial $P(z)$ has all its zeros in $|z| \leq 1$, then Turán [24] proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

As an extension of (1.2), Malik [14] proved that if $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.4}$$

whereas, if the polynomial $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, it was shown by Govil [7] that (1.3) can be replaced by

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.5}$$

For the class of polynomials not vanishing in $|z| < k,\ k \leq 1$, the precise estimate of maximum $|P'(z)|$ on $|z| = 1$ is not easily obtainable. For quite some time it was believed that if $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, then the inequality analogous to (1.4) should be

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{1.6}$$

till Professor Saff gave the example $P(z) = (z - \frac{1}{2})(z + \frac{1}{3})$ to counter this belief. Finally, in 1980, it was shown by Govil [6] that (1.6) holds with an additional hypothesis and proved the following result.

Theorem A *Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is best possible and equality in (1.7) holds for $P(z) = z^n + k^n$.

In 1997, Aziz and Ahmad [2] improved the bound in (1.7) and proved the following result.

Theorem B *Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\}. \tag{1.8}$$

The result is best possible and equality in (1.8) holds for $P(z) = z^n + k^n$.

For a polynomial $P(z)$ of degree n, we define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point α. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} = P'(z),$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

Aziz [1] was among the first to extend some of the above inequalities by replacing the derivative with the polar derivative of polynomial. In fact, in 1988, Aziz proved that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \le n\left(\frac{|\alpha| + k}{1+k}\right)\max_{|z|=1}|P(z)|. \tag{1.9}$$

If we divide both sides of (1.9) by $|\alpha|$ and make $|\alpha| \to \infty$, we get (1.4). For more information on polar derivative of polynomial, one can consult the books of Milovanović et al. [16], Marden [15] and Rahman and Schmeisser [23]. Recently, Kumar [12] had used the new version of the Schwarz lemma and proved a polar derivative inequality for the class of polynomials $P(z)$ having all zeros in $|z| \le k,\ k \ge 1$, that involve the modulus of each zero of the underlying polynomial. More precisely, Kumar proved that if $P(z) = \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge k$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| &\ge 2\left(\frac{|\alpha| - k}{1+k^n}\right)\left[1 + \frac{(|a_n|k^n - |a_0|)(k-1)}{2(|a_n|k^n + k|a_0|)}\right] \\ &\times \sum_{j=1}^{n}\frac{k}{k+|z_j|}\max_{|z|=1}|P(z)|. \end{aligned}\tag{1.10}$$

Since $k \ge 1$, therefore $\frac{k}{k+|z_j|} \ge \frac{1}{2}$ for $1 \le j \le n$, the above inequality (1.10) gives, in particular, the following result.

Theorem C *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for every complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n\left(\frac{|\alpha| - k}{1+k^n}\right)\left[1 + \frac{(|a_n|k^n - |a_0|)(k-1)}{2(|a_n|k^n + k|a_0|)}\right]\max_{|z|=1}|P(z)|. \tag{1.11}$$

If we divide both sides of (1.11) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following refinement of (1.5):

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n}\left[1 + \frac{(|a_n|k^n - |a_0|)(k-1)}{2(|a_n|k^n + k|a_0|)}\right]\max_{|z|=1}|P(z)|. \tag{1.12}$$

For the latest research and development in this direction, one can see some of the papers [8,12,17–22] and also a recently published book chapter by Govil and Kumar [9]. The present paper is mainly motivated by the desire to establish some improved bounds for the polar derivative of a polynomial by using some of its coefficients. The obtained results besides yield polar derivative generalizations of (1.7) and (1.8), also produce refinements of (1.11), (1.12) and related inequalities.

2 Main results

We begin by presenting the following Bernstein-type inequality for the polar derivative of a polynomial. The obtained result not only yields the polar derivative generalization of (1.7), but also provides a sharpening of it .

