THE ARGUMENT PRINCIPLE & ITS IMPLICATIONS

Dissertation submitted to the Department of Mathematics

in partial fulfilment of the requirements for the award of

Master's Degree in Mathematics

by

Zubair Ahmmad Dar	(21068120018)
Zahida Banoo	(21068120041)
Syed Khushboo Farooq	(21068120071)
Ishtiyaq Ahmad Dar	(21068120074)

(Batch 2021)

under the supervision of

DR. MOHAMMAD ABDULLAH MIR

Associate Professor

Department of Mathematics

UNIVERSITY OF KASHMIR, HAZRATBAL, SRINAGAR

JAMMU & KASHMIR, 190006.

Department of Mathematics

University of Kashmir

(*NAAC Accredited "A+" University*)

Hazratbal, Srinagar, Jammu & Kashmir, 190006

CERTIFICATE

This is to certify that the dissertation entitled, **"THE ARGUMENT PRINCIPLE & ITS IMPLICATIONS"** *being submitted by the students with the enrollments* ***21068120018, 21068120041, 21068120071, 21068120074*** *to the* ***Department of Mathematics, University of Kashmir, Srinagar,*** *for the award of Master's degree in Mathematics, is an original project work carried out by them under my guidance and supervision.*

The project dissertation meets the standard of fulfilling the requirements of regulations related to the award of the Master's degree in Mathematics. The material embodied in the project dissertation has not been submitted to any other institute, or to this university for the award of Master's degree in Mathematics or any other degree.

DR. M. A. KHANDAY

PROFESSOR AND HEAD

Department of Mathematics

University of Kashmir, Srinagar

DR. MOHAMMAD ABDULLAH MIR

ASSOCIATE PROFESSOR

Department of Mathematics

University of Kashmir, Srinagar

(Supervisor)

Acknowledgement

All praise be to Almighty Allah, the Beneficent, the Merciful and the Lord of all that exists. It is by His grace and blessings that the present work has been accomplished.

We are extremely grateful to all those who have supported us throughout our project journey in completing the Master's degree in Mathematics. We would like to express our sincere gratitude to our worthy supervisor, Dr. Mohammad Abdullah Mir, for his unwaverig support, encouragement, cooperation, and the freedom he granted us throughout the project. Dr. Mohammad Abdullah Mir's demeanour, supervision and support during challenging times, the conducive environment he fostered, and the space he provided for our project work are truly commendable. Working with him has been a truly enriching experience. His guidance, patience and understanding have been instrumental in shaping our academic path and have played a crucial role in the development of the content presented in this project dissertation. We have been consistently inspired, enlightened, and motivated to pursue a complex and rewarding project journey under his mentorship.

We are deeply grateful to the faculty members of the department, especially Prof. N.A. Rather, Prof. S. Pirzada, Prof. M.H. Gulzar, Prof. B.A. Zargar, and Prof. M.A Khanday, for their warm companionship, which greatly constributed to making our stay a pleasant and an enriching experience in the department. We are also thankful to Dr. Ishfaq Ahmad Malik, Dr. M. Saleem Lone, Dr. Aijaz A Malla and Dr. Shahnawaz A Rather, who have been always there for their constant support and insightful guidance. Their collective wisdom and invaluable advice have served as a constant source of enlightenment and motivation throughout the journey.

We are grateful to the Head of Department of Mathematics, University of Kashmir and non-teaching staff for providing the required infrastructural facilities in the department and facilitating us in the official procedures.

Lastly, we express our deepest gratitude and utmost respect to our families whose tremendous moral support and unfaltering belief in us have been a constant source of strength throughout this entire endeavour. Their unwavering faith, love, care and sacrifices has been always invaluable for us.

October 2023 Zubair Ahmmad Dar (21068120018)

Zahida Banoo (21068120041)

Syed Khushboo Farooq (21068120071)

Ishtiyaq Ahmad Dar (21068120074)

Contents

Contents

Introduction

In the realm of complex analysis, the argument principle establishes a connection between the difference in the number of zeros and poles and the closed integral of a logarithmic derivative of an analytic function. It is a consequence of the residue theorem. It connects the winding number of a curve with the number of zeros and poles inside the curve. This is useful for applications (mathematical and otherwise) where we want to know the location of zeros and poles. The argument principle provides a tool in the form of Rouche's theorem to see how the number of zeros of analytic functions remains constant under small perturbations. We also look at a very important theorem in complex analysis, the Fundamental Theorem of Algebra and prove it using the Argument Principle. In addition, some more important applications are discussed in this project.

Chapter 1

Basic Notations and Preliminaries

In this chapter, we give some basic concepts and preliminaries which will be helpful for the reader to understand the subsequent chapter.

Section 1.1

Important Definitions

Differentiable functions: If $f(z)$ is single-valued in some region $\mathscr{R}$ of the z plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f(z)$ is differentiable at z.

Analytic functions: If the derivative $f'(z)$ exists at all points z of a region $\mathscr{R}$, then $f(z)$ is said to be analytic in $\mathscr{R}$ and is referred to as an analytic function in $\mathscr{R}$ or a function analytic in $\mathscr{R}$. The terms regular and holomorphic are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighborhood $\mid z - z_0 \mid < \delta$, at all points of which $f'(z)$ exists.

Entire Functions: A function that is analytic everywhere in the finite plane [i.e. everywhere except at ∞] is called an entire function or integral function. The functions e^z, $sinz$, $cosz$ are entire functions.

Meromorphic Functions: A function that is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Simply and Multiply Connected Domain: A pathwise-connected domain is said to be simply connected (also called 1-connected) if any simple closed curve can be shrunk to a point continuously in the set. If the domain is connected but not simply, it is said to be multiply connected.

Jordan Curve: Any continuous, closed curve that does not intersect itself and may or may not have a finite length, is called a Jordan curve.

Jordan Curve Theorem: A Jordan curve divides the plane into two regions having the curve as a common boundary. That region, which is bounded [i.e., is such that all points of it satisfy $|z| < M$ where M is some positive constant], is called the interior or inside of the curve, while the other region is called the exterior or outside of the curve.

