Partial Differential Equations

T. Muthukumar
tmk@iitk.ac.in

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Notations

Symbols

Δ $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

∇ $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$

Ω denotes an open subset of $\mathbb{R}^n$, not necessarily bounded

$\partial\Omega$ denotes the boundary of Ω

D^α $\frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$

Function Spaces

$C^k(X)$ is the class of all k-times ($k \geq 1$) continuously differentiable functions on X

$C^\infty(X)$ is the class of all infinitely differentiable functions on X

$C_c^\infty(X)$ is the class of all infinitely differentiable functions on X with compact support

General Conventions

$B_r(x)$ denotes the open disk with centre at x and radius r

$S_r(x)$ denotes the circle or sphere with centre at x and radius r

w_n denotes the surface area of a n-dimensional sphere of radius 1.

Chapter 1

PDE: An Introduction

A partial differential equation (PDE) is an equation involving an unknown function u of two or more variables and some or all of its partial derivatives. The partial differential equation is usually a mathematical representation of problems arising in nature, around us. The process of understanding physical systems can be divided in to three stages:

(i) Modelling the problem or deriving the mathematical equation (in our case it would be formulating PDE). The derivation process is usually a result of conservation laws or balancing forces.

(ii) Solving the equation (PDE). What do we mean by a solution of the PDE?

(iii) Studying properties of the solution. Usually, we do not end up with a definite formula for the solution. Thus, how much information about the solution can one extract without any knowledge of the formula?

1.1 Definitions

Recall that the ordinary differential equations (ODE) dealt with functions of one variable, $u : \Omega \subset \mathbb{R} \to \mathbb{R}$. The subset Ω could have the interval form (a, b). The derivative of u at $x \in \Omega$ is defined as

$$u'(x) := \lim_{h \to 0} \frac{u(x+h) - u(x)}{h},$$

provided the limit exists. The derivative gives the slope of the tangent line at $x \in \Omega$. How to generalise this notion of derivative to a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$? These concepts are introduced in a course on multi-variable calculus. However, we shall jump directly to concepts necessary for us to begin this course.

Let Ω be an open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a given function. We denote the *directional derivative* of u at $x \in \Omega$, along a vector $\xi \in \mathbb{R}^n$, as

$$\frac{\partial u}{\partial \xi}(x) = \lim_{h\to 0} \frac{u(x+h\xi) - u(x)}{h},$$

provided the limit exists. The directional derivative of u at $x \in \Omega$, along the standard basis vectors $e_i = (0,0,\ldots,1,0,\ldots,0)$ is called the i-th *partial derivative* of u at x and is given as

$$u_{x_i} = \frac{\partial u}{\partial x_i}(x) = \lim_{h\to 0} \frac{u(x+he_i) - u(x)}{h}.$$

Similarly, one can consider higher order derivatives, as well. We now introduce Schwartz's multi-index notation for derivative, which will be used to denote a PDE in a concise form. A multi-index $\alpha \in \mathbb{Z}^n_+$ is a n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers and let $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If α and β are two multi-indices, then $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for all $1 \leq i \leq n$. Also, $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n)$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for any $x \in \mathbb{R}^n$. A k-degree polynomial in n variables can be represented as

$$\sum_{|\alpha|\leq k} a_\alpha x^\alpha.$$

The partial differential operator of *order* α is denoted as,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \ldots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}.$$

For $k \in \mathbb{N}$, we define $D^k u(x) := \{D^\alpha u(x) \mid |\alpha| = k\}$. Thus, for $k = 1$, we regard Du as being arranged in a vector,

$$\nabla = \left(D^{(1,0,\ldots,0)}, D^{(0,1,0,\ldots,0)}, \ldots, D^{(0,0,\ldots,0,1)}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

We call this the *gradient* vector. The dot product of the gradient vector with itself $\Delta := \nabla \cdot \nabla$ is called the *Laplacian* and $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Similarly, for