Theorem 1 *Let $P(z)=\sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having no zeros in $|z|<k,\ k\leq 1$, and let $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for every complex number α with $|\alpha|\geq 1$,*

$$\max_{|z|=1}|D_\alpha P(z)|\leq n\left(\frac{|\alpha|+k^{n-1}S_0(k)}{1+k^{n-1}S_0(k)}\right)\max_{|z|=1}|P(z)|, \tag{2.1}$$

where

$$S_0(k)=\frac{|c_n|k^{n+1}+|c_0|}{|c_n|k^{n-1}+|c_0|}.$$

The result is best possible and equality in (2.1) holds for $P(z)=z^n+k^n$, with real $\alpha\geq 1$.

Remark 1 Since $P(z)=\sum_{j=0}^{n} c_j z^j\neq 0$ in $|z|<k,\ k\leq 1$, and if $z_1, z_2, \ldots z_n$, are the zeros of $P(z)$ then

$$\left|\frac{c_0}{c_n}\right|=|z_1z_2\ldots z_n|\geq k^n. \tag{2.2}$$

Using this, one can easily check that

$$S_0(k)=\frac{|c_n|k^{n+1}+|c_0|}{|c_n|k^{n-1}+|c_0|}\geq k.$$

Also, for every $\alpha\in\mathbb{C}$, the function

$$x\mapsto\frac{|\alpha|+x}{1+x},\ \ (x\geq 0)$$

is decreasing for $|\alpha|\geq 1$, it follows that

$$\frac{|\alpha|+k^{n-1}S_0(k)}{1+k^{n-1}S_0(k)}\leq\frac{|\alpha|+k^n}{1+k^n}.$$

Hence, from Theorem 1, we get the following polar derivative generalization of Theorem A.

Corollary 1 *Let $P(z)=\sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having no zeros in $|z|<k,\ k\leq 1$, and let $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for every complex number α with $|\alpha|\geq 1$,*

$$\max_{|z|=1}|D_\alpha P(z)|\leq n\left(\frac{|\alpha|+k^n}{1+k^n}\right)\max_{|z|=1}|P(z)|. \tag{2.3}$$

The result is best possible and equality in (2.3) holds for $P(z)=z^n+k^n$, with real $\alpha\geq 1$.

Remark 2 Clearly (2.3) is a generalization of (1.7) and to obtain (1.7) from the above corollary, simply divide both sides of (2.3) by $|\alpha|$ and let $|\alpha|\to\infty$.

Remark 3 Dividing both sides of (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get under the same hypothesis as in Theorem 1, that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n-1} S_0(k)} \max_{|z|=1} |P(z)|. \tag{2.4}$$

Equality in (2.4) holds for $P(z) = z^n + k^n$.

In fact, excepting the case when all the zeros of $P(z)$ lie on $|z| = k$, the bound obtained in (2.4) is always sharper than the bound obtained in Theorem A as $S_0(k) \geq k$. Next, we shall prove the following refinement of Theorem 1, which in turn sharpen the bounds in Theorems A and B.

Theorem 2 *Let $P(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{1 + k^n S_m(k)} \left(\left(|\alpha| + k^n S_m(k)\right) \max_{|z|=1} |P(z)| - m(|\alpha| - 1) S_m(k) \right), \tag{2.5}$$

where

$$S_m(k) = \frac{|c_0| + |c_n| k^{n+1} + km}{|c_0| k + |c_n| k^n + m} \quad \text{and} \quad m = \min_{|z|=k} |P(z)|.$$

The result is best possible and equality in (2.5) holds for $P(z) = z^n + k^n$, with real $\alpha \geq 1$.