Winding number: The winding number of a point with respect to a closed curve is an integer that represents the number of times the curve winds around the point in a counterclockwise direction. It indicates how many times the function evaluates to zero at that point.

Cauchy's Integral Theorem: Let $f(z)$ be analytic inside and on a simple closed curve C and let $f'(z)$ be continuous there. Then,

$$\int_C f(z)\,dz = 0.$$

Cauchy–Goursat Theorem: Let $f(z)$ be analytic in a region $\mathscr{R}$ and on its boundary C. Then

$$\int_C f(z)\,dz = 0.$$

This fundamental theorem, often called Cauchy's integral theorem or simply Cauchy's theorem, is valid for both simply and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f(z)$ be continuous in $\mathscr{R}$. However, Goursat gave a proof which removed this restriction. For this reason, the theorem is sometimes called the Cauchy–Goursat theorem when one desires to emphasize the removal of this restriction.

Section 1.2

Laurent's Theorem

Statement: Let $f(w)$ be analytic in the ring-shaped region between and on the two concentric circles $C_1 = \mid w-a \mid = r_1$ and $C_2 = \mid w-a \mid = r_2$ i.e. analytic for $r_1 \leq \mid w-a \mid \leq r_2$. Then for any point z in the ring-shaped region, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$
The importance of Laurent's Theorem is due to the fact that it enables us to classify various types of singularities of the function $f(z)$ as follows:

Types of singularities

The first series in the Laurent expansion of $f(z)$ i.e. $\sum_{n=0}^{\infty} a_n(z-a)^n$ is clearly analytic at $z=a$ and is therefore called the **analytic part of** $f(z)$, but the second series i.e. $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is not analytic at $z=a$. This part is, therefore, called the **singular part or the principal part of** $f(z)$ and depends on the coefficients b_n.

- If $b_n = 0 \; \forall \; n = 1,2,\ldots$, then the point $z=a$ is called a **removable singularity** of $f(z)$ and $f(z)$ becomes analytic at $z=a$ by defining $f(a)$ suitably.
- If $b_n = 0 \; \forall \; n > m$ i.e. $b_{m+1} = 0 = b_{m+2} = b_{m+3} = \ldots$, but $b_m \neq 0$, then $z=a$ is called a **pole of order or multiplicity m of** $f(z)$ and the principal part of $f(z)$

reduces to

$$\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_m}{(z-a)^m}.$$

- A pole of order 1 is called a **simple pole**.
- The first coefficient b_1 i.e. the coefficient of $\frac{1}{z-a}$ is then called the **residue** of $f(z)$ at the pole $z=a$ and is given by

$$b_1=\frac{1}{2\pi\imath}\int_{C_1} f(w)dw.$$

- If $b_n \neq 0$ for an infinite number of values of n, then $z=a$ is called an **essential singularity of** $f(z)$. This singularity is the most complicated singularity.

Section 1.3

The Residue Theorem

Statement: If $f(z)$ is analytic within and on a simple closed curve C, except at a finite number 'n' of poles within C, then

$$\int_C f(z)dz=2\pi\imath\sum_{j=1}^{n} R_j,$$

where R_j is the residue of $f(z)$ at a pole within C.

Proof. Let $a_1,a_2,\ldots,a_n$ be n poles of $f(z)$ within C. With each pole a_j as centre, draw circles γ_j of radius r so small that these circles lie entirely within C and does not touch each other. Then $f(z)$ is analytic in the region lying between C and the circles. So by Cauchy's theorem

$$\int_{C-(\gamma_1+\gamma_2+\cdots+\gamma_n)} f(z)dz=0,$$

or

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz. \tag{3.1}$$

Now

$$b_1 = \frac{1}{2\pi\imath}\int_C f(z)dz$$
$$\implies \int_C f(z)dz = (2\pi\imath)b_1.$$

Therefore,

$$\int_{\gamma_1} f(z)dz = (2\pi\imath)R_1, \text{ where } R_1 \text{ is the residue of } f(z) \text{ at } z = a.$$

Similarly

$$\int_{\gamma_n} f(z)dz = (2\pi\imath)R_n,\ n = 1,2,\ldots$$

From equation(3.1) we obtain

$$\begin{aligned}\int_C f(z)dz &= (2\pi\imath)R_1 + (2\pi\imath)R_2 + \cdots + (2\pi\imath)R_n \\ &= (2\pi\imath)\sum_{j=1}^{n} R_j.\end{aligned}$$

This completes the proof. □

Chapter 2

Argument Principle

Section 2.1

Argument Principle, First Version

To state and prove argument principle, we first state following theorem

***Theorem* 2.1.1.** *Given a closed rectifiable Jordan curve L, suppose $\phi(z)$ is analytic on $\overline{I(L)}$, while $f(z)$ is analytic on $\overline{I(L)}$ except for poles in $I(L)$ at the points $b_1, b_2, \ldots, b_n$. Moreover, suppose $f(z)$ has A-points $a_1, a_2, \ldots, a_m$ in $I(L)$, but none on L itself. Then*

$$\frac{1}{2\pi\imath}\int_L \phi(z)\frac{f'(z)}{f(z)-A}\,dz = \sum_{k=1}^{m}\alpha_k\phi(a_k) - \sum_{k=1}^{n}\beta_k\phi(b_k) \tag{1.1}$$

where α_k is the order of a_k and β_k the order of b_k.

The first term in the right-hand side of (1.1) is the sum of the values of $\phi(z)$ at the A-points of $f(z)$, where each value is repeated a number of times equal to the order of the A-point. If we assume that each A-point of $f(z)$ is counted a number of times equal to its order, then the sum

$$\sum_{k=1}^{m}\alpha_k\phi(a_k)$$

can be regarded as just the sum of the values of $\phi(z)$ at the A-points of $f(z)$. Similarly, the sum

$$\sum_{k=1}^{n} \beta_k \phi(b_k)$$

can be regarded as the sum of the values of $\phi(z)$ at the poles of $f(z)$, if each pole of $f(z)$ is counted a number of times equal to its order. With this convention (which will be in force from now on), we can paraphrase Theorem (2.1.1) as follows:

***Theorem* 2.1.2.** *Given a closed rectifiable Jordan curve L, suppose $\phi(z)$ is analytic on $\overline{I(L)}$, while $f(z)$ is analytic on $\overline{I(L)}$ except for poles in $I(L)$ and has no A-points on L. Then the integral*

$$\frac{1}{2\pi\imath}\int_L \phi(z)\frac{f'(z)}{f(z)-A}\,dz \tag{1.2}$$

equals the sum of the values of $\phi(z)$ at the A-points of $f(z)$ minus the sum of the values of $\phi(z)$ at the poles of $f(z)$.

