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ORIGINAL PAPER

Inequalities of Turán-type for algebraic polynomials

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Abstract

In this paper, we establish some new inequalities in the complex plane that are inspired by some classical Turán-type inequalities that relate the norm of a univariate complex coefficient polynomial and its derivative on the unit disk. The obtained results produce various inequalities in the supremum-norm and in the integral-norm of a polynomial that are sharper than the previous ones while taking into account the placement of the zeros and some of the extremal coefficients of the underlying polynomial. Moreover, our results besides derive polar derivative analogues of some classical Turán-type inequalities also include several interesting generalizations and refinements of some integral inequalities for polynomial as well. Some numerical examples are given in order to graphically illustrate and compare the obtained inequalities with some classical results.

Keywords Turán's classical inequality · Minimum Modulus Principle · Polar derivative of a polynomial

Mathematics Subject Classification 30A10 · 30C10 · 30C15

1 Introduction to Turán type inequalities

Let $\mathcal{P}_n$ be the class of all complex polynomials $P(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ of degree n and $P'(z)$ its derivative. The study of Turán-type inequalities relating the norm of the derivative and the polynomial itself as well as generalizing the classical polynomial inequalities is a fertile area in analysis for researchers. Here, we study some of the new inequalities centered around Turán-type inequalities that relate the norm of the polar derivative (derivative) and the poly-

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nomial under some conditions. One of the Turán's classical inequality [26] provides a lower bound estimate to the size of the derivative of a polynomial on the unit circle in the plane relative to the size of the polynomial itself when there is a restriction on its zeros. It states that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

As a refinement of (1.1), Aziz and Dawood [2] under the same hypothesis proved the following result:

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\}. \tag{1.2}$$

Equality in (1.1) and (1.2) holds for any polynomial which has all its zeros on $|z| = 1$.

Over the years, Turán's inequality (1.1) has been generalized and extended in several directions. Before proceeding towards some specific generalizations of (1.1) for a particular operator, we find it useful to mention few of its extensions for the ordinary derivative. As an extension of (1.1), Malik [13] established that

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.3}$$

when the polynomial $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k,\ k \le 1$.

As a generalization of (1.1), Govil [10] proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Equality in (1.3) holds for $P(z) = (z+k)^n$, where as equality in (1.4) holds for $P(z) = z^n + k^n$.

Although the inequality (1.4) is sharp but it has a drawback. The bound in this inequality depends on the zero of largest modulus and not on the other zeros even if some of them are very close to the origin. This was taken into consideration by Aziz [1], who proved that if $P(z) = c_n \prod_{\nu=1}^{n} (z - z_\nu)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{2}{1+k^n} \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} \max_{|z|=1} |P(z)|, \tag{1.5}$$

whereas in 2008, Dewan and Upadhye [5] strengthened (1.5) by proving under the same hypothesis that

$$\max_{|z|=1} |P'(z)| \ge \left[\frac{2}{1+k^n} \max_{|z|=1} |P(z)| + \frac{1}{k^n} \cdot \frac{k^n - 1}{k^n + 1} \min_{|z|=k} |P(z)| \right] \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|}. \tag{1.6}$$

Equality in (1.5) and (1.6) holds for $P(z) = z^n + k^n$.

In 2007, Dubinin [7] used the classical Schwarz lemma and obtained an interesting refinement of (1.1). Precisely, Dubinin proved that if $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \left\{ \frac{n}{2} + \frac{1}{2} \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right\} \max_{|z|=1} |P(z)|. \tag{1.7}$$

It is worth mentioning that different versions of Turán's inequality (cf. [15, pp. 664–674]) have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. These inequalities have their own significance and importance in Approximation theory. Before proceeding to some other results, let us introduce the concept of the polar derivative involved. For $P \in \mathcal{P}_n$, we define

$$D_\beta P(z) := nP(z) + (\beta - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point β (see [14] and [9, Chap. 6]). The polynomial $D_\beta P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta P(z)}{\beta}\right\} = P'(z),$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

Various results of majorization on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [15], Marden [14] and Rahman and Schmeisser [24], where some approaches to obtaining polynomial inequalities are developed on applying the methods and results of the geometric function theory. For the latest research and development in this direction, one can see some of papers and monographs (see [9, 12, 16–20, 22]).

In 1996, Shah [25] established the polar derivative analogue of (1.1) by proving that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \ge 1$,

$$\max_{|z|=1} |D_\beta P(z)| \ge n\frac{|\beta|-1}{2}\max_{|z|=1}|P(z)|. \tag{1.8}$$

If we divide both sides of the above inequality (1.8) by $|\beta|$ and let $|\beta| \to +\infty$, we obtain the inequality (1.1). Aziz and Rather [3] extended (1.3) to the polar derivative of a polynomial by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \ge k$,

$$\max_{|z|=1} |D_\beta P(z)| \ge n\frac{|\beta|-k}{1+k}\max_{|z|=1}|P(z)|, \tag{1.9}$$

whereas, in the same paper, Aziz and Rather showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \ge 1$, then for $|\beta| \ge k$,

$$\max_{|z|=1} |D_\beta P(z)| \ge n\frac{|\beta|-k}{1+k^n}\max_{|z|=1}|P(z)|, \tag{1.10}$$

thereby giving a polar derivative generalization of (1.4).

Govil and McTume [11] proved under the same hypothesis as in (1.9), that

$$\max_{|z|=1} |D_\beta P(z)| \ge n\frac{|\beta|-L}{1+k}\max_{|z|=1}|P(z)|, \tag{1.11}$$

for every $\beta \in \mathbb{C}$ with $|\beta| \ge L$, where

$$L = \frac{nk^2|c_n| + |c_{n-1}|}{|c_{n-1}| + n|c_n|}. \tag{1.12}$$

A similar type of modification as in (1.6) to the inequality (1.10) was given by Dewan and Upadhye [5], who proved that if $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu = c_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for $|\beta| \ge k$,

$$\max_{|z|=1}|D_\beta P(z)| \geq (|\beta|-k)\left[\frac{2}{1+k^n}\max_{|z|=1}|P(z)|+\frac{1}{k^n}\cdot\frac{k^n-1}{k^n+1}\min_{|z|=k}|P(z)|\right]\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}. \tag{1.13}$$

In 2018, Mir et al. [23] extended (1.7) to the polar derivative of a polynomial and thereby obtained a generalization of it. More precisely, they proved that if $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z|\leq 1$, then for every $\beta\in\mathbb{C}$ with $|\beta|\geq 1$ and $|z|=1$,

$$|D_\beta P(z)| \geq \frac{|\beta|-1}{2}\left\{n+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{1.14}$$

They also proved the following more general result that if $P\in\mathcal{P}_n$ has all its zeros in $|z|\leq 1$, then for every β with $|\beta|\geq 1$, $0\leq t<1$ and $|z|=1$,

$$\begin{aligned}\left|D_\beta P(z)\right| \geq{}& \frac{n}{2}\left\{(|\beta|-1)\max_{|z|=1}|P(z)|+(|\beta|+1)\,tm_1\right\}\\ &+n\,\frac{|\beta|-1}{2}\cdot\frac{|c_n|-tm_1-|c_0|}{|c_n|-tm_1+|c_0|}\left\{\max_{|z|=1}|P(z)|-tm_1\right\},\end{aligned} \tag{1.15}$$

where $m_1=\min_{|z|=1}|P(z)|$.

Equality in (1.15) holds for $P(z)=(z-1)^n$ with real $\beta\geq 1$.

Very recently, Chanam et al. [4] obtained an inequality which is the L^r-analogue of (1.14). In fact, they first proved the following result:

Theorem A *If $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu\in\mathcal{P}_n$ has all its zeros in $|z|\leq 1$, then for every $\beta\in\mathbb{C}$ with $|\beta|\geq 1$ and for each $r>0$,*

$$\left\{\int_0^{2\pi}|D_\beta P(\mathrm{e}^{\mathrm{i}\theta})|^r\mathrm{d}\theta\right\}^{1/r} \geq \frac{n(|\beta|-1)}{2}\left\{1+\frac{1}{n}\cdot\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\left\{\int_0^{2\pi}|P(\mathrm{e}^{\mathrm{i}\theta})|^r\mathrm{d}\theta\right\}^{1/r}. \tag{1.16}$$

As an application of Theorem A, they also proved the following L^r-analogue of (1.15).