Remark 4 Recall that $P(z) = \sum_{j=0}^{n} c_j z^j \neq 0$ in $|z| < k,\ k \leq 1$. We first show that

$$k^n |c_n| \leq |c_0| - m. \tag{2.6}$$

We can assume without loss of generality that $P(z)$ has no zeros on $|z| = k$, for otherwise (2.6) holds trivially by (2.2). Now, as in the proof of Lemma 3 (given in next section), we have for every λ with $|\lambda| < 1$, the polynomial

$$P(z) + \frac{\lambda m z^n}{k^n} = \left(c_n + \frac{\lambda m}{k^n}\right) z^n + \sum_{j=0}^{n-1} c_j z^j$$

does not vanish in $|z| < k,\ k \leq 1$, hence

$$\left| \frac{c_0}{c_n + \frac{\lambda m}{k^n}} \right| \geq k^n. \tag{2.7}$$

If in (2.7), we choose the argument of λ suitably, so that

$$\left| c_n + \frac{\lambda m}{k^n} \right| = |c_n| + \frac{|\lambda| m}{k^n},$$

we get

$$k^n |c_n| + |\lambda| m \leq |c_0|. \tag{2.8}$$

The inequality (2.6) follows by letting $|\lambda| \to 1$ in (2.8).

By using the inequality (2.6), one can see that $S_m(k) \geq 1$ for every $k \leq 1$. Also, it is easy to verify that the function

$$x \mapsto \left(\frac{|\alpha| + k^n x}{1 + k^n x}\right) \max_{|z|=1} |P(z)| - \left(\frac{x(|\alpha| - 1)}{1 + k^n x}\right) \min_{|z|=k} |P(z)|, \ \ (x \geq 0)$$

is decreasing for $|\alpha| \geq 1$, we get from Theorem 2, the following polar derivative analogue of Theorem B.

Corollary 2 *Let $P(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{1 + k^n} \left(\left(|\alpha| + k^n\right) \max_{|z|=1} |P(z)| - \left(|\alpha| - 1\right) \min_{|z|=k} |P(z)| \right). \tag{2.9}$$

The result is best possible and equality in (2.9) holds for $P(z) = z^n + k^n$, with real $\alpha \geq 1$.

Remark 5 Dividing both sides of (2.9) by $|\alpha|$ and let $|\alpha| \to \infty$, we get (1.8).

Remark 6 Dividing both sides of (2.5) by $|\alpha|$ and let $|\alpha| \to \infty$, we get under the same hypothesis as in Theorem 2, the following refinement of Theorem B.

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^n S_m(k)} \left(\max_{|z|=1} |P(z)| - m S_m(k) \right), \tag{2.10}$$

where m and $S_m(k)$ are as defined in Theorem 2.
Equality in (2.10) holds for $P(z) = z^n + k^n$.
We shall also prove the following Turán-type inequality for a polynomial that sharpens Theorem C.

Theorem 3 *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every complex number α with $|\alpha| \geq k$, we have*

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k}{1 + k^n}\right) \left(1 + \frac{|a_n| k^n - |a_0|}{n(|a_n| k^n + |a_0|)}\right) \\ \times \left(1 + \frac{(|a_n| k^n - |a_0|)(k - 1)}{2(|a_n| k^n + k |a_0|)}\right) \max_{|z|=1} |P(z)|. \end{aligned} \tag{2.11}$$

Remark 7 If we divide both sides of (2.11) by $|\alpha|$ and let $|\alpha| \to \infty$, we get

$$\begin{aligned} \max_{|z|=1} |P'(z)| \geq \left(\frac{n}{1 + k^n}\right) \left(1 + \frac{|a_n| k^n - |a_0|}{n(|a_n| k^n + |a_0|)}\right) \\ \times \left(1 + \frac{(|a_n| k^n - |a_0|)(k - 1)}{2(|a_n| k^n + k |a_0|)}\right) \max_{|z|=1} |P(z)|, \end{aligned}$$

thereby, sharpening the inequality (1.12).

Now, we compare the bounds of the obtained results by means of the following examples.