Example 1: If $\phi(z) = z$, then

$$\frac{1}{2\pi\imath}\int_L \frac{zf'(z)}{f(z)-A}\,dz = \sum_{k=1}^{m}\alpha_k a_k - \sum_{k=1}^{n}\beta_k b_k, \tag{1.3}$$

i.e. the integral (1.2) is just the sum of the A-points of $f(z)$ inside L minus the sum of the poles of $f(z)$ inside L.

Example 2: If $\phi(z) = 1$, then

$$\frac{1}{2\pi\imath}\int_L \frac{f'(z)}{f(z)-A}\,dz = \sum_{k=1}^{m}\alpha_k - \sum_{k=1}^{n}\beta_k, \tag{1.4}$$

where the quantity on the right equals the number of A-points of $f(z)$ inside L minus the number of poles of $f(z)$ inside L.

Now suppose $f(z)$ has N zeros and P poles inside L, where each zero and pole is counted a number of times equal to its order, as already described. (In particular, N or P may vanish). Then, setting $A = 0$ in equation (1.4), we have

$$\frac{1}{2\pi\imath}\int_L \frac{\mathrm{d}}{\mathrm{dz}}\ln f(z)\,dz = \frac{1}{2\pi\imath}\int_L \frac{f'(z)}{f(z)}\,dz = N - P. \tag{1.5}$$

The integral on the left is called the logarithmic residue of $f(z)$ relative to the cantour L. In other words, the number of zeros of $f(z)$ inside L minus the number of poles of $f(z)$ inside L equals the logarithmic residue of $f(z)$ relative to L.

The logarithmic residue of $f(z)$ relative to L has a simple **geometric interpretation**. Choosing any point $z_0 \in L$ as the initial and final point of the path of integration, we make one circuit around L in the positive(i.e., counterclockwise) direction. Then $\ln f(z)$ varies continuously, and in general returns to z_0 with a value different from its original value at z_0. In fact, since

$$\ln f(z) = \log \mid f(z) \mid + \imath Arg f(z),$$

the change in $\ln f(z)$ is entirely due to the change in $Arg f(z)$. Letting Ω_0 denote the original value of $Arg f(z_0)$ and Ω_1 its value after the circuit around L, we have

$$\frac{1}{2\pi\imath}\int_L \frac{\mathrm{d}}{\mathrm{d}z}\ln f(z)\,dz = \frac{1}{2\pi\imath}[\ln \mid f(z) \mid + \imath\Omega_1] - \frac{1}{2\pi\imath}[\ln \mid f(z_0) \mid + \imath\Omega_0] = \frac{\Omega_1 - \Omega_0}{2\pi} \quad (1.6)$$

comparing equation(1.5) and equation(1.6), we find that

$$N - P = \frac{\Omega_1 - \Omega_0}{2\pi} = \frac{1}{2\pi}\Delta_L Arg f(z). \quad (1.7)$$

In other words, the number of zeros of $f(z)$ inside L minus the number of poles of $f(z)$ inside L equals $\frac{1}{2\pi}$ times the change in $Arg f(z)$ when the contour L is traversed once in the positive direction.

There is still another way of interpreting this result: As a variable point z describes the closed curve L once in the positive direction, the image point $w = f(z)$ describes a closed curve $L^* = f(L)$ in the w-plane, making some number v of complete circuits around the origin made in the positive direction is counted as $+1$ and every circuit made in the negative direction is counted as -1.

Making the obvious generalization of the case where $A \neq 0$ in equation(1.4), we summarize these results in the form of

Theorem* 2.1.3. *(Argument Principle) *Given a closed rectifiable Jordan curve L, suppose $f(z)$ is analytic on $\overline{I(L)}$ except for poles in $I(L)$ and has no A-points on L. Then the number of A-points of $f(z)$ inside L equals the number of circuits around the point $w = A$ made by the point $w = f(z)$ as the point z traverses the curve L once in the positive direction.*

Section 2.2

Argument Principle, Second Version

Theorem* 2.2.1. *(Argument Principle) *Let ϕ be meromorphic in a domain $D \subseteq \mathbb{C}$ and have only finitely many zeros and poles in D. If C is a simple closed contour in D such that no poles of ϕ lie on C, then*

$$\frac{1}{2\pi\imath}\int_C \frac{\phi'(z)}{\phi(z)}dz = N - P. \tag{2.8}$$

where N and P denote, respectively, the number of zeros and poles of ϕ inside C, each counted according to their order.

Proof. Define $F(z) = \frac{\phi'(z)}{\phi(z)}$. Then, the only possible singularities of F inside C are the zeros and poles of ϕ. Therefore

$$\frac{1}{2\pi\imath}\int_C F(z)\,dz = \sum Res\,[F(z);C]\,. \tag{2.9}$$

if a_j is a zero of order n_j of ϕ and if b_k is a pole of order p_k of ϕ, then it follows that

$$Res\left[\frac{\phi'(z)}{\phi(z)};a_j\right] = n_j \text{ and } Res\left[\frac{\phi'(z)}{\phi(z)};b_k\right] = -p_k.$$

Thus equation(2.9) becomes

$$\frac{1}{2\pi\imath}\int_C \frac{\phi'(z)}{\phi(z)}\,dz = \sum_j n_j - \sum_k p_k = N - P.$$

□

Remark 2.2.2. *If in addition, ψ is analytic on D, then under the hypothesis of Theorem (2.2.1) we easily see that*

$$\frac{1}{2\pi\imath}\int_C \psi(z)\frac{\phi'(z)}{\phi(z)}dz = \sum_j n_j\psi(a_j) - \sum_k p_k\psi(b_k),$$

where a_j and b_k are the zeros of order n_j and the poles of order p_k for ϕ, respectively.