Theorem B *If $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu\in\mathcal{P}_n$ has all its zeros in $|z|\leq 1$, then for every $\beta\in\mathbb{C}$ with $|\beta|\geq 1$, $0\leq t<1$ and for each $r>0$,*

$$\begin{aligned}&\left\{\int_0^{2\pi}\Big(|D_\beta P(\mathrm{e}^{\mathrm{i}\theta})|-m_1nt|\beta|\Big)^r\mathrm{d}\theta\right\}^{1/r}\\ &\quad\geq\frac{n(|\beta|-1)}{2}\left\{1+\frac{1}{n}\cdot\frac{|c_n|-tm_1-|c_0|}{|c_n|-tm_1+|c_0|}\right\}\left\{\int_0^{2\pi}\Big(|P(\mathrm{e}^{\mathrm{i}\theta})|-tm_1\Big)^r\mathrm{d}\theta\right\}^{1/r},\end{aligned} \tag{1.17}$$

where $m_1=\min_{|z|=1}|P(z)|$.

Taking limits as $r\to+\infty$ in (1.16) and (1.17), we get (1.14) and (1.15), respectively.

The present paper is mainly motivated by the desire to establish some improved bounds for the polar derivative (derivative) of a polynomial both in the supremum-norm and in the integral norm. The obtained results produce various Turán-type inequalities that are sharper than the previous ones while taking into account the placement of the zeros and some of the extremal coefficients of the underlying polynomial.

The paper is organized as follows. In Sect. 2, we present some auxiliary results necessary in proving the main results. Section 3 is devoted to the main results in the supremum-norm and in the integral norm. Some numerical results are given in Sect. 4 in order to graphically illustrate and compare our inequalities with some classical results. Finally, Sect. 5 contains some conclusions.

2 Auxiliary results

For the proofs of our main results, we use the lemmas presented in this section.

Lemma 2.1 *If* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$, *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for* $R \geq 1$ *and* $0 \leq t \leq 1$, *we have*

$$\max_{|z|=R} |P(z)| \leq \frac{(1+R^n)(|c_0| + R|c_n| - tm_1)}{(1+R)(|c_0| + |c_n| - tm_1)} \max_{|z|=1} |P(z)| - \left(\frac{(1+R^n)(|c_0| + R|c_n| - tm_1)}{(1+R)(|c_0| + |c_n| - tm_1)} - 1 \right) tm_1, \tag{2.1}$$

where $m_1 = \min_{|z|=1} |P(z)|$.

Equality in (2.1) *holds for* $P(z) = (\mu + \nu z^n)/2$, $|\mu| = |\nu| = 1$.

This lemma is due to Mir et al. [21].

Lemma 2.2 *If* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ *has all its zeros in* $|z| \leq k$, $k \geq 1$, *then for* $0 \leq t \leq 1$, *we have*

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} \left\{ \left[1 + \frac{k-1}{2} S_t(k) \right] \max_{|z|=1} |P(z)| + \frac{k-1}{2k^n} \left[k^{n-1} + k^{n-2} + \cdots + k + 1 - S_t(k) \right] tm \right\}, \tag{2.2}$$

where $S_t(k)$ *and* m *are given by*

$$S_t(k) = \frac{k^n|c_n| - |c_0| - tm}{k^n|c_n| + k|c_0| - tm} \quad \text{and} \quad m = \min_{|z|=k} |P(z)|.$$

Equality in (2.2) *holds for* $P(z) = z^n + k^n$.

Proof Let $T(z) = P(kz)$. Since $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, the polynomial $T(z)$ has all its zeros in $|z| \leq 1$. Let $H(z) = z^n T(1/z)$ be the reciprocal polynomial of $T(z)$, then $H(z)$ is a polynomial of degree n having no zeros in $|z| < 1$. Hence applying (2.1) of Lemma 2.1 to the polynomial $H(z)$, we get for $k \geq 1$, and $0 \leq t \leq 1$,

$$\max_{|z|=k} |H(z)| \leq \frac{(1+k^n)(k^n|c_n| + k|c_0| - tm^*)}{(1+k)(k^n|c_n| + |c_0| - tm^*)} \max_{|z|=1} |H(z)| - \left(\frac{(1+k^n)(k^n|c_n| + k|c_0| - tm^*)}{(1+k)(k^n|c_n| + |c_0| - tm^*)} - 1 \right) tm^*, \tag{2.3}$$

where $m^* = \min_{|z|=1} |H(z)|$.

Since $|H(z)| = |T(z)|$ on $|z| = 1$, therefore,

$$m^* = \min_{|z|=1} |H(z)| = \min_{|z|=1} \left|z^n P\left(\tfrac{k}{z}\right)\right| = \min_{|z|=k} |P(z)| = m,$$
$$\max_{|z|=1} |H(z)| = \max_{|z|=1} |T(z)| = \max_{|z|=k} |P(z)|,$$

and

$$\max_{|z|=k} |H(z)| = \max_{|z|=k} \left|z^n P\left(\frac{k}{z}\right)\right| = k^n \max_{|z|=1} |P(z)|.$$

The above when substituted in (2.3) gives

$$\max_{|z|=k} |P(z)| \geq \left(\frac{(1+k)(k^n|c_n| + |c_0| - tm)}{(1+k^n)(k^n|c_n| + k|c_0| - tm)}\right) k^n \max_{|z|=1} |P(z)| + \left(1 - \frac{(1+k)(k^n|c_n| + |c_0| - tm)}{(1+k^n)(k^n|c_n| + k|c_0| - tm)}\right) tm. \tag{2.4}$$

Using the fact that

$$\frac{(1+k)(k^n|c_n| + |c_0| - tm)}{(1+k^n)(k^n|c_n| + k|c_0| - tm)} = \frac{2}{1+k^n} + \frac{(k^n|c_n| - |c_0| - tm)(k-1)}{(1+k^n)(k^n|c_n| + k|c_0| - tm)},$$

in (2.4), we get (2.2) and this completes the proof of Lemma 2.2. □

Lemma 2.3 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$, then for all $R \geq 1$,*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - \left(R^n - R^{n-2}\right) |P(0)|, \qquad \text{if } n \geq 2, \tag{2.5}$$

and

$$\max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R-1)|P(0)|, \qquad \text{if } n = 1. \tag{2.6}$$

The above lemma is due to Frappier et al. [8].

Lemma 2.4 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then on $|z| = 1$, we have*

$$|Q'(z)| \leq \frac{nk^2|c_n| + |c_{n-1}|}{|c_{n-1}| + n|c_n|} \left|P'(z)\right|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

This lemma is due to Govil and McTume [11].

Lemma 2.5 *If x_ν, $\nu = 1, 2, \ldots, n$, is a sequence of real numbers such that for all $\nu \in \mathbb{N}$, we have $0 \leq x_\nu \leq 1$, then for all $n \in \mathbb{N}$*

$$\sum_{\nu=1}^{n} \frac{1 - x_\nu}{1 + x_\nu} \geq \frac{1 - \prod_{\nu=1}^{n} x_\nu}{1 + \prod_{\nu=1}^{n} x_\nu}.$$

Proof We prove this result with the help of mathematical induction and we use induction on n. The result is trivially true for $n = 1$.