Example 1 Consider the polynomial $P(z) = z^3 - z^2 + z - 1$. Clearly $P(z)$ has all its zeros $\{1, i, -i\}$ which all lie on $|z| = 1$. Further

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = -P(z),$$

so that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. We take $k = \frac{1}{2}$, so that $P(z) \neq 0$ in $|z| < k = \frac{1}{2}$. We find numerically that $\max_{|z|=1} |P(z)| = 4$, $\min_{|z|=\frac{1}{2}} |P(z)| = \frac{5}{8}$. Taking $\alpha = \frac{3+i\sqrt{7}}{2}$, so that $|\alpha| = 2$, we obtain the following estimates:

$$\max_{|z|=1} |D_\alpha P(z)| \leq 21.89 \quad \text{(by Theorem 1)},$$
$$\max_{|z|=1} |D_\alpha P(z)| \leq 22.66 \quad \text{(by (2.3))},$$
$$\max_{|z|=1} |D_\alpha P(z)| \leq 20.73 \quad \text{(by Theorem 2)},$$
$$\max_{|z|=1} |D_\alpha P(z)| \leq 21.0 \quad \text{(by (2.9))}.$$

For the same polynomial, we can compare the obtained bounds with Theorems A and B as follows:

$$\max_{|z|=1} |P'(z)| \leq 10.7 \quad \text{(by Theorem } A\text{)},$$
$$\max_{|z|=1} |P'(z)| \leq 9.0 \quad \text{(by Theorem } B\text{)},$$
$$\max_{|z|=1} |P'(z)| \leq 9.89 \quad \text{(by (2.4))},$$
$$\max_{|z|=1} |P'(z)| \leq 8.7 \quad \text{(by (2.10))}.$$

In the same way, Theorem 3 in general provides much better information than Theorem C regrading the maximum of $|D_\alpha P(z)|$ on $|z| = 1$. We illustrate this with the help of following example.

Example 2 Consider $P(z) = z^2 + 3z + \frac{5}{4}$. Clearly, $P(z)$ is a polynomial of degree 2 having all its zeros in $|z| \leq \frac{5}{2}$. We take $k = 3$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 4$. We find that $\max_{|z|=1} |P(z)| = \frac{21}{4}$. Using Theorem C for this polynomial with $k = 3$, we see that

$$\max_{|z|=1} |D_\alpha P(z)| \geq 1.69,$$

while as Theorem 3, gives

$$\max_{|z|=1} |D_\alpha P(z)| \geq 2.33,$$

which is much better than the bound given by (1.11).

3 Auxiliary results

In order to prove our main results, we need the following lemmas.

Lemma 1 *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$ and $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then on $|z| = 1$,*

$$|Q'(z)| \leq k^{n+1} \left(\frac{|c_n| k^{n-1} + |c_0|}{|c_n| k^{n+1} + |c_0|} \right) \max_{|z|=1} |P'(z)|.$$

The above lemma is due to Jain ([11], inequality (3.13)).

Lemma 2 *If $P(z)=\sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having no zeros in $|z|<k,\ k\le 1$ and, let $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$, then on $|z|=1$,*

$$k^{n-1}S_0(k)|P'(z)|\le \max_{|z|=1}|Q'(z)|, \tag{3.1}$$

where $S_0(k)$ is as defined in Theorem 1.

Proof Since $P(z)=\sum_{j=0}^{n} c_j z^j\neq 0$ in $|z|<k,\ k\le 1$, it follows that all the zeros of the polynomial $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$ of degree n lie in $|z|\le\frac{1}{k},\ \frac{1}{k}\ge 1$. Also $P(z)=z^n\overline{Q\left(\frac{1}{\bar z}\right)}$, we get on applying Lemma 1 to $Q(z)$ for $|z|=1$,

$$\begin{aligned}|P'(z)| &\le \frac{1}{k^{n+1}}\left(\frac{|c_0|\frac{1}{k^{n-1}}+|c_n|}{|c_0|\frac{1}{k^{n+1}}+|c_n|}\right)\max_{|z|=1}|Q'(z)|\\ &=\frac{1}{k^{n-1}}\left(\frac{|c_0|+|c_n|k^{n-1}}{|c_0|+|c_n|k^{n+1}}\right)\max_{|z|=1}|Q'(z)|,\end{aligned}$$

which is equivalent to (3.1) and this proves Lemma 2 completely.

Lemma 3 *If $P(z)=\sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having no zeros in $|z|<k,\ k\le 1$ and, let $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$, then on $|z|=1$,*

$$k^n S_m(k)\left(|P'(z)|+\frac{mn}{k^n}\right)\le \max_{|z|=1}|Q'(z)|,$$

where $S_m(k)$ and m are as defined in Theorem 2.