Why is Theorem (2.2.1) known as an **argument principle**? Let us now restate Theorem (2.2.1) in terms of the properties of the logarithmic function $\log\phi(z)$. For this, under the hypotheses of Theorem (2.2.1), consider the transformation $w = \log\phi(z)$. Note that ϕ is analytic on C and $\phi(z) \neq 0$ in a neighborhood of C. For any analytic branch $\log\phi(z)$ of logarithm of $\phi(z)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(\log\phi(z)) = \frac{\phi'(z)}{\phi(z)}$$

and therefore,

$$\frac{1}{2\pi\imath}\int_C \frac{\phi'(z)}{\phi(z)}dz = \frac{1}{2\pi\imath}\int_C \mathrm{d}(log\phi(z))\,dz = \frac{1}{2\pi\imath}\Delta_C\log\phi(z). \tag{2.10}$$

We refer to this integral as the **logarithmic integral** of $\phi(z)$ along C. Here $\Delta_C\log\phi(z)$ denotes the increase in $\log\phi(z)$ when C is traversed once in the positive direction, and we say that the logarithmic integral measures the change of $\log\phi(z)$ along the cantour C. Now, we express

$$\log\phi(z) = \ln\mid\phi(z)\mid + \imath arg\,\phi(z),$$

where $\ln\mid\phi(z)\mid$ is single-valued and hence, $\Delta_C\ln\mid\phi(z)\mid$ returns to its original value when C is traversed. This observation implies that

$$\Delta_C\ln\mid\phi(z)\mid = \imath\Delta_C arg\,\phi(z).$$

Therefore, equation(2.10) yields

$$\frac{1}{2\pi\imath}\int_C \frac{\phi'(z)}{\phi(z)}dz = \frac{1}{2\pi}\Delta_C arg\phi(z).$$

where $\Delta_C arg\, \phi(z)$ is referred to as the increase in the argument of $\phi(z)$ along C. Thus, the **argument principle** can be restated as follows.

***Corollary* 2.2.3.** *Under the hypothesis of Theorem (2.8), we have*

$$\frac{1}{2\pi}\Delta_C arg \phi(z) = N - P.$$

***Corollary* 2.2.4.** *If ϕ is analytic inside and on a simple closed contour C and $\phi(z) \neq 0$ on C, then*

$$\frac{1}{2\pi}\Delta_C arg\, \phi(z) = N.$$

Section 2.3

Argument Principle, Third Version

***Theorem* 2.3.1.** *Let $f(z)$ be analytic inside and on simple closed curve C except at a finite number of poles $b_1, b_2, \ldots, b_m$ inside C, but no zero or pole on C. Then*

$$\frac{1}{2\pi\imath}\int_C \frac{f'(z)}{f(z)}dz = N - P,$$

where N is the number of zero and P is the number of poles , the zeros and the poles being counted according to their multiplicities.

Proof. Let $f(z)$ have zeros $a_1, a_2, \ldots, a_n$ of multiplicities $k_1, k_2, \ldots, k_n$ and poles $b_1, b_2, \ldots, b_m$ multiplicities $l_1, l_2, \ldots, l_m$ inside C and no zero or pole on C. Then we can write

$$f(z) = \frac{(z-a_1)^{k_1}(z-a_2)^{k_2}\ldots(z-a_n)^{k_n}}{(z-b_1)^{l_1}(z-b_2)^{l_2}\ldots(z-b_m)^{l_m}}\theta(z),$$

where $\theta(z)$ is analytic inside and on C with no zeros or poles.

Taking log on both sides, we get

$$\log f(z) = \sum_{r=1}^{n} k_r \log(z-a_r) - \sum_{s=1}^{m} l_s \log(z-b_s) + \log\theta(z).$$

Differentiating both sides w.r.t. z we get

$$\frac{f'(z)}{f(z)} = \sum_{r=1}^{n} \frac{k_r}{z-a_r} - \sum_{s=1}^{m} \frac{l_s}{z-b_s} + \frac{\theta'(z)}{\theta(z)}.$$

Multiplying both sides by $\frac{1}{2\pi\iota}$ and integrating over C, we get

$$\frac{1}{2\pi\iota}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi\iota}\sum_{r=1}^{n} k_r \int_C \frac{dz}{z-a_r} - \frac{1}{2\pi\iota}\sum_{s=1}^{m} l_s \int_C \frac{dz}{z-b_s} + \frac{1}{2\pi\iota}\int_C \frac{\theta'(z)}{\theta(z)}dz.$$

Since $\int_C \frac{dz}{z-\alpha} = \int_\Gamma \frac{dz}{z-\alpha} = 2\pi\iota$ (Γ being a circle of small radius inside C) for any α inside C and since $\theta(z) \neq 0$ and $\theta'(z)$ are analytic inside and on C so that $\frac{\theta'(z)}{\theta(z)}$ is analytic inside and on C and hence equals to zero by Cauchy's theorem, it follows that

$$\frac{1}{2\pi\iota}\int_C \frac{f'(z)}{f(z)}dz = \sum_{r=1}^{n} k_r - \sum_{s=1}^{m} l_s + 0 = \sum_{r=1}^{n} k_r - \sum_{s=1}^{m} l_s = N - P.$$

□

Remark 2.3.2. *If in the above theorem $f(z)$ has no poles inside or on C and no zeros on C, then*

$$\frac{1}{2\pi\iota}\int_C \frac{f'(z)}{f(z)}dz = N = \textit{Number of zeros of f(z) inside C.}$$

In other words, we have the following result known as the **Argument Principle**.

Theorem 2.3.3. ***(Argument Principle)*** *Let $f(z)$ be analytic inside and on a simple closed curve C having no zero on C. Then*

$$\frac{1}{2\pi\iota}\int_C \frac{f'(z)}{f(z)}dz = \textit{Number of zeros of f(z) inside C.}$$

Proof. Let $a_1, a_2, \ldots, a_m$ be the zeros of $f(z)$ inside C of multiplicity $n_1, n_2, \cdots, n_m$. Therefore total number of zeros of $f(z)$ inside C is equal to $n_1 + n_2 + \cdots + n_m$.
Now $z = a_r$ is a zero of $f(z)$ of multiplicity n_r, it follows by factor theorem that $(z-a_r)^{n_r}$ is a factor of $f(z)$, $\;r = 1, 2, \ldots, m$

Hence we write $f(z) = (z-a_1)^{n_1}(z-a_2)^{n_2}\cdots(z-a_m)^{n_m}\theta(z)$, where $\theta(z)$ is analytic inside & non zero inside & on C.