For $n = 2$

$$\frac{1 - x_1}{1 + x_1} + \frac{1 - x_2}{1 + x_2} \geq \frac{1 - x_1x_2}{1 + x_1x_2}$$

if

$$\frac{2(1-x_1x_2)}{1+x_1+x_2+x_1x_2} \geq \frac{1-x_1x_2}{1+x_1x_2},$$

i.e., if $(1-x_1)(1-x_2) \geq 0$, which is true, since $x_1, x_2 \leq 1$. This shows that the result holds for $n = 2$. Assume the result is true for $n = s \in \mathbb{N}$. Now, since $\prod_{\nu=1}^{s} x_\nu \leq 1$, we have

$$\begin{aligned}\sum_{\nu=1}^{s+1} \frac{1-x_\nu}{1+x_\nu} &= \sum_{\nu=1}^{s} \frac{1-x_\nu}{1+x_\nu} + \frac{1-x_{s+1}}{1+x_{s+1}} \\ &\geq \frac{1-\prod_{\nu=1}^{s} x_\nu}{1+\prod_{\nu=1}^{s} x_\nu} + \frac{1-x_{s+1}}{1+x_{s+1}} \qquad \text{(by induction hypothesis)} \\ &\geq \frac{1-\prod_{\nu=1}^{s+1} x_\nu}{1+\prod_{\nu=1}^{s+1} x_\nu} \qquad \text{(by the case } n=2\text{)}.\end{aligned}$$

This shows that the result holds for $n = s + 1$ as well. Therefore, by the principle of mathematical induction, it follows that the result holds for all $n \in \mathbb{N}$. This completes the proof of Lemma 2.5. □

Lemma 2.6 *If* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ *and* $P(z) \neq 0$ *in* $|z| > k$, $k \leq 1$, *then for each point* z *on* $|z| = 1$ *at which* $P(z) \neq 0$, *we have*

$$\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left\{1 + \frac{k}{n} \cdot \frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}. \tag{2.7}$$

Proof Recall that $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$. If $z_1, z_2, \ldots, z_n$ are the zeros of $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ of degree n, then $|z_\nu| \leq k$, $k \leq 1$, and we can write $P(z) = c_n \prod_{\nu=1}^{n}(z - z_\nu)$. This gives

$$\frac{zP'(z)}{P(z)} = \sum_{\nu=1}^{n} \frac{z}{z - z_\nu}.$$

Now for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, with $P(e^{i\theta}) \neq 0$, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) &= \sum_{\nu=1}^{n} \mathrm{Re}\left(\frac{e^{i\theta}}{e^{i\theta} - z_\nu}\right) \\ &\geq \sum_{\nu=1}^{n} \frac{1}{1+|z_\nu|} \\ &= \frac{n}{1+k} + \frac{k}{1+k}\sum_{\nu=1}^{n} \frac{k - |z_\nu|}{k + k|z_\nu|} \\ &\geq \frac{n}{1+k} + \frac{k}{1+k}\sum_{\nu=1}^{n} \frac{1 - |z_\nu|/k}{1 + |z_\nu|/k}.\end{aligned}$$

Since $|z_\nu|/k \leq 1$, $\nu = 1, 2, \ldots, n$, we get on using Lemma 2.5 for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, with $P(e^{i\theta}) \neq 0$,

$$\mathrm{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) \geq \frac{n}{1+k} + \frac{k}{1+k} \cdot \frac{1 - \prod_{\nu=1}^{n} |z_\nu|/k}{1 + \prod_{\nu=1}^{n} |z_\nu|/k}$$

$$= \frac{n}{1+k} + \frac{k}{1+k} \cdot \frac{1 - |c_0|/k^n|c_n|}{1 + |c_0|/k^n|c_n|},$$

which is equivalent to (2.7). This completes the proof of Lemma 2.6. □

Lemma 2.7 *If* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ *has all its zeros in* $|z| \le k$, $k > 0$, *then*

$$|c_n| > \frac{m}{k^n},$$

where $m = \min_{|z|=k} |P(z)|$.

Proof Since $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ has all its zeros in $0 < |z| \le k$, then $Q(z) = z^n \overline{P(1/\overline{z})} \ne 0$ for $|z| < 1/k$. We can assume without loss of generality that $Q(z)$ has no zero on $|z| = 1/k$, for otherwise the result holds trivially. Since $Q(z)$ is analytic in $|z| \le 1/k$ and has no zero in $|z| \le 1/k$, by the Minimum Modulus Principle,

$$\min_{|z|=1/k} |Q(z)| \le |Q(z)| \quad \text{for } |z| \le \frac{1}{k},$$

which implies

$$\frac{1}{k^n} \min_{|z|=k} |P(z)| \le |Q(z)| \quad \text{for } |z| \le \frac{1}{k},$$

which in particular gives

$$\frac{m}{k^n} < |Q(0)| = |c_n|.$$

This proves Lemma 2.7. □

Lemma 2.8 *The function*

$$x \mapsto S(x) = \frac{k^2 x + c}{x + c},$$

where $k \le 1$ *and* $c > 0$, *is a non-increasing function for* $x \ge 0$.

Proof It is clear from

$$S'(x) = -\frac{c\left(1 - k^2\right)}{(c + x)^2} \le 0.$$

□

3 Main results

In this section, we present our main results.

We begin by proving the following Turán-type inequality for the class of polynomials having all zeros in $|z| \le k$, $k \ge 1$, by obtaining a bound which involves the modulus of each zero of the underlying polynomial, and at the same time our result sharpens (1.5), (1.6) and several of the earlier related inequalities considerably.

Theorem 3.1 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu = c_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree $n \geq 2$, having all its zeros in $|z| \leq k$, $k \geq 1$, then for $0 \leq t \leq 1$, we have*

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1+k^n} \left\{ \left[1 + \frac{k-1}{2} S_t(k)\right] \max_{|z|=1} |P(z)| + \frac{1}{2k^n}\left[k^n - 1 - (k-1)S_t(k)\right] tm \right\} \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} + |c_1|\psi(k), \tag{3.1}$$

where

$$S_t(k) = \frac{k^n|c_n| - |c_0| - tm}{k^n|c_n| + k|c_0| - tm}, \qquad \psi(k) = \begin{cases} 1 - \frac{1}{k^2}, & \text{if } \; n > 2, \\ 1 - \frac{1}{k}, & \text{if } \; n = 2, \end{cases}$$

and

$$m = \min_{|z|=k} |P(z)|.$$

Equality in (3.1) holds for $P(z) = z^n + k^n$.

Proof First, we suppose that

$$P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu = c_n \prod_{\nu=1}^{n}(z - z_\nu)$$

is a polynomial of degree $n > 2$. Recall that $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, therefore, all the zeros of

$$F(z) = P(kz) = c_n k^n \prod_{\nu=1}^{n} \left(z - \frac{z_\nu}{k}\right)$$

lie in $|z| \leq 1$. Since for all z on $|z| = 1$ for which $F(z) \neq 0$, we have

$$\frac{zF'(z)}{F(z)} = \sum_{\nu=1}^{n} \frac{z}{z - (z_\nu/k)},$$

therefore,

$$\begin{aligned} \operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) &= \sum_{\nu=1}^{n} \operatorname{Re}\left(\frac{z}{z-(z_\nu/k)}\right) \\ &\geq \sum_{\nu=1}^{n} \frac{1}{1+|z_\nu/k|} \\ &= \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|}, \end{aligned}$$

for all z on $|z| = 1$ for which $F(z) \neq 0$.

This gives

$$\left|\frac{zF'(z)}{F(z)}\right| \geq \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|},$$

for all z on $|z| = 1$ for which $F(z) \neq 0$. Therefore

$$\max_{|z|=1} |F'(z)| \geq \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |F(z)|, \tag{3.2}$$

which is equivalent to

$$k \max_{|z|=1} |P'(kz)| \geq \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=k} |P(z)|. \tag{3.3}$$

Since $P'(z)$ is a polynomial of degree $n - 1$ and $k \geq 1$, and hence applying inequality (2.5) of Lemma 2.3 to the polynomial $P'(z)$, we get

$$\max_{|z|=1} |P'(kz)| = \max_{|z|=k} |P'(z)| \leq k^{n-1} \max_{|z|=1} |P'(z)| - \left(k^{n-1} - k^{n-3}\right) |c_1|.$$

Using this and Lemma 2.2 in (3.3), we get for $k \geq 1$ and $0 \leq t \leq 1$,

$$k^n \left\{ \max_{|z|=1} |P'(z)| - \left(1 - \frac{1}{k^2}\right) |c_1| \right\} \geq \frac{2k^n}{1 + k^n} \left\{ \left[1 + \frac{k-1}{2} S_t(k)\right] \max_{|z|=1} |P(z)| \right.$$
$$\left. + \frac{k-1}{2k^n} \left[k^{n-1} + k^{n-2} + \cdots + k + 1 - S_t(k)\right] tm \right\} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|},$$

which is equivalent to (3.1), and thus Theorem 3.1, in the case $n > 2$ is proved.

The proof of the theorem in the case $n = 2$ follows on the same lines as above except that instead of inequality (2.5) of Lemma 2.3, we use (2.6) of the same lemma. □

If we do not have the knowledge of $\min_{|z|=k} |P(z)|$ or $t = 0$, we get the following result from Theorem 3.1 which represents a refinement of (1.5).