Proof Since $m\le |P(z)|$ for $|z|=k$. Observe first that if $P(z)$ has some zeros on $|z|=k$, then $m=\min_{|z|=k}|P(z)|=0$ and in this case the result follows from Lemma 2. So, henceforth, we assume that all the zeros of $P(z)$ lie in $|z|>k,\ k\le 1$. Clearly $m>0$, and we have

$$\left|\lambda m\frac{z^n}{k^n}\right|<|P(z)| \text{ on } |z|=k \text{ for any } \lambda \text{ with } |\lambda|<1.$$

By Rouché's theorem, it follows that the polynomial $F(z)=P(z)+\lambda m\frac{z^n}{k^n}$ has no zeros in $|z|<k,\ k\le 1$. Let $G(z)=z^n\overline{F\left(\frac{1}{\bar z}\right)}=z^n\overline{P\left(\frac{1}{\bar z}\right)}+\bar\lambda\frac{m}{k^n}=Q(z)+\bar\lambda\frac{m}{k^n}$.
Applying Lemma 2 to $F(z)$, we get for any λ with $|\lambda|<1$, and $|z|=1$,

$$k^{n-1}\left(\frac{|c_0|+|c_n+\frac{\lambda m}{k^n}|k^{n+1}}{|c_0|+|c_n+\frac{\lambda m}{k^n}|k^{n-1}}\right)|F'(z)|\le\max_{|z|=1}|G'(z)|,$$

implying that

$$k^{n-1}\left(\frac{|c_0|+|c_n+\frac{\lambda m}{k^n}|k^{n+1}}{|c_0|+|c_n+\frac{\lambda m}{k^n}|k^{n-1}}\right)\left|P'(z)+\frac{\lambda mnz^{n-1}}{k^n}\right|\le\max_{|z|=1}|Q'(z)|, \tag{3.2}$$

for $|z|=1$ and $|\lambda|<1$.
For every $\lambda\in\mathbb{C}$, we have

$$\left|c_n+\frac{\lambda m}{k^n}\right|\le|c_n|+\frac{|\lambda|m}{k^n},$$

and since the function

$$x \mapsto \frac{|c_0| + k^{n+1}x}{|c_0| + k^{n-1}x}, \ (x \geq 0)$$

is non-increasing for $k \leq 1$, it follows from (3.2) that for every $|\lambda| < 1$ and $|z| = 1$,

$$k^{n-1}\left(\frac{|c_0| + \left(|c_n| + \frac{|\lambda|m}{k^n}\right)k^{n+1}}{|c_0| + \left(|c_n| + \frac{|\lambda|m}{k^n}\right)k^{n-1}}\right)\left|P'(z) + \frac{\lambda m n z^{n-1}}{k^n}\right| \leq \max_{|z|=1}|Q'(z)|. \tag{3.3}$$

Choosing the argument of λ on the left hand side of (3.3) suitably, we get for $|z| = 1$,

$$k^n\left(\frac{|c_0| + |c_n|k^{n+1} + |\lambda|mk}{|c_0|k + |c_n|k^n + |\lambda|m}\right)\left(|P'(z)| + \frac{|\lambda|mn}{k^n}\right) \leq \max_{|z|=1}|Q'(z)|. \tag{3.4}$$

Finally, letting $|\lambda| \to 1$ in (3.4), we get the required assertion. Hence Lemma 3 is proved.

Lemma 4 *If $P(z)$ is a polynomial of degree n and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1}|P(z)|.$$

This lemma is a special case of a result due to Govil and Rahman [10].

Lemma 5 *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$ and $|z| = 1$,*

$$|D_\alpha P(z)| \geq (|\alpha| - 1)\left(\frac{n}{2} + \frac{|c_n| - |c_0|}{2(|c_n| + |c_0|)}\right)\max_{|z|=1}|P(z)|.$$

The above lemma is due to Mir, Wani and Gulzar [22].