Taking log on both sides and using their properties we get

$$\log f(z) = n_1\log(z-a_1)+n_2\log(z-a_2)+\cdots+n_m log(z-a_m)+\log\theta(z).$$

Differentiating both sides w.r.t. z over C we get

$$\frac{f'(z)}{f(z)} = \frac{n_1}{z-a_1}+\frac{n_2}{z-a_2}+\cdots+\frac{n_m}{z-a_m}+\frac{\theta'(z)}{\theta(z)}.$$

Integrating both sides w.r.t z over C we get

$$\int_C \frac{f'(z)}{f(z)}dz = \sum_{r=1}^{m}\int_C \frac{n_r}{z-a_r}dz+\int_C \frac{\theta'(z)}{\theta(z)}dz. \tag{3.11}$$

Since $\theta(z)$ is analytic inside & on C. Therefore by Cauchy's integral formula $\theta'(z)$ is also analytic inside & on C.

Therefore, $\frac{\theta'(z)}{\theta(z)}$ being the quotient of two analytic functions with non zero denominator inside & on C, is analytic inside & on C. Hence by Cauchy's theorem

$$\int_C \frac{\theta'(z)}{\theta(z)}dz = 0.$$

Also for $r = 1,2,\ldots,m$ we have

$$\int_C \frac{n_r}{z-a_r}dz = \int_\Gamma \frac{n_r}{z-a_r}dz,$$

where Γ is a circle of radius ρ with centre a_r lying entirely inside C. Therefore

$$\int_C \frac{n_r}{z-a_r}dz = \int_0^{2\pi} \frac{n_r\rho\imath e^{\imath\theta}d\theta}{z-a_r}dz.$$

Hence it follows from equation(3.11) that

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}dz &= \sum_{r=1}^{m} 2\pi\imath.n_m\\ &= 2\pi\imath(n_1+n_2+\ldots+n_r)\end{aligned}$$

$$\implies \frac{1}{2\pi\imath}\int_C \frac{f'(z)}{f(z)}dz = \text{Number of zeros of f(z) inside C.}$$

□

Chapter 3

Applications

Some of the important applications of Argument principle are:

Section 3.1

Rouche's Theorem

Statement: let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $| g(z) |<| f(z) |$ on C. Then $f(z)$ and $f(z) \pm g(z)$ have the same number of zeros inside C.

Proof. We first note that $f(z) \neq 0$ on C. For if $f(z) = 0$ on C, then according to hypothesis $| g(z) |< |f(z)| = 0$ i.e. $| g(z) |< 0$ on C, which is absurd.
Similarly, $f(z) + g(z) \neq 0$ on C, because otherwise $g(z) = -f(z)$ on C so that $| g(z) |=| -f(z) |=| f(z) |$ on C, which is a contradiction to the hypothesis that $| g(z) |<| f(z) |$ on C.
Thus, we have by the Argument Principle applied to the analytic functions $f(z)$ and $f(z)+g(z)$,

$$\frac{1}{2\pi\imath} \int_C \frac{f'(z)}{f(z)} dz = \text{Number of zeros of } f(z) \text{ inside } C.$$

and

$$\frac{1}{2\pi\imath}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \text{Number of zeros of } f(z)+g(z) \text{ inside } C.$$

To prove the theorem, it is suffice to show that

$$\frac{1}{2\pi\imath}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi\imath}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz.$$

For any $\lambda \in [0,1]$ i.e. $0 \leq \lambda \leq 1$. We have on C

$$\begin{aligned} \mid f(z)+\lambda g(z) \mid &\geq \mid f(z) \mid - \mid \lambda g(z) \mid \\ &= \mid f(z) \mid - \lambda \mid g(z) \mid \\ &\geq \mid f(z) \mid - \mid g(z) \mid \\ &> 0. \end{aligned}$$

This shows that $f(z)+\lambda g(z) \neq 0$ on C. Therefore, by Argument Principle applied to the analytic function $f(z)+\lambda g(z)$, We have

$$\frac{1}{2\pi\imath}\int_C \frac{f'(z)+\lambda g'(z)}{f(z)+\lambda g(z)}\,dz = I(\lambda) \text{ (say)} = \text{Number of zeros of } f(z)+\lambda g(z) \text{ inside } C.$$

Clearly $I(\lambda)$ is a continous function of λ in [0,1] and it assumes only positive integral values. Since a continuous function in a closed interval assuming only rational values (and hence only positive integral values) is a constant, it follows that $I(\lambda)$ is a constant. Therefore, $I(0) = I(1)$ i.e.

$$\frac{1}{2\pi\imath}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi\imath}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz,$$

thereby proving that $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

Since $\breve{}g(z)$ is also analytic and $\mid -g(z) \mid = \mid g(z) \mid$, the above arguments shows that $f(z)$ and $f(z)-g(z)$ also have the same number of zeros inside C.

$\square$

Section 3.2

Evaluation of integrals

With the help of the Argument Principle we can evaluate certain integrals. For example consider the integral $\int_{|z|=2} \frac{3z^2}{z^3-1}\,dz$. If we take $f(z) = z^3 - 1$, then $f(z)$ is analytic $|\, z\,| \leq 2$, $f'(z) = 3z^2$ and $f(z)$ has 3 zeros $1, \frac{-1+\imath\sqrt{3}}{2}, \frac{-1-\imath\sqrt{3}}{2}$ in $|\, z\,| < 2$.
Therefore, by the Argument Principle,

$$\frac{1}{2\pi\imath}\int_{|z|=2} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi\imath}\int_{|z|=2} \frac{3z^2}{z^3-1}\,dz = \text{Number of zeros of } f(z) = z^3 - 1 \text{ in } |\, z\,| < 2.$$

which gives

$$\int_{|z|=2} \frac{3z^2}{z^3-1}\,dz = 2\pi\imath.3 = 6\pi\imath.$$

We can use Rouche's Theorem to prove following results

Section 3.3

Fundamental Theorem of Algebra

Statement: Every complex polynomial of degree n has exactly n zeros in C.