Corollary 3.1 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu = c_n \prod_{\nu=1}^{n} (z - z_\nu)$ is a polynomial of degree $n \geq 2$, having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \left\{ \frac{2}{1 + k^n} + \frac{(k^n|c_n| - |c_0|)(k-1)}{(1 + k^n)(k^n|c_n| + k|c_0|)} \right\} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(z)| + |c_1| \psi(k), \tag{3.4}$$

where $\psi(k)$ is as defined in Theorem 3.1. Equality in (3.4) holds for $P(z) = z^n + k^n$.

Remark 3.1 Recall that the polynomial $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and in this case Theorem 3.1 reduces to Corollary 3.1. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k$, $k \geq 1$. Let $H(z) = P(kz)$ and $G(z) = z^n \overline{H(1/\overline{z})} = z^n \overline{P(k/\overline{z})}$, then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$. This gives $m \leq |P(kz)|$ for $|z| = 1$, and since $m/P(kz)$ is not a constant, it follows by the Minimum Modulus Principle, that

$$\left| z^n \overline{P\left(\frac{k}{\overline{z}}\right)} \right| = |P(kz)| \geq m \quad \text{for } |z| \leq 1.$$

Replacing z by $1/\overline{z}$, it implies that

$$|P(kz)| \geq m|z|^n \quad \text{for } |z| \geq 1,$$

or

$$|P(z)| \geq m \left|\frac{z}{k}\right|^n \quad \text{for} \ \ |z| \geq k. \tag{3.5}$$

Consider the polynomial $F(z) = P(z) + \lambda m$, where $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, then all the zeros of $F(z)$ lie in $|z| \leq k$. Because, if for some $z = z_0$ with $|z_0| > k$, we have

$$F(z_0) = P(z_0) + \lambda m = 0,$$

then

$$|P(z_0)| = |\lambda m| \leq m < m \left|\frac{z_0}{k}\right|^n,$$

which is a contradiction to (3.5). Hence all the zeros of $P(z) + \lambda m$ lie in $|z| \leq k$, $k \geq 1$ for every λ with $|\lambda| \leq 1$. If $z_1, z_2, \ldots, z_n$, are the zeros of

$$P(z) + \lambda m = (c_0 + \lambda m) + \sum_{\nu=1}^{n} c_\nu z^\nu,$$

then

$$\left|\frac{c_0 + \lambda m}{c_n}\right| = |z_1 z_2 \cdots z_n| \leq k^n. \tag{3.6}$$

If in (3.6), we choose the argument of λ suitably, we get

$$|c_0| + |\lambda| m \leq k^n |c_n|. \tag{3.7}$$

If we take $|\lambda| = t$ in (3.7), so that $0 \leq t \leq 1$, we get $|c_0| + tm \leq k^n |c_n|$. Using this, it easily follows that

$$\frac{k^n |c_n| - |c_0| - tm}{k^n |c_n| + k|c_0| - tm} = S_t(k) \geq 0. \tag{3.8}$$

Also, by Lemma 2.8 (see Sect. 2), we have $|c_n| \geq m/k^n$, which further implies that

$$\max_{|z|=1} |P(z)| \geq |c_n| \geq \frac{m}{k^n} \geq \frac{tm}{k^n}, \quad 0 \leq t \leq 1.$$

Using this and $k \geq 1$, one can easily check that

$$\phi(x) = \left(1 + \frac{k-1}{2} x\right) \max_{|z|=1} |P(z)| + \frac{k-1}{2k^n} \left(k^{n-1} + k^{n-2} + \cdots + k + 1 - x\right) tm,$$

is an increasing function of x. Thus, from Theorem 3.1, we get the following refinement of (1.6).

Corollary 3.2 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu = c_n \prod_{\nu=1}^{n} (z - z_\nu)$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, $k \geq 1$, then for $0 \leq t \leq 1$, we have*

$$\max_{|z|=1} |P'(z)| \geq \left\{\frac{2}{1+k^n} \max_{|z|=1} |P(z)| + \frac{1}{k^n} \cdot \frac{k^n - 1}{k^n + 1} tm\right\} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} + |c_1| \psi(k), \tag{3.9}$$

where $\psi(k)$ and m are as defined in Theorem 3.1. Equality in (3.9) holds for $P(z) = z^n + k^n$.

Our next result is a polar derivative generalization of Theorem 3.1, which also provides a strengthening of (1.13).

Theorem 3.2 *Let $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu=c_n\prod_{\nu=1}^{n}(z-z_\nu)$ be a polynomial of degree $n\geq 2$, having all its zeros in $|z|\leq k$, $k\geq 1$. Then for every complex number β with $|\beta|\geq k$, and $0\leq t\leq 1$, we have*

$$\max_{|z|=1}|D_\beta P(z)|\geq 2\,\frac{|\beta|-k}{1+k^n}\left\{\left[1+\frac{k-1}{2}S_t(k)\right]\max_{|z|=1}|P(z)|\right.$$
$$\left.+\frac{1}{2k^n}\left[k^n-1-(k-1)S_t(k)\right]tm\right\}\sum_{\nu=0}^{n}\frac{k}{k+|z_\nu|}$$
$$+|nc_0+\beta c_1|\psi(k), \tag{3.10}$$

where $S_t(k)$, $\psi(k)$ and m are defined in Theorem 3.1. Equality in (3.10) holds for $P(z)=z^n+k^n$.

Proof Let $F(z)=P(kz)$, where $P(z)$ is a polynomial of degree $n>2$ having all its zeros in $|z|\leq k$, $k\geq 1$. Therefore, all the zeros of $F(z)$ lie in $|z|\leq 1$, hence by (3.2), we get

$$\max_{|z|=1}|F'(z)|\geq\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|F(z)|. \tag{3.11}$$

Let $H(z)=z^n\overline{F(1/\overline{z})}$. Then it can be easily verified that

$$|H'(z)|=|nF(z)-zF'(z)|\quad\text{for}\quad |z|=1. \tag{3.12}$$

The polynomial $H(z)$ has all its zeros in $|z|\geq 1$ and $|H(z)|=|F(z)|$ for $|z|=1$, therefore on applying Lemma 2.4 for $k=1$ with $P(z)$ replaced by $F(z)$ and $Q(z)$ by $H(z)$, we get

$$|H'(z)|\leq|F'(z)|\quad\text{for}\quad |z|=1. \tag{3.13}$$

Now, noting that by hypothesis, we have $|\beta|/k\geq 1$, hence on using definition of polar derivative of a polynomial, we get

$$\left|D_{\beta/k}F(z)\right|=\left|nF(z)+\frac{\beta}{k}F'(z)-zF'(z)\right|$$
$$\geq\left|\frac{\beta}{k}\right||F'(z)|-|nF(z)-zF'(z)|,$$

which on using (3.12) and (3.13), gives

$$\max_{|z|=1}\left|D_{\beta/k}F(z)\right|\geq\frac{|\beta|-k}{k}\max_{|z|=1}|F'(z)|. \tag{3.14}$$

Using (3.11) in (3.14) and replace $F(z)$ by $P(kz)$, we get

$$\max_{|z|=1}\left|D_{\beta/k}P(kz)\right|\geq\frac{|\beta|-k}{k}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|P(kz)|,$$

which implies

$$\max_{|z|=1}\left|nP(kz)+\left(\frac{\beta}{k}-z\right)kP'(kz)\right|\geq\frac{|\beta|-k}{k}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=k}|P(z)|.$$

Thus on using the fact that

$$\max_{|z|=1}\left|nP(kz)+\left(\frac{\beta}{k}-z\right)kP'(kz)\right|=\max_{|z|=k}|D_\beta P(z)|,$$

gives

$$\max_{|z|=k}|D_\beta P(z)|\geq\frac{|\beta|-k}{k}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=k}|P(z)|. \tag{3.15}$$