$k = 2$, we regard D^2 as being arranged in a matrix form (called the *Hessian* matrix),

$$D^2 = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2}{\partial x_2 \partial x_n} \\ & \ddots & \\ \frac{\partial^2}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2}{\partial x_n^2} \end{pmatrix}_{n \times n}.$$

The trace of the Hessian matrix is called the Laplace operator, denoted as $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Note that under some prescribed order on multi-indices α, $D^k u(x)$ can be regarded as a vector in R^{n^k}. Then $|D^k u| := (\sum_{|\alpha|=k} |D^\alpha u|^2)^{1/2}$. In particular, $|\nabla u| = (\sum_{i=1}^n u_{x_i}^2)^{1/2}$ and $|D^2 u| = (\sum_{i,j=1}^n u_{x_i x_j}^2)^{1/2}$.

Example 1.1. Let $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be $u(x, y) = ax^2 + by^2$. Then

$$\nabla u = (u_x, u_y) = (2ax, 2by).$$

$$D^2 u = \begin{pmatrix} u_{xx} & u_{yx} \\ u_{xy} & u_{yy} \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

Note that, for convenience, we can view $\nabla u : \mathbb{R}^2 \to \mathbb{R}^2$ and $D^2 u : \mathbb{R}^2 \to \mathbb{R}^4 = \mathbb{R}^{2^2}$, by assigning some ordering to the partial derivatives .

Definition 1.1.1. *Let Ω be an open subset of $\mathbb{R}^n$. A k-th* order *PDE F is a given map $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ having the form*

$$F\left(D^k u(x), D^{k-1} u(x), \ldots Du(x), u(x), x\right) = 0, \tag{1.1.1}$$

for each $x \in \Omega$ and $u : \Omega \to \mathbb{R}$ is the unknown.

A general first order PDE is of the form $F(Du(x), u(x), x) = 0$ and, in particular, for a two variable function $u(x, y)$ the PDE will be of the form $F(u_x, u_y, u, x, y) = 0$. If $u(x, y, z)$ is a three variable function, then the PDE is of the form $F(u_x, u_y, u_z, u, x, y, z) = 0$. A general second order PDE is of the form $F(D^2 u(x), Du(x), u(x), x) = 0$.

As we know the PDE is a mathematical description of the behaviour of the associated system. Thus, our foremost aim is to solve the PDEs for the unknown function u, usually called the solution of the PDE. The first expected notion of solution is as follows:

Definition 1.1.2. *We say $u : \Omega \to \mathbb{R}$ is a* solution *(in the classical sense) to the k-th order PDE (1.1.1),*

- *if u is k-times differentiable with the k-th derivative being continuous*
- *and u satisfies the equation* (1.1.1).

Example 1.2. Consider the equation $u_x(x,y) = 0$. If we freeze the variable y, the equation is very much like an ODE. Integrating both sides, we would get $u(x,y) = f(y)$ as a solution, where f is any arbitrary function of y. Thus, the family of solution depends on the choice of $f \in C^1$. Similarly, the solution of $u_y(x,y) = 0$ is $u(x,y) = g(x)$ for any choice of $g \in C^1$.

Example 1.3. Consider the equation $u_t(x,t) = u(x,t)$. If we freeze the variable x, the equation is very much like an ODE. Integrating both sides, we would get $u(x,t) = f(x)e^t$ as a solution, where f is any arbitrary function of x. Thus, the family of solution depends on the choice of f.

Example 1.4. Let us solve for u in the equation $u_{xy} = 4x^2y$. Unlike previous example, the PDE here involves derivatives in both the variable. Still one can solve this PDE for a general solution. We first integrate w.r.t x both sides to get $u_y = (4/3)x^3y + f(y)$. Then, integrating again w.r.t y, we get $u(x,y) = (2/3)x^3y^2 + F(y) + g(x)$, where $F(y) = \int f(y)\,dy$.