Lemma 6 *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k, \ k \geq 1$, then*

$$\max_{|z|=k}|P(z)| \geq \left(\frac{2k^n}{1 + k^n} + \frac{k^n(|c_n|k^n - |c_0|)(k-1)}{(1+k^n)(|c_n|k^n + k|c_0|)}\right)\max_{|z|=1}|P(z)|.$$

The above lemma is due to Kumar [12].

4 Proofs of main results

Proof of Theorem 1 Recall that $P(z) = \sum_{j=0}^{n} c_j z^j \neq 0$ in $|z| < k, \ k \leq 1$. By Lemma 2, we have

$$k^{n-1}S_0(k)\max_{|z|=1}|P'(z)| \leq \max_{|z|=1}|Q'(z)|. \tag{4.1}$$

Also, $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. Let

$$\max_{|z|=1}|P'(z)| = |P'(e^{i\alpha})|, \ 0 \leq \alpha < 2\pi \tag{4.2}$$

then

$$\max_{|z|=1}|Q'(z)| = |Q'(e^{i\alpha})|. \tag{4.3}$$

On combining (4.1), (4.2), (4.3) and Lemma 4, we get

$$\max_{|z|=1}|P'(z)| + k^{n-1}S_0(k)\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)|,$$

which is equivalent to

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k^{n-1}S_0(k)}\max_{|z|=1}|P(z)|. \tag{4.4}$$

Since $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, it is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.5}$$

Also, for every complex number α with $|\alpha| \ge 1$, we have by (4.5) and $|z| = 1$,

$$\begin{aligned}
|D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\
&= |nP(z) - zP'(z) + \alpha P'(z)| \\
&\le |nP(z) - zP'(z)| + |\alpha||P'(z)| \\
&= |Q'(z)| + |\alpha||P'(z)| \\
&= |Q'(z)| + |P'(z)| - |P'(z)| + |\alpha||P'(z)| \\
&\le n\max_{|z|=1}|P(z)| + (|\alpha| - 1)|P'(z)| \quad \text{(by Lemma 4)} \\
&\le n\max_{|z|=1}|P(z)| + (|\alpha| - 1)\max_{|z|=1}|P'(z)|.
\end{aligned} \tag{4.6}$$

Inequality (4.6) in conjunction with inequality (4.4), gives for $|z| = 1$,

$$|D_\alpha P(z)| \le n\max_{|z|=1}|P(z)| + (|\alpha| - 1)\left\{\frac{n}{1+k^{n-1}S_0(k)}\max_{|z|=1}|P(z)|\right\},$$

from which we can obtain (2.1). This completes the proof of Theorem 1.

Proof of Theorem 2 Recall that $P(z) = \sum_{j=0}^{n} c_j z^j \ne 0$ in $|z| < k$, $k \le 1$, therefore, by Lemma 3, we have

$$k^n S_m(k)\max_{|z|=1}|P'(z)| + mnS_m(k) \le \max_{|z|=1}|Q'(z)|. \tag{4.7}$$

By the given hypothesis, $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. Let

$$\max_{|z|=1}|P'(z)| = |P'(e^{i\alpha})|,\ 0 \le \alpha < 2\pi \tag{4.8}$$

then

$$\max_{|z|=1}|Q'(z)| = |Q'(e^{i\alpha})|. \tag{4.9}$$

Also, by Lemma 4, we have

$$|P'(e^{i\alpha})| + |Q'(e^{i\alpha})| \le n\max_{|z|=1}|P(z)|,$$

which gives with the help of (4.7), (4.8) and (4.9), that

$$n\max_{|z|=1}|P(z)| \ge \max_{|z|=1}|P'(z)| + k^n S_m(k)\max_{|z|=1}|P'(z)| + mnS_m(k),$$

which implies

$$\max_{|z|=1}|P'(z)| \le \frac{n}{(1+k^n S_m(k))}\left(\max_{|z|=1}|P(z)| - mS_m(k)\right). \tag{4.10}$$