Proof. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be any complex polynomial of degree n so that $a_n \neq 0$, $a_{n-1}, \ldots, a_1, a_0$ are complex numbers.
We take $f(z) = a_n z^n$ and $g(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Then $f(z)$ and $g(z)$ being entire are analytic for $|\, z\,| \leq r$ for any $r > 0$, howsoever large.
Further, for $|\, z\,| = r$, we have

$$|\, f(z)\,| = |\, a_n z^n\,| = |\, a_n\,||\, z\,|^n = |\, a_n\,|\, r^n$$

and

$$\begin{aligned}
|g(z)| &= |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0| \\
&= |a_{n-1}|r^{n-1} + \cdots + |a_1|r + |a_0|\,.
\end{aligned}$$

Thus for $|z| = r$, We have

$$\begin{aligned}
\frac{|g(z)|}{|f(z)|} &\leq \frac{|a_{n-1}|r^{n-1} + \cdots + |a_1|r + |a_0|}{|a_n|r^n} \\
&= \frac{|a_{n-1}|}{|a_n|}\frac{1}{r} + \frac{|a_{n-2}|}{|a_n|}\frac{1}{r^2} + \cdots + \frac{|a_1|}{|a_n|} + \frac{|a_0|}{|a_n|}\frac{1}{r^n}.
\end{aligned}$$

letting $r \to \infty$ and noting that the RHS of above inequality tends to zero as $r \to \infty$, it follows that $\frac{|g(z)|}{|f(z)|} < 1$ for $|z| = r$ and for suitably chosen large r. Thus $|g(z)| < |f(z)|$ for $|z| = r$ for that r.
Hence, it follows by Rouche's theorem that $f(z)$ and $f(z)+g(z)$ have the same number of zeros in $|z| < r$. Since $f(z)$ has n zeros in $|z| < r$, all at $z = 0$. It follows that $f(z)+g(z)$ i.e, $p(z)$ also has n zeros in $|z| < r$ and, hence in the complex plane C. □

Section 3.4

First Part of Maximum Modulus Theorem

Statement: If $f(z)$ is analytic inside and on a simple closed curve C and $|f(z)| \leq M$ on C, then $|f(z)| \leq M$ inside C.

Proof. Suppose that $|f(z)| \leq M$ on C and $|f(z)| > M$ inside C. Then there exists a point z_0 inside C such that $|f(z_0)| > M$. Thus on C, $|f(z)| \leq M < |f(z_0)|$
Since $f(z)$ and $f(z_0)$ are analytic inside and on C, it follows by Rouche's theorem that $f(z_0)$ and $f(z_0) - f(z)$ have the same number of zeros inside C.
Since $|f(z_0)| > M > 0$, we have $|f(z_0)| \neq 0$.

Therefore, $f(z_0)-f(z)$ also has no zeros inside C which is contradiction, since $f(z_0)-f(z)$ has a zero $z=z_0$ inside C.
This contradiction shows that $\mid f(z) \mid \leq M$ inside C also. □

Section 3.5

Schwarz Lemma

Statement: Let $f(z)$ be analytic for $\mid z \mid \leq r$, $\mid f(z) \mid \leq M$ for $\mid z \mid = r$ and $f(0)=0$. Then $\mid f(z) \mid \leq \frac{M|z|}{r}$ for $\mid z \mid \leq r$.

Proof. Consider the function $F(z)=\frac{f(z)}{z}$. Then a possible singularity of $F(z)$ is the point $z=0$. But since $f(0)=0$, $z=0$ is a zero of $f(z)$ and hence $z-0=z$ is a factor of $f(z)$ so that the term z in the denominator of $F(z)$ cancels with the same factor of $f(z)$ in the numerator. Therefore $F(z)$ is analytic wherever $f(z)$ is analytic. Since $f(z)$ is analytic for $\mid z \mid \leq r$ by hypothesis , it follows that $F(z)$ is analytic for $\mid z \mid \leq r$.
Also for $\mid z \mid = r$, since $\mid f(z) \mid \leq M$ by hypothesis,

$$\mid F(z) \mid = \left| \frac{f(z)}{z} \right| = \frac{\mid f(z) \mid}{\mid z \mid} = \frac{\mid f(z) \mid}{r} \leq \frac{M}{r}.$$

Therefore, by Maximum Modulus Theorem

$$\begin{aligned} \mid F(z) \mid &\leq \frac{M}{r} \quad \text{for} \quad \mid z \mid < r, \\ \text{i.e.} \quad \frac{\mid f(z) \mid}{\mid z \mid} &\leq \frac{M}{r} \quad \text{for} \quad \mid z \mid < r, \\ \text{or} \quad \mid f(z) \mid &\leq \frac{M \mid z \mid}{r} \quad \text{for} \quad \mid z \mid < r. \end{aligned}$$

Since for $\mid z \mid = r$, $\frac{M|z|}{r} = M$ and $\mid f(z) \mid \leq M$ for $\mid z \mid = r$ by hypothesis, it follows that

$$\mid f(z) \mid \leq \frac{M \mid z \mid}{r} \quad \text{for} \quad \mid z \mid = r$$

Thus,

$$| f(z) | \leq \frac{M | z |}{r} \text{ for } | z | \leq r$$

and the result follows. □

Generalization of Schwarz lemma for unit circle

Statement: Let $f(z)$ be analytic for $| z | \leq 1$ and $| f(z) | \leq M$ for $| z |= 1$ and $f(a) = 0$, $| a |< 1$. Then $| f(z) | \leq M \left| \frac{z-a}{1-\bar{a}z} \right|$ for $| z | \leq 1$

Proof. For $a = 0$, the result reduces to Schwarz lemma.