For $|\beta|=k$, the coefficient of z^{n-1} in $D_\beta P(z)$ vanishes if $\beta=z_1=z_2=\cdots=z_n=ke^{i\theta}$, that is, when $P(z)$ is of the form $(z-\beta)^n$, for which inequality (3.10) is obvious. Therefore, let $|\beta|>k$, and since $P(z)$ is of degree $n>2$, and so the polynomial $D_\beta P(z)$ is of degree $n-1$, where $n-1\geq 2$, and hence on applying inequality (2.5) of Lemma 2.3 to the polynomial $D_\beta P(z)$, we get for $k\geq 1$,

$$\max_{|z|=k}\left|D_\beta P(z)\right|\leq k^{n-1}\max_{|z|=1}\left|D_\beta P(z)\right|-(k^{n-1}-k^{n-3})\left|D_\beta P(0)\right|,$$

or

$$\max_{|z|=k}\left|D_\beta P(z)\right|\leq k^{n-1}\max_{|z|=1}\left|D_\beta P(z)\right|-(k^{n-1}-k^{n-3})\left|nc_0+\beta c_1\right|. \tag{3.16}$$

On using (3.16) and Lemma 2.2 in (3.15), we get

$$\begin{aligned}
k^{n-1}&\left\{\max_{|z|=1}\left|D_\beta P(z)\right|-\left(1-\frac{1}{k^2}\right)|nc_0+\beta c_1|\right\}\geq\max_{|z|=k}\left|D_\beta P(z)\right|\\
&\geq\frac{2k^n(|\beta|-k)}{k(1+k^n)}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\left\{\left[1+\frac{k-1}{2}S_t(k)\right]\max_{|z|=1}|P(z)|\right.\\
&\left.\quad+\frac{k-1}{2k^n}\left[k^{n-1}+k^{n-2}+\cdots+k+1-S_t(k)\right]tm\right\},
\end{aligned}$$

which is equivalent to (3.10) for $n>2$.

For the case $n=2$, the proof follows along the same lines as that of $n>2$, but instead of (2.5) of Lemma 2.3, we use (2.6) of the same lemma. This proves Theorem 3.2 completely. □

Remark 3.2 If we divide both sides of (3.10) by $|\beta|$ and let $|\beta|\to+\infty$, we recover (3.1). If we use the same arguments as in Remark 3.1, we get from Theorem 3.2 the following refinement of (1.13).

Corollary 3.3 *Let $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu=c_n\prod_{\nu=1}^{n}(z-z_\nu)$ be a polynomial of degree $n\geq 2$, having all its zeros in $|z|\leq k$, $k\geq 1$. Then for every complex number β with $|\beta|\geq k$, and $0\leq t\leq 1$, we have*

$$\begin{aligned}
\max_{|z|=1}|D_\beta P(z)|\geq\ &(|\beta|-k)\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\left[\frac{2}{1+k^n}\max_{|z|=1}|P(z)|+\frac{1}{k^n}\cdot\frac{k^n-1}{k^n+1}tm\right]\\
&+|nc_0+\beta c_1|\psi(k),
\end{aligned} \tag{3.17}$$

where $\psi(k)$ and m are defined in Theorem 3.1. Equality in (3.17) holds for $P(z)=z^n+k^n$.

We now turn to study some integral inequalities of Turán-type for the class of polynomials having all zeros in $|z| \leq k,\ k \leq 1$. In this direction, we first establish the following generalization of Theorem A.

Theorem 3.3 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z| \leq k,\ k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \geq k$ and for each $r > 0$,*

$$\left\{\int_0^{2\pi} |D_\beta P(\mathrm{e}^{\mathrm{i}\theta})|^r \mathrm{d}\theta\right\}^{1/r} \geq \frac{n(|\beta| - L)}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\left\{\int_0^{2\pi} |P(\mathrm{e}^{\mathrm{i}\theta})|^r \mathrm{d}\theta\right\}^{1/r}, \tag{3.18}$$

where L is defined as in (1.12), i.e.,

$$L = \frac{nk^2|c_n| + |c_{n-1}|}{n|c_n| + |c_{n-1}|}.$$

Proof If $Q(z) = z^n\overline{P(1/\overline{z})}$, then $P(z) = z^n\overline{Q(1/\overline{z})}$. It is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{3.19}$$

Using the definition of polar derivative of a polynomial $P \in \mathcal{P}_n$ with respect to complex number β with $|\beta| \geq k \geq L$, we have

$$\begin{aligned}\left|D_\beta P(z)\right| &= |nP(z) + (\beta - z)P'(z)| \\ &\geq |\beta|\left|P'(z)\right| - \left|nP(z) - zP'(z)\right|,\end{aligned} \tag{3.20}$$

which gives by using (3.19) for $|z| = 1$, that

$$\begin{aligned}\left|D_\beta P(z)\right| &\geq |\beta|\left|P'(z)\right| - \left|Q'(z)\right| \\ &\geq (|\beta| - L)|P'(z)|. \qquad \text{(by Lemma 2.4)}\end{aligned} \tag{3.21}$$

For any $r > 0$ and $0 \leq \theta < 2\pi$, from (3.21) we have

$$\left|D_\beta P(\mathrm{e}^{\mathrm{i}\theta})\right|^r \geq (|\beta| - L)^r \left|P'(\mathrm{e}^{\mathrm{i}\theta})\right|^r,$$

which equivalently gives,

$$\left\{\int_0^{2\pi} |D_\beta P(\mathrm{e}^{\mathrm{i}\theta})|^r d\theta\right\}^{1/r} \geq (|\beta| - L)\left\{\int_0^{2\pi} |P'(\mathrm{e}^{\mathrm{i}\theta})|^r d\theta\right\}^{1/r}. \tag{3.22}$$

By Lemma 2.6, we have for each z on $|z| = 1$ at which $P(z) \neq 0$,

$$|P'(z)| \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}|P(z)|. \tag{3.23}$$

Further, it is evident that this inequality follows trivially for those z on $|z| = 1$ at which $P(z) = 0$ as well. Also, from (3.23), we have for $0 \leq \theta < 2\pi$ and $r > 0$,

$$\left\{\int_0^{2\pi} |P'(\mathrm{e}^{\mathrm{i}\theta})|^r d\theta\right\}^{1/r} \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\left\{\int_0^{2\pi} |P(\mathrm{e}^{\mathrm{i}\theta})|^r d\theta\right\}^{1/r}. \tag{3.24}$$

The above inequality (3.24) in conjunction with (3.22) yields (3.18). This completes the proof of Theorem 3.3. □

Remark 3.3 If we take $k = 1$ in (3.18), we recover (1.16). Thus, Theorem 3.3 generalizes Theorem A. Note that for $0 < r < 1$ we work with quasi-norm.

Remark 3.4 Since $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ has all its zeros in $|z| \le k$, $k \le 1$, then

$$\frac{1}{n}\left|\frac{c_{n-1}}{c_n}\right| \le k,$$

which can also be taken as equivalent to

$$L = \frac{nk^2|c_n| + |c_{n-1}|}{n|c_n| + |c_{n-1}|} \le k. \tag{3.25}$$

Hence from Theorem 3.3, we get the following result.

Corollary 3.4 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \ge k$ and for each $r > 0$,*

$$\left\{\int_0^{2\pi} \left|D_\beta P(e^{i\theta})\right|^r d\theta\right\}^{1/r} \ge \frac{n(|\beta| - k)}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r}. \tag{3.26}$$

If we divide both sides of inequality (3.26) by $|\beta|$ and let $|\beta| \to +\infty$, we get the following result:

Corollary 3.5 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for each $r > 0$,*

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^r d\theta\right\}^{1/r} \ge \frac{n}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r}. \tag{3.27}$$

If we take $k = 1$ in (3.27), we get L^r-norm version of (1.7). Further, letting $r \to \infty$ in (3.26), we get the following refinement of (1.9).

Corollary 3.6 *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \ge k$,*

$$\max_{|z|=1}\left|D_\beta P(z)\right| \ge \frac{n(|\beta| - k)}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{3.28}$$

Remark 3.5 Since $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \ne 0$ in $|z| > k$, $k \le 1$, and if $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then

$$\left|\frac{c_0}{c_n}\right| = |z_1 z_2 \cdots z_n| = |z_1||z_2|\cdots|z_n| \le k^n.$$

In view of this fact inequality (3.28) refines (1.9). Furthermore, (3.28) reduces to (1.14) for $k = 1$.

As an application of Theorem 3.3, we prove the following more general result. One can observe that the above inequality (3.18) will be a consequence of a more fundamental inequality presented by the following theorem.