Using the definition of polar derivative, we have from (4.6) that for every α with $|\alpha| \ge 1$ and $|z| = 1$,

$$|D_\alpha P(z)| \le n\max_{|z|=1}|P(z)| + (|\alpha|-1)\max_{|z|=1}|P'(z)|. \tag{4.11}$$

Inequality (4.11) in conjunction with inequality (4.10), gives for $|z| = 1$,

$$|D_\alpha P(z)| \le n\max_{|z|=1}|P(z)| + (|\alpha|-1)\left\{\frac{n}{1+k^n S_m(k)}\left(\max_{|z|=1}|P(z)| - mS_m(k)\right)\right\},$$

from which we can obtain (2.5). This completes the proof of Theorem 2.

Proof of Theorem 3 By hypothesis $P(z) = \sum_{j=0}^{n} c_j z^j$ has all its zeros in $|z| \le k,\ k \ge 1$, hence all the zeros of $G(z) = P(kz)$ lie in $|z| \le 1$. Also, noting by hypothesis we have $\frac{|\alpha|}{k} \ge 1$, it follows by using Lemma 5, that

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}G(z)\right| \ge \left(\frac{|\alpha|-k}{k}\right)\left(\frac{n}{2} + \frac{|c_n|k^n - |c_0|}{2(|c_n|k^n + |c_0|)}\right)\max_{|z|=1}|G(z)|,$$

which is equivalent to

$$\max_{|z|=1}\left|nP(kz) + \left(\frac{\alpha}{k} - z\right)kP'(kz)\right| \ge \left(\frac{|\alpha|-k}{k}\right)\left(\frac{n}{2} + \frac{|c_n|k^n - |c_0|}{2(|c_n|k^n + |c_0|)}\right) \times \max_{|z|=k}|P(z)|. \tag{4.12}$$

Using the fact that $\max_{|z|=1}\left|nP(kz) + \left(\frac{\alpha}{k} - z\right)kP'(kz)\right| = \max_{|z|=k}|D_\alpha P(z)|$ and the Lemma 6, in (4.12), we get

$$\max_{|z|=k}|D_\alpha P(z)| \ge \left(\frac{|\alpha|-k}{k}\right)\left(\frac{n}{2} + \frac{|c_n|k^n - |c_0|}{2(|c_n|k^n + |c_0|)}\right) \times \left(\frac{2k^n}{1+k^n} + \frac{k^n(|c_n|k^n - |c_0|)(k-1)}{(1+k^n)(|c_n|k^n + k|c_0|)}\right)\max_{|z|=1}|P(z)|. \tag{4.13}$$

Since $D_\alpha P(z)$ is a polynomial of degree $n-1$ and $k \ge 1$, hence by Bernstein-inequality, we have $\max_{|z|=k}|D_\alpha P(z)| \le k^{n-1}\max_{|z|=1}|D_\alpha P(z)|$, which on using in (4.13), gives

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)| \ge nk^{n-1}\left(\frac{|\alpha|-k}{1+k^n}\right)\left(1 + \frac{|c_n|k^n - |c_0|}{n(|c_n|k^n + |c_0|)}\right) \times \left(1 + \frac{(|c_n|k^n - |c_0|)(k-1)}{2(|c_n|k^n + k|c_0|)}\right)\max_{|z|=1}|P(z)|,$$

which in particular gives (2.11) and this completes the proof of Theorem 3.

Remark 8 In the Theorems 1, 2 and Corollaries 1, 2, the condition that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, is needed only for $0 < k < 1$. For $k = 1$, all these results hold without this condition, for example see Dewan et al. [5, Theorem 1] for $k = t = 1$).

Remark 9 It may be remarked here that Theorem 3 also improves a result of Govil and Kumar ([8, Theorem 1.1] for $m = 0$) considerably when $k > 1$. In fact, excepting the case when all the zeros of the polynomial $P(z)$ lie on $|z| = k$, the considerable improvement can be observed over here by virtue of Lemma 6 which improves the corresponding lemma used by Govil and Kumar in that paper for the case $k \neq 1$.