Consider the function

$$g(z) = f(z) \left(\frac{1-\bar{a}z}{z-a} \right).$$

Since by hypothesis $f(a) = 0$, therefore $z = a$ is zero of $f(z)$. So by factor theorem $z = a$ is factor of $f(z)$. Therefore the factor $z - a$ in the denominator of $g(z)$ will cancel with the same factor of $f(z)$ in the numerator. Hence it follows that $g(z)$ is analytic whenever $f(z)$ is so.

Since by hypothesis $f(z)$ is analytic for $| z | \leq 1$, it follows that $g(z)$ is also analytic for $| z | \leq 1$.

Further for $| z |= 1$,

$$\begin{aligned} | g(z) | &= | f(z) | . \left| \frac{1-\bar{a}z}{z-a} \right| \\ &\leq M \left| \frac{1-\bar{a}z}{z-a} \right|. \end{aligned} \tag{5.1}$$

We will show that

$$\begin{aligned} \left| \frac{z-a}{1-\bar{a}z} \right| &= 1 \text{ for } | z |= 1 \\ &< 1 \text{ for } | z |< 1. \end{aligned} \tag{5.2}$$

To prove this, let us take $z = x + \imath y$, $a = \alpha + \imath\beta$ so that $| z |^2 = x^2 + y^2$, $| a |^2 = \alpha^2 + \beta^2$ and $\bar{a} = \alpha - \imath\beta$

Then

$$\begin{aligned}
\left|\frac{z-a}{1-\bar{a}z}\right|^2 &= \left|\frac{(x+\imath y)-(\alpha+\imath\beta)}{1-(\alpha-\imath\beta)(x+\imath y)}\right|^2 \\
&= \left|\frac{(x-\alpha)+\imath(y-\beta)}{(1-\alpha x-\beta y)+\imath(\beta x-\alpha y)}\right|^2 \\
&= \frac{(x-\alpha)^2+(y-\beta)^2}{(1-\alpha x-\beta y)^2+(\beta x-\alpha y)^2} \\
&= \frac{x^2-2\alpha x+\alpha^2+y^2-2\beta y+\beta^2}{1+\alpha^2x^2+\beta^2y^2-2\alpha x-2\beta y+2\alpha\beta xy+\beta^2x^2+\alpha^2y^2-2\alpha\beta xy} \\
&= \frac{(x^2+y^2)+(\alpha^2+\beta^2-2\alpha x-2\beta y)}{1+(\alpha^2+\beta^2)(x^2+y^2)-2\alpha x-2\beta y)} \\
&= \frac{|z|^2+|a|^2-2\alpha x-2\beta y}{1+|a|^2|z|^2-2\alpha x-2\beta y}.
\end{aligned}$$

Put $|z|=1$ we have

$$\begin{aligned}
\left|\frac{z-a}{1-\bar{a}z}\right|^2 &= \frac{1+|a|^2-2\alpha x-2\beta y}{1+|a|^2-2\alpha x-2\beta y} \\
&= 1.
\end{aligned}$$

Also for $|z|<1$, we have

$$\begin{aligned}
\left|\frac{z-a}{1-\bar{a}z}\right|^2 &< 1, \\
\text{if } \frac{|z|^2+|a|^2-2\alpha x-2\beta y}{1+|a|^2|z|^2-2\alpha x-2\beta y} &< 1, \\
\text{i.e. if } |z|^2+|a|^2-2\alpha x-2\beta y &< 1+|a|^2|z|^2-2\alpha x-2\beta y, \\
\text{i.e. if } |z|^2(1-|a|^2)z &< (1-|a|^2), \\
\text{i.e. if } |z|^2 &< 1 \quad (\text{because}|z|<1), \\
\text{i.e. if } |z| &< 1,
\end{aligned}$$

which is true by our supposition.

Thus

$$\begin{aligned}
\left|\frac{z-a}{1-\bar{a}z}\right| &= 1 \text{ for } z=1 \\
&< 1 \text{ for } |z|<1.
\end{aligned}$$

Using in equation(5.1) it follows that $\mid g(z) \mid \leq M$ for $\mid z \mid = 1$.

Therefore, it follows by Maximum Modulus Theorem that

$$|g(z)| \leq M \quad \text{for} |z| < 1,$$

$$\text{i.e.} \quad |f(z)| \left| \frac{1-\bar{a}z}{z-a} \right| \leq M,$$

$$\text{i.e.} \quad |f(z)| \leq M \left| \frac{z-a}{1-\bar{a}z} \right| \quad \text{for} \quad |z| < 1,$$

Since for $|z| = 1$, we have

$$\left| \frac{z-a}{1-\bar{a}z} \right| = 1 \quad \& \quad |f(z)| \leq M \quad \text{for} \quad |z| = 1.$$

Therefore

$$|f(z)| \leq M | \frac{z-a}{1-\bar{a}z} | \quad \text{for} \quad |z| = 1.$$

Hence, it follows that

$$|f(z)| \leq M \left| \frac{z-a}{1-\bar{a}z} \right| \quad \text{for} \quad |z| \leq 1.$$

□

Second Generalization

Statement: Let $f(z)$ be analytic for $\mid z \mid \leq 1$ and $\mid f(z) \mid \leq 1$ for $\mid z \mid = 1$ and $f(a) = 0$, $0 \leq a < 1$. Then $\mid f(z) \mid \leq \frac{|z|+a}{1+a|z|}$ for $\mid z \mid \leq 1$.

Proof. For $a = 0$, the result reduces to Schwarz lemma.

So let $0 < a < 1$.

Consider the function

$$g(z) = \frac{f(z)-a}{1-af(z)}. \tag{5.3}$$

Then

$$\begin{aligned} & 1 - af(z) = 0 \\ & \Longrightarrow f(z) = \frac{1}{a} \\ \Longrightarrow & |f(z)| = \frac{1}{|a|} = \frac{1}{a} > 0, \quad (\text{because} \quad 0 < a < 1), \end{aligned}$$

Since for $|z| \leq 1$, $|f(z)| \leq 1$. Therefore, it follows that $1 - af(z) \neq 0$ for $|z| \leq 1$

Thus $g(z)$ is the quotient of analytic functions with non zero determinant in $|z| \leq 1$.