Example 1.5. Consider the equation $u_x(x,y) = u_y(x,y)$. At first look this doesn't look simple for solving directly. But a change of coordinates rewrites the equation in a simpler form. If we choose the coordinates $w = x + y$ and $z = x - y$, we have $u_x = u_w + u_z$ and $u_y = u_w - u_z$. Substituting this, we get $u_z(w,z) = 0$ which is the form considered in Example 1.2. Hence its solution is $u(w,z) = g(w)$ for $g \in C^1$ and, hence, $u(x,y) = g(x+y)$.

Observe in the above examples that, in contrast to ODE, the family of solutions of a PDE may be indexed by a function rather than a constant.

Exercise 1. Determine a and b so that $u(x,y) = e^{ax+by}$ is a solution to

$$u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0.$$

Exercise 2. Determine the relation between a and b if $u(x,y) = f(ax+by)$ is a solution to $3u_x - 7u_y = 0$ for any differentiable function f such that $f'(z) \neq 0$ for all real z. (Answer: $a = 7b/3$).

It is not necessary that the general form a solution has unique represenation. Note that in the example below we have three different family of solutions for the same PDE and we may have more!

Example 1.6. Consider the PDE $u_t(x,t) = u_{xx}(x,t)$.

(i) Note that $u(x,t) = c$ is a solution of the PDE, for any constant $c \in \mathbb{R}$. Thus, we have a family of solutions depending on c.

(ii) The function $u : \mathbb{R}^2 \to \mathbb{R}$ defined as $u(x,t) = (1/2)x^2 + t + c$, for a given constant $c \in \mathbb{R}$, is a solution of the PDE $u_t = u_{xx}$. Because $u_t = 1$, $u_x = x$ and $u_{xx} = 1$. We have another family of solutions for the same PDE.

(iii) Note that $u(x,t) = e^{ax+bt}$ is a solution to the PDE $u_t = u_{xx}$ if $b = a^2$. Note that $u_t = bu$, $u_x = au$ and $u_{xx} = a^2 u$.

As of today, there is no universal way of solving a given PDE. Thus, the PDE's have to be categorised based on some common properties, all of which may be expected to have a common technique to solve. One such classification is given below.

Definition 1.1.3. *We say F is* linear *if* (1.1.1) *has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

for given functions f and a_α ($|\alpha| \leq k$). If $f \equiv 0$, we say F is homogeneous. F is said to be semilinear*, if it is linear only in the highest order, i.e., F has the form*

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

We say F is quasilinear *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x), \ldots, Du(x), u(x), x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0,$$

i.e., the highest order derivative coefficient contains derivative only upto the previous order. Finally, we say F is fully nonlinear *if it depends nonlinearly on the highest order derivatives.*

Note that the classification is heirarchical, i.e., we have the inclusion

$$\text{linear} \subset \text{semilinear} \subset \text{quasilinear} \subset \text{fully nonlinear}.$$

Thus, common sense tells that we classify a PDE based on the smallest class it sits in the heirarchy.

Example 1.7. (i) $a_1(x)u_{xx} + a_2(x)u_{xy} + a_3(x)u_{yy} + a_4(x)u_x + a_5(x)u_y = a_6(x)u$ is linear.

(ii) $xu_y - yu_x = u$ is linear.

(iii) $xu_x + yu_y = x^2 + y^2$ is linear.

(iv) $u_{tt} - c^2 u_{xx} = f(x, t)$ is linear.

(v) $y^2 u_{xx} + xu_{yy} = 0$ is linear.

(vi) $u_x + u_y - u^2 = 0$ is semilinear.

(vii) $u_t + uu_x + u_{xxx} = 0$ is semilinear.

(viii) $u_{tt}^2 + u_{xxxx} = 0$ is semilinear.

(ix) $u_x + uu_y - u^2 = 0$ is quasilinear.

(x) $uu_x + u_y = 2$ is quasilinear.

(xi) $u_x u_y - u = 0$ is nonlinear.

Example 1.8 (Examples of Linear PDE). **Transport Equation** $u_t(x, t) + b \cdot \nabla_x u(x, t) = 0$ for some given $b \in \mathbb{R}^n$ assuming that $x \in \mathbb{R}^n$.

Laplace Equation $\Delta u = 0$.