Acknowledgements The authors are indebted to the anonymous referee for valuable comments and suggestions which indeed improved the quality of the paper.

References

1. Aziz, A.: Inequalities for the polar derivative of a polynomial. J. Approx. Theory **55**, 183–193 (1988)
2. Aziz, A., Ahmad, N.: Inequalities for the derivative of a polynomial. Proc. Indian Acad. Sci. (Math. Sci.) **107**, 189–196 (1997)
3. Bernstein, S.: Sur l'ordre de la meilleure approximation des functions continues par des polynômes de degré donné. Mem. Acad. R. Belg. **4**, 1–103 (1912)
4. Bernstein, S.: Sur la limitation des dérivées des polynômes. C. R. Acad. Sci. Paris **190**, 338–340 (1930)
5. Dewan, K.K., Singh, N., Mir, A.: Extensions of some polynomial inequalities to the polar derivative. J. Math. Anal. Appl. **350**, 807–815 (2009)
6. Govil, N.K.: On a theorem of S. Bernstein. Proc. Natl. Acad. Sci. **50**, 50–52 (1980)
7. Govil, N.K.: On the derivative of a polynomial. Proc. Am. Math. Soc. **41**, 543–546 (1973)
8. Govil, N.K., Kumar, P.: On sharpening of an inequality of Turán. Appl. Anal. Discret. Math. **13**, 711–720 (2019)
9. Govil, N.K., Kumar, P.: On Bernstein-type inequality for the polar derivative of a polynomial. Prog. Approx. Theory Appl. Complex Anal. Springer Optim. Appl. **117**, 41–74 (2017)
10. Govil, N.K., Rahman, Q.I.: Functions of exponential type not vanishing in a half plane and related polynomials. Trans. Am. Math. Soc. **137**, 501–517 (1969)
11. Jain, V.K.: On the derivative of a polynomial. Bull. Math. Soc. Sci. Math. Roumanie Tome **59**, 339–347 (2016)
12. Kumar, P.: On the inequalities concerning polynomials. Complex Anal. Oper. Theory **14**, 1–11 (2020)
13. Lax, P.D.: Proof of a conjecture of P. Erdös on the derivative of a polynomial. Bull. Am. Math. Soc **50**, 509–513 (1944)
14. Malik, M.A.: On the derivative of a polynomial. J. Lond. Math. Soc. **1**, 57–60 (1969)
15. Marden, M.: Geometry of Polynomials, Mathematical Surveys, vol. 3. American Mathematical Society, Providence (1966)
16. Milovanović, G.V., Mitrinović, D.S., Rassias, ThM: Topics in Polynomials, Extremal Properties, Inequalities, Zeros. World Scientific Publishing Co., Singapore (1994)
17. Mir, A.: Bernstein-type integral inequalities for a certain class of polynomials. Mediterr. J. Math. **16**(Art. 143), 1–11 (2019)
18. Mir, A.: Generalizations of Bernstein and Turán-type inequalities for the polar derivative of a complex polynomial. Mediterr. J. Math. **17**(Art. 14), 1–12 (2020)
19. Mir, A.: On an operator preserving inequalities between polynomials. Ukrain. Math. J. **69**, 1234–1247 (2018)
20. Mir, A., Hussain, I.: On the Erdös–Lax inequality concerning polynomials. C. R. Acad. Sci. Paris Ser. **I**(355), 1055–1062 (2017)
21. Mir, A., Wani, A.: A note on two recent results about polynomials with restricted zeros. J. Math. Inequal. **14**, 47–52 (2020)
22. Mir, A., Wani, A., Gulzar, M.H.: Some inequalities concerning the polar derivative of a polynomial. J. Interdiscip. Math. **21**, 1387–1393 (2018)
23. Rahman, Q.I., Schmeisser, G.: Analytic Theory of Polynomials. Oxford University Press Inc., New York (2002)
24. Turán, P.: Über die Ableitung von Polynomen. Compos. Math. **7**, 89–95 (1939)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.