Theorem 3.4 *If* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \in \mathcal{P}_n$ *having all its zeros in* $|z| \le k,\ k \le 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \ge k$, $0 \le t \le 1$ *and for each* $r > 0$,

$$\left\{ \int_0^{2\pi} \left(\left| D_\beta P(\mathrm{e}^{\mathrm{i}\theta}) \right| - \frac{mnt|\beta|}{k^n} \right)^r \mathrm{d}\theta \right\}^{1/r} \ge \frac{n(|\beta| - A(t))}{1+k} \left\{ 1 + \frac{k}{n} \cdot \frac{k^n|c_n| - tm - |c_0|}{k^n|c_n| - tm + |c_0|} \right\} \left\{ \int_0^{2\pi} \left(\left| P(\mathrm{e}^{\mathrm{i}\theta}) \right| - \frac{tm}{k^n} \right)^r \mathrm{d}\theta \right\}^{1/r}, \tag{3.29}$$

where

$$A(t) = \frac{n\left(|c_n| - \frac{tm}{k^n}\right)k^2 + |c_{n-1}|}{n\left(|c_n| - \frac{tm}{k^n}\right) + |c_{n-1}|} \quad \textit{and} \quad m = \min_{|z|=k} |P(z)|.$$

Proof Since $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le k$, $k \le 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and the result follows from Theorem 3.3 in this case. Henceforth, we assume that all the zeros of $P(z)$ lie in $|z| < k$, so that $m > 0$. Now $m \le |P(z)|$ for $|z| = k$, therefore, if μ is any complex number such that $|\mu| < 1$, then

$$\left| m\mu \left(\frac{z}{k} \right)^n \right| < |P(z)| \quad \text{for } |z| = k.$$

Since, all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouché's theorem that all the zeros of $T(z) = P(z) - m\mu(z/k)^n$ also lie in $|z| < k$.

Applying Theorem 3.3 to $T(z)$, we have for $|\beta| \ge k \ge L'$, $0 \le \theta < 2\pi$ and for every $r > 0$,

$$\left\{ \int_0^{2\pi} \left| D_\beta T(\mathrm{e}^{\mathrm{i}\theta}) \right|^r \mathrm{d}\theta \right\}^{1/r} \ge \frac{n\left(|\beta| - L'\right)}{1+k} \left(1 + \frac{k}{n} \cdot \frac{k^n \left| c_n - \frac{\mu m}{k^n} \right| - |c_0|}{k^n \left| c_n - \frac{\mu m}{k^n} \right| + |c_0|} \right) \left\{ \int_0^{2\pi} \left| T(\mathrm{e}^{\mathrm{i}\theta}) \right|^r \mathrm{d}\theta \right\}^{1/r}, \tag{3.30}$$

where

$$L' = \frac{n\left|c_n - \frac{\mu m}{k^n}\right| k^2 + |c_{n-1}|}{n\left|c_n - \frac{\mu m}{k^n}\right| + |c_{n-1}|}.$$

Replacing $T(\mathrm{e}^{\mathrm{i}\theta})$ by $P(\mathrm{e}^{\mathrm{i}\theta}) - \mu m \mathrm{e}^{\mathrm{i}n\theta}/k^n$ and using the fact that

$$D_\beta T(\mathrm{e}^{\mathrm{i}\theta}) = D_\beta \left(P(\mathrm{e}^{\mathrm{i}\theta}) - \frac{\mu m \mathrm{e}^{\mathrm{i}n\theta}}{k^n} \right) = D_\beta P(\mathrm{e}^{\mathrm{i}\theta}) - \frac{1}{k^n} \mu m n \beta \mathrm{e}^{\mathrm{i}(n-1)\theta},$$

we have from (3.30) for $|\beta| \ge k$, $0 \le \theta < 2\pi$ and for every $r > 0$,

$$\left\{ \int_0^{2\pi} \left| D_\beta P(\mathrm{e}^{\mathrm{i}\theta}) - \frac{1}{k^n} \mu m n \beta \mathrm{e}^{\mathrm{i}(n-1)\theta} \right|^r \mathrm{d}\theta \right\}^{1/r} \ge \frac{n\left(|\beta| - L'\right)}{1+k} \left(1 + \frac{k}{n} \cdot \frac{k^n \left| c_n - \frac{\mu m}{k^n} \right| - |c_0|}{k^n \left| c_n - \frac{\mu m}{k^n} \right| + |c_0|} \right) \left\{ \int_0^{2\pi} \left| P(\mathrm{e}^{\mathrm{i}\theta}) - \frac{1}{k^n} \mu m \mathrm{e}^{\mathrm{i}n\theta} \right|^r \mathrm{d}\theta \right\}^{1/r}. \tag{3.31}$$

For every $\mu \in \mathbb{C}$ with $|\mu| < 1$, we have

$$\left| c_n - \frac{\mu m}{k^n} \right| \ge |c_n| - \frac{|\mu| m}{k^n} > |c_n| - \frac{m}{k^n}, \tag{3.32}$$

and $|c_n| > m/k^n$ by Lemma 2.7. Now, combining (3.32) and Lemma 2.8, we have for every $|\mu| < 1$,

$$L' = \frac{n\left|c_n - \frac{\mu m}{k^n}\right|k^2 + |c_{n-1}|}{n\left|c_n - \frac{\mu m}{k^n}\right| + |c_{n-1}|} \leq \frac{n\left(|c_n| - \frac{|\mu|m}{k^n}\right)k^2 + |c_{n-1}|}{n\left(|c_n| - \frac{|\mu|m}{k^n}\right) + |c_{n-1}|} = A(|\mu|). \tag{3.33}$$

Also, since the function

$$x \mapsto \frac{k^n x - |c_0|}{k^n x + |c_0|} \qquad (x \geq 0),$$

is a non-decreasing function of x, we get

$$\frac{k^n\left|c_n - \frac{\mu m}{k^n}\right| - |c_0|}{k^n\left|c_n - \frac{\mu m}{k^n}\right| + |c_0|} \geq \frac{k^n|c_n| - |\mu|m - |c_0|}{k^n|c_n| - |\mu|m + |c_0|}. \tag{3.34}$$

On using (3.33) and (3.34) in (3.31), we get for $|\beta| \geq k$, $0 \leq \theta < 2\pi$, and for every $r > 0$,

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|D_\beta P(\mathrm{e}^{\mathrm{i}\theta}) - \frac{1}{k^n}\mu m n\beta \mathrm{e}^{\mathrm{i}(n-1)\theta}\right|^r \mathrm{d}\theta\right\}^{1/r}\\ &\quad\geq \frac{n\left(|\beta| - A(|\mu|)\right)}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |\mu|m - |c_0|}{k^n|c_n| - |\mu|m + |c_0|}\right\}\\ &\quad\times\left\{\int_0^{2\pi}\left(\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| - \frac{1}{k^n}|\mu|m\right)^r \mathrm{d}\theta\right\}^{1/r}. \end{aligned} \tag{3.35}$$

It is a simple consequence of Laguerre theorem (cf. [14, p. 52]) on the polar derivative of a polynomial that for every β, with $|\beta| \geq k$, the polynomial

$$D_\beta\left(P(z) - \left(\frac{z}{k}\right)^n \mu m\right) = D_\beta P(z) - \frac{1}{k^n}\mu m n\beta z^{n-1} \tag{3.36}$$

has all its zeros in $|z| < k$, $k \leq 1$. This implies

$$\left|D_\beta P(z)\right| \geq \frac{1}{k^n} m n|\beta||z|^{n-1} \qquad \text{for} \quad |z| \geq k. \tag{3.37}$$

Because if (3.37) is not true, then there is a point $z = z_0$ with $|z_0| \geq k$, such that

$$\left|D_\beta P(z_0)\right| < \left|\frac{m n\beta z_0^{n-1}}{k^n}\right|.$$

We choose

$$\mu = \frac{k^n D_\beta P(z_0)}{m n\beta z_0^{n-1}},$$

so that $|\mu| < 1$, and with this choice of μ, from (3.36), we get

$$D_\beta\left(P(z_0) - \left(\frac{z_0}{k}\right)^n \mu m\right) = 0,$$

where $|z_0| \geq k$, which contradicts the fact that all the zeros of

$$D_\beta\left(P(z) - \left(\frac{z}{k}\right)^n \mu m\right)$$

lie in $|z| < k$, $k \leq 1$, for every μ with $|\mu| < 1$.