Hence it follows that $g(z)$ is analytic for $|z| \leq 1$

Also

$$\begin{aligned} g(0) &= \frac{f(0)-a}{1-af(0)} \\ &= \frac{a-a}{1-a^2} \\ &= \frac{0}{1-a^2} \\ &= 0. \end{aligned}$$

Further, for $|z| = 1$

$$\begin{aligned} |g(z)| &= \left|\frac{f(z)-a}{1-af(z)}\right| \\ &= \left|\frac{f(z)-a}{1-\bar{a}f(z)}\right| \quad \text{(because a is real)} \\ &\leq 1 \ \text{ for } \ |f(z)| \leq 1. \end{aligned}$$

Thus $g(z)$ satisfies all the three conditions of Schwarz lemma for $|z| \leq 1$.

Therefore we conclude that

$$\begin{aligned} |g(z)| &\leq \frac{1.|z|}{1} \quad (\text{because } \ M = 1, r = 1) \\ \implies |g(z)| &\leq z \ \text{ for} |z| \leq 1. \end{aligned}$$

Let

$$g(z) = \rho e^{\imath\theta} = \rho\cos\theta + \imath\rho\sin\theta.$$

Then

$$|g(z)| = \rho$$

Hence

$$\rho \leq |z| \text{ for } |z| \leq 1. \tag{5.4}$$

Solving equation(5.3) for $g(z)$ interms of $f(z)$ we get

$$\begin{aligned}
g(z) - af(z)g(z) &= f(z) - a \\
\implies f(z)(1+g(z)) &= g(z) + a \\
\implies f(z) &= \frac{g(z)+a}{1+g(z)}. \\
\text{Therefore } |f(z)|^2 &= \left|\frac{g(z)+a}{1+g(z)}\right|^2 \\
\implies |f(z)|^2 &= \left|\frac{\rho\cos\theta + \iota\rho\sin\theta}{1+a\rho\cos\theta+\iota a\rho\sin\theta}\right|^2 \\
\implies |f(z)|^2 &= \frac{(\rho\cos\theta+a)^2+\rho^2\sin^2\theta}{(1+\rho\cos\theta)^2+a^2\rho^2\sin^2\theta} \\
&= \frac{\rho^2+a^2+2\rho a\cos\theta}{1+a^2\rho^2+2a\rho\cos\theta} \\
&= p(\theta)(\text{say}).
\end{aligned}$$

For maximum value of θ we put $p'(\theta) = 0$

$$\begin{aligned}
\frac{(1+a^2\rho^2+2a\rho\cos\theta)(-2a\rho\sin\theta)-(\rho^2+a^2+2\rho a\cos\theta)(-2a\rho\sin\theta)}{(1+a^2\rho^2+2a\rho\cos\theta))^2} &= 0 \\
\implies \frac{-2a\rho\sin\theta(1+a^2\rho^2-\rho^2-a^2)}{(1+a^2\rho^2+2a\rho\cos\theta)^2} &= 0 \\
\implies \sin\theta &= 0 \\
\implies \theta &= 0, \pi, 2\pi.
\end{aligned}$$

For $\theta = 0$, we have $p''(\theta) < 0$. Therefore, $p(\theta)$ is maximum for $\theta = 0$ and the maximum value is

$$p(0) = \frac{\rho^2+a^2+2\rho a}{1+a^2\rho^2+2a\rho} = \left(\frac{\rho+a}{1+a\rho}\right)^2$$

Thus $p(\theta) \leq p(0)$ i.e.

$$p(\theta) \leq \left(\frac{\rho+a}{1+a\rho}\right)^2$$

Hence it follows from above

$$|f(z)|^2 \leq \left(\frac{\rho+a}{1+a\rho}\right)^2$$

This implies

$$|f(z)| \le \frac{\rho + a}{1 + a\rho} \quad \text{for} \quad |z| \le 1. \tag{5.5}$$

Now the function

$$q(\rho) = \frac{\rho + a}{1 + a\rho}.$$

Therefore

$$\begin{aligned} q'(\rho) &= \frac{(1 + a\rho) - (\rho + a)a}{1 + a\rho} \\ &= \frac{1 - a^2}{(1 + a\rho)^2} \\ &> 0 \quad (\text{because} \quad a < 1). \end{aligned}$$

Therefore the function $q(\rho)$ is an increasing function of ρ. since by equation(5.4) $\rho \le |z|$ for $|z| \le 1$.

Therefore

$$q(\rho) \le q(z).$$

This implies

$$\frac{\rho + a}{1 + a\rho} \le \frac{|z| + a}{1 + a|z|}. \tag{5.6}$$

Combining equation(5.4) and equation(5.5) we obtain

$$|f(z)| \le \frac{|z| + a}{1 + a|z|} \text{ for } |z| \le 1.$$

Hence proved. $\square$

Chapter 4

Conclusion

Throughout the project we discussed three different versions and the proofs of Argument Principle. We also discussed different applications of Argument Principle. The proof of Rouche's Theorem and the evaluation of integrals saw the next major applications of the Argument Principle. We also looked at the Fundamental Theorem of Algebra which tells us that a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $a_n \neq 0$, $a_{n-1}, \ldots, a_1, a_0$ are complex, has n roots in the complex plane. There are various ways of proving this theorem but in this project we looked specifically at the proof using the Argument Principle. We also proved first part of Maximum Modulus Theorem by Rouche's theorem. Schwarz lemma and its generalizations are also discussed. These are just a few examples of the applications of the Argument principle which are discussed in project. In general, its utility extends to various areas of mathematics, physics, engineering, and beyond, where the study of complex functions and their properties is crucial.

Bibliography

[1] James Ward Brown and Ruel V Churchill. *Complex variables and applications*. McGraw-Hill,, 2009.

[2] John B Conway. *Functions of one complex variable II*, volume 159. Springer Science & Business Media, 2012.

[3] Harvir Singh Kasana. *Complex variables: theory and applications*. PHI Learning Pvt. Ltd., 2005.

[4] S. Ponnusamy. *Foundations of Complex Analysis*. Narosa Publishing House, 1995.

[5] Richard A Silverman. *Introductory complex analysis*. Courier Corporation, 2013.

[6] Murray R Spiegel. *Complex Variables with an Introduction to conformal Mapping and its Applications*. 2009.