Now choosing the argument of μ suitably on the left hand side of (3.35) such that

$$\left|D_\beta P(z) - \frac{1}{k^n}\mu mn\beta z^{n-1}\right| = |D_\beta P(z)| - \frac{1}{k^n}mn|\mu||\beta| \qquad \text{for } |z| = 1,$$

which is possible by (3.37), we get for $|\beta| \geq k$, $|\mu| < 1$, $0 \leq \theta < 2\pi$, and for every $r > 0$,

$$\left\{\int_0^{2\pi}\left(\left|D_\beta P(e^{i\theta})\right| - \frac{1}{k^n}mn|\mu||\beta|\right)^r d\theta\right\}^{1/r}$$
$$\geq \frac{n\left(|\beta| - A(|\mu|)\right)}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - |\mu|m - |c_0|}{k^n|c_n| - |\mu|m + |c_0|}\right\}\left\{\int_0^{2\pi}\left(\left|P(e^{i\theta})\right| - \frac{1}{k^n}|\mu|m\right)^r d\theta\right\}^{1/r}. \tag{3.38}$$

For μ with $|\mu| = 1$, the above inequality follows by continuity. Putting $|\mu| = t$ in (3.38), we get

$$\left\{\int_0^{2\pi}\left(\left|D_\beta P(e^{i\theta})\right| - \frac{1}{k^n}mnt|\beta|\right)^r d\theta\right\}^{1/r}$$
$$\geq \frac{n\left(|\beta| - A(t)\right)}{1+k}\left\{1 + \frac{k}{n}\cdot\frac{k^n|c_n| - tm - |c_0|}{k^n|c_n| - tm + |c_0|}\right\}\left\{\int\limits_0^{2\pi}\left(\left|P\left(e^{i\theta}\right)\right| - \frac{1}{k^n}tm\right)^r d\theta\right\}^{1/r},$$

where $0 \leq t \leq 1$. This completes the proof of Theorem 3.4. □

Remark 3.6 Recall that $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ for $|z| > k,\ k \leq 1$. Here, we first show that

$$A(t) = \frac{n\left(|c_n| - \frac{tm}{k^n}\right)k^2 + |c_{n-1}|}{n\left(|c_n| - \frac{tm}{k^n}\right) + |c_{n-1}|} \leq k, \tag{3.39}$$

for $0 \leq t \leq 1$. We can assume without loss of generality that $P(z)$ has no zeros on $|z| = k$, for otherwise (3.39) holds trivially by (3.25). Now, as in the proof of Theorem 3.4, we have for every μ with $|\mu| < 1$, the polynomial

$$T(z) = P(z) - m\mu\left(\frac{z}{k}\right)^n = \left(c_n - \frac{\mu m}{k^n}\right)z^n + \sum_{\nu=0}^{n-1} c_\nu z^\nu$$

has all its zeros in $|z| < k,\ k \leq 1$, hence on applying (3.25) gives

$$\frac{n\left|c_n - \frac{\mu m}{k^n}\right|k^2 + |c_{n-1}|}{n\left|c_n - \frac{\mu m}{k^n}\right| + |c_{n-1}|} \leq k. \tag{3.40}$$

For every $\mu \in \mathbb{C}$ with $|\mu| < 1$, we have

$$\left|c_n - \frac{\mu m}{k^n}\right| \geq |c_n| - \frac{|\mu|m}{k^n} > |c_n| - \frac{m}{k^n}$$

and $|c_n| > m/k^n$ by Lemma 2.7. Therefore, it is possible to choose the argument of μ suitably, so that

$$\left|c_n - \frac{\mu m}{k^n}\right| = |c_n| - \frac{|\mu|m}{k^n},$$

we get from (3.40), that

$$\frac{n\left(|c_n| - \frac{|\mu|m}{k^n}\right)k^2 + |c_{n-1}|}{n\left(|c_n| - \frac{|\mu|m}{k^n}\right) + |c_{n-1}|} \leq k. \tag{3.41}$$

For μ with $|\mu| = 1$, the above inequality follows by continuity.

The inequality (3.39) follows by taking $|\mu| = t$ in (3.41), so that $0 \leq t \leq 1$.

Remark 3.7 Letting $r \to +\infty$ in (3.29), we get for $|\beta| \geq k$, and $0 \leq t \leq 1$,

$$\max_{|z|=1} \left|D_\beta P(z)\right| \geq n\, \frac{|\beta| - A(t)}{1+k} \max_{|z|=1} |P(z)| + \frac{mnt}{k^n} \cdot \frac{|\beta|k + A(t)}{1+k} + k\, \frac{|\beta| - A(t)}{1+k} \cdot \frac{k^n|c_n| - tm - |c_0|}{k^n|c_n| - tm + |c_0|} \left(\max_{|z|=1} |P(z)| - \frac{tm}{k^n}\right),$$

which is equivalent to

$$\begin{aligned}\max_{|z|=1} \left|D_\beta P(z)\right| \geq\ & n\, \frac{|\beta| - k}{1+k} \max_{|z|=1} |P(z)| + n\left(\frac{|\beta|+1}{k^{n-1}(1+k)}\right)tm \\ & + n\, \frac{k - A(t)}{1+k} \max_{|z|=1} |P(z)| + n\, \frac{A(t) - k}{k^n(1+k)}\, tm \\ & + k\, \frac{|\beta| - A(t)}{1+k} \cdot \frac{k^n|c_n| - tm - |c_0|}{k^n|c_n| - tm + |c_0|} \left(\max_{|z|=1} |P(z)| - \frac{tm}{k^n}\right).\end{aligned} \tag{3.42}$$

It is easy to show that

$$n\, \frac{k - A(t)}{1+k} \max_{|z|=1} |P(z)| + n\, \frac{A(t) - k}{k^n(1+k)}\, tm \geq 0,$$

which is equivalent to showing

$$n\, \frac{k - A(t)}{1+k} \max_{|z|=1} |P(z)| \geq \frac{n}{k^n} \cdot \frac{k - A(t)}{1+k}\, tm.$$

In view of (3.39), the above inequality becomes equivalent to

$$\max_{|z|=1} |P(z)| \geq \frac{tm}{k^n},$$

which is true by Lemma 2.7, and the fact that

$$\max_{|z|=1} |P(z)| = \max_{|z|=1} \left|\sum_{\nu=0}^{n} c_\nu z^\nu\right| \geq |c_n|.$$

Hence, inequality (3.42) represents an interesting refinement of the following result due to Dewan et al. [6].

Theorem C *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \geq k$, we have*

$$\max_{|z|=1} \left|D_\beta P(z)\right| \geq n\, \frac{|\beta| - k}{1+k} \max_{|z|=1} |P(z)| + n\, \frac{|\beta|+1}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)|.$$

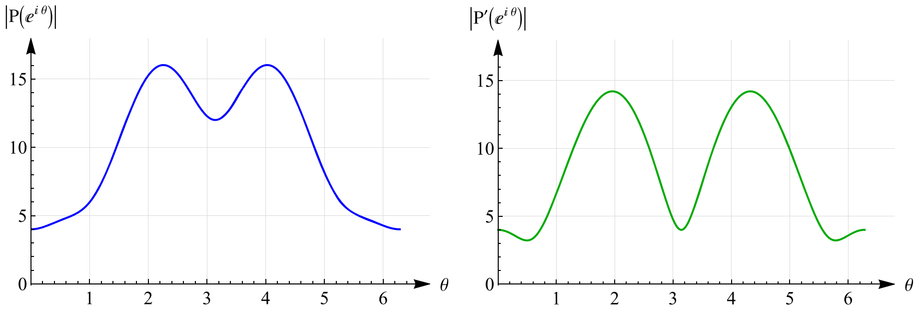

Fig. 1 Graphics of the functions $\theta \mapsto P\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (left) and $\theta \mapsto P'\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (right) for $0 \le \theta < 2\pi$

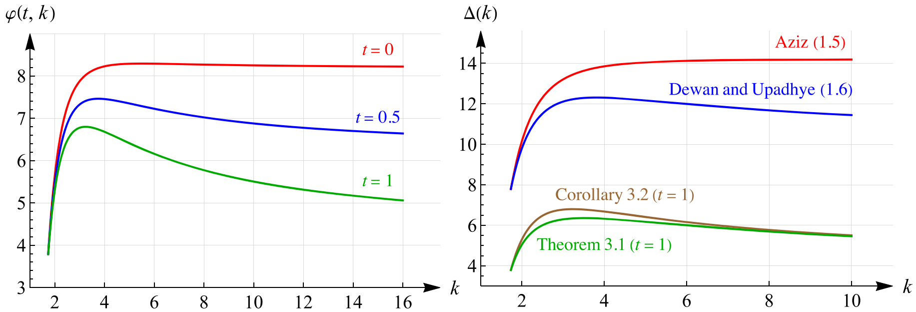

Fig. 2 (Left) The function $k \mapsto \varphi(t,k)$, $\sqrt{3} \le k \le 16$, for $t = 0$, $t = 1/2$, and $t = 1$; (Right) Comparison of the differences $k \mapsto \Delta(k)$ in the inequalities (1.5), (1.6), and (3.9) and (3.1) for $t = 1$

Remark 3.8 It is important to mention that the bound obtained in Theorem 3.4 is optimal when $t = 1$. However, the parameter t plays a vital role for making Theorem 3.4 more general, and to get different bounds from it by simply giving different values to t while varying in the closed unit interval $[0, 1]$ and without changing the hypothesis of this theorem. For example, for $t = 0$ (without assuming that $P(z)$ has a zero on $|z| = k$), we get (3.18).

4 Numerical examples

As an illustration of the obtained results, in this section we consider the following two examples.

Example 4.1 Let $P(z) = z^4 - 2z^3 + 6z - 9$, with all zeros $\left\{-\sqrt{3},\ \sqrt{3},\ 1-\sqrt{2}\,\mathrm{i},\ 1+\sqrt{2}\,\mathrm{i}\right\}$ on the circle $|z| = \sqrt{3}$, so that Theorem 3.1 holds for $k \ge \sqrt{3}$.

On the unit circle we have

$$|P(\mathrm{e}^{\mathrm{i}\theta})| = \sqrt{2[61 - 56\cos(\theta) - 12\cos(2\theta) + 24\cos(3\theta) - 9\cos(4\theta)]}$$

and

$$|P'(\mathrm{e}^{\mathrm{i}\theta})| = \sqrt{8[11 - 6\cos(\theta) - 9\cos(2\theta) + 6\cos(3\theta)]}$$

and their graphics for $0 \le \theta < 2\pi$ are presented in Fig. 1.

Since

$$M=\max_{|z|=1}|P(z)|=\max_{0\le\theta<2\pi}\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|=16.0217581269879\ldots$$

and

$$M_1=\max_{|z|=1}|P'(z)|=\max_{0\le\theta<2\pi}\left|P'\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|=14.199691411439\ldots,$$

as well as $m=\min_{|z|=k}|P(z)|=P(k)=(k^2-3)\big((k-1)^2+2\big)$, $k\ge\sqrt{3}$, we can consider the difference between the left and the right hand side in (3.9)

$$\varphi(t,k)=M_1-\left\{\frac{2M}{1+k^4}+\frac{1}{k^4}\cdot\frac{k^4-1}{k^4+1}(k^2-3)\big((k-1)^2+2\big)t\right\}\sum_{\nu=1}^{4}\frac{k}{k+|z_\nu|}-6\left(1-\frac{1}{k^2}\right),$$

Graphics of the functions $k\mapsto\varphi(t,k)$ for $t=0,1/2,1$ are presented in Fig. 2 (left).

In the same figure (right) we show the difference $k\mapsto\Delta(k)$ between the left and the right hand side in the inequalities (1.5) given by Aziz [1] and in (1.6) given by Dewan and Upadhye [5], as well as in the inequalities (3.9) for $(t=1)$ and in (3.1) from Theorem 3.1 for $t=1$.

Example 4.2 Let $P(z)=z^4+4$, with all zeros $\{1-\mathrm{i},\ 1+\mathrm{i},\ -1-\mathrm{i},\ -1+\mathrm{i}\}$ on the circle $|z|=\sqrt{2}$, so that Theorem 3.2 holds for $|z|\ge\sqrt{2}$. We take $\beta\in\mathbb{C}$ with $|\beta|=8$. Since, in this case,

$$M=\max_{|z|=1}|P(z)|=5\quad\text{and}\quad M_\beta=\max_{|z|=1}|D_\beta P(z)|=\max_{|z|=1}|4(4+z^3\beta)|=4(4+|\beta|)=48,$$

we consider the difference between M_β and the right hand side in the inequality (3.10),

$$\psi(t,k)=M_\beta-2\frac{8-k}{k^4+1}\left\{\left(1+\frac{k-1}{2}S_t(k)\right)M+\frac{1}{2k^4}\big(k^4-1-(k-1)S_t(k)\big)m\,t\right\}\frac{4k}{\sqrt{2}+k}$$
$$-16\left(1-\frac{1}{k^2}\right),$$

where $c_0=4,\ c_1=c_2=c_3=0,\ c_4=1$,

$$m=\min_{|z|=k}|P(z)|=k^4-4,\quad S_t(k)=\frac{(k^4-4)(1-t)}{k^4(1-t)+4(k+t)},\quad\sum_{\nu=0}^{4}\frac{k}{k+|z_\nu|}=\frac{4k}{\sqrt{2}+k}.$$

In Fig. 3 we present the difference $\Delta(k)$ between the left and right sides in the inequalities (1.10), (1.13) and (3.10) for $t=0$, $t=1/2$ and $t=1$, i.e.,

$$\Delta(k)=\begin{cases}M_\beta-\dfrac{4(|\beta|-k)}{k^4+1}M, & \text{Inequality (1.10)},\\[2mm] M_\beta-\dfrac{2(|\beta|-k)}{k^4+1}\left[M+\dfrac{1}{2k^4}(k^4-1)m\right]\dfrac{4k}{\sqrt{2}+k}, & \text{Inequality (1.13)},\\ \psi(0,k), & \text{Inequality (3.10)},\\ \psi(1/2,k), & \text{Inequality (3.10)},\\ \psi(1,k), & \text{Inequality (3.10)}.\end{cases}$$

Among these considered inequalities, from Fig. 3 we see that (3.10) is the best one obtained by Theorem 3.2 for $t=1$.

We mention here that some graphical illustrations can be also done for the L^r-inequalities.

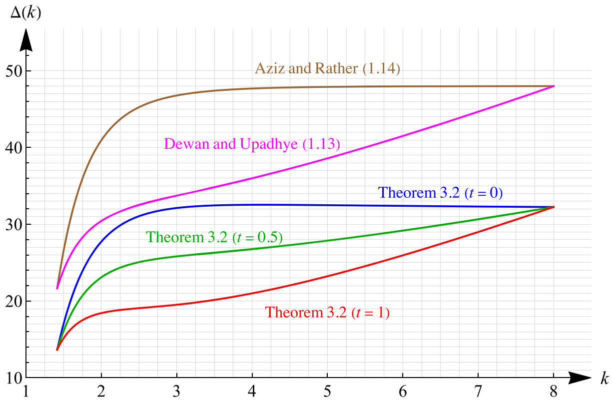

Fig. 3 Graphics $k \mapsto \Delta(k)$ for $\sqrt{2} \le k \le |\beta| = 8$ in different inequalities

5 Conclusions

In the past few years, a series of papers related to Turán-type inequalities for algebraic polynomials has been published and significant advances have been achieved. In this paper, we continue the study of this type of inequalities for polynomials, following up on a study started by various authors in the recent past. More precisely, some new inequalities of Turán-type are established that relate the supremum-norm of a univariate complex coefficient polynomial and its derivative on the unit disk. Besides, some classical Turán-type inequalities that relate the supremum-norm of the derivative and the polynomial on the unit disk are generalized to the L^r-norm of the polar derivative and the polynomial. Our results also include several interesting generalizations and refinements of some integral inequalities for algebraic polynomials. Two numerical examples are given in order to graphically illustrate and compare the obtained inequalities with some classical results.

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Declarations

Conflict of interest The authors declare no potential conflict of interests.

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