Poisson Equation $\Delta u(x) = f(x)$.

Poisson Equation $\Delta u(x) = f(u)$.

Helmholtz Equation $\Delta u + k^2 u = 0$, for a given constant k.

Heat Equation $u_t - \Delta u = 0$.

Kolmogorov's Equation $u_t - A \cdot D^2 u + b \cdot \nabla u = 0$, for given $n \times n$ matrix $A = (a_{ij})$ and $b \in \mathbb{R}^n$. The first scalar product is in R^{n^2} and the second is in $\mathbb{R}^n$.

Wave Equation $u_{tt} - \Delta u = 0$

General Wave Equation $u_{tt} - A \cdot D^2 u + b \cdot \nabla u = 0$, for given $n \times n$ matrix $A = (a_{ij})$ and $b \in \mathbb{R}^n$. The first scalar product is in R^{n^2} and the second is in $\mathbb{R}^n$.

Schrödinger Equation $iu_t + \Delta u = 0$.

Airy's Equation $u_t + u_{xxx} = 0$.

Beam Equation $u_t + u_{xxxx} = 0$.

Example 1.9 (Examples of Nonlinear PDE). **Inviscid Burgers' Equation** $u_t + uu_x = 0$, for $x \in \mathbb{R}$

Eikonal Equation $|\nabla u(x)| = f(x)$ is a first order nonlinear equation. This equation arises in geometrical optics, optimal control and computer vision etc. In fact, the name "eikon" is a greek word for image.

Hamilton-Jacobi Equation $u_t + H(\nabla u, x) = 0$.

Minimal Surface Equation

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = f(x)$$

arises in geometry. The graph of the solution u defined on the domain Ω (say convex domain, for simplicity) has the given mean curvature $f : \Omega \to \mathbb{R}$. When $f \equiv 0$ the equation is called *minimal surface equation*. This is a second order elliptic type PDE.

Image Processing A degenerate elliptic equation

$$\mathbf{u} - \lambda \nabla \cdot \left(\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} \right) = \mathbf{f}(x)$$

is a system of three equations where the solution $\mathbf{u} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$, which measures the intensity of red, green and blue pixels iin a coloured image. The problem is given a noisy image $\mathbf{f} : \Omega \to \mathbb{R}^3$, we seek a de-noised image $\mathbf{u}$ by diffusing the noise in directions parallel to the image edges. The $\lambda > 0$ is a diffusive scaling.

Monge-Ampére Equation $\det(D^2 u) = f(x, u, \nabla u)$ is a fully nonlinear PDE encountered in optimal transport problems. The gradient of the solution u, ∇u, maps optimal transportation path.

Schrödinger Equation $iu_t + \Delta u - V(u)u = 0$. The solution $u : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{C}$, a wavefunction, is associated with a particle of mass m and driven by a potential $V(u)$.

Korteweg de Vries (KdV) Equation $u_t + u_x + uu_x + u_{xxx} = 0$.

Exercise 3. Classify all the important PDE listed in Example 1.8.

Exercise 4. Classify the PDEs in the hierarchy of linearity:

(i) $(y - u)u_x + xu_y = xy + u^2$.

(ii) $uu_x^2 - xu_y = \frac{2}{x}u^3$.

(iii) $x^2 u_x + (y - x)u_y = y \sin u$.

(iv) $(\sin y)u_x - e^x u_y = e^y u$.

(v) $u_x + \sin(u_y) = u$.

(vi) $uu_x + x^2 u_{yyy} + \sin x = 0$.

(vii) $u_x + e^{x^2} u_y = 0$.

(viii) $u_{tt} + (\sin y)u_{yy} - e^t \cos y = 0$.

(ix) $x^2 u_{xx} + e^x u = xu_{xyy}$.

(x) $e^y u_{xxx} + e^x u = -\sin y + 10xu_y$.

(xi) $y^2 u_{xx} + e^x uu_x = 2xu_y + u$.

(xii) $u_x u_{xxy} + e^x uu_y = 5x^2 u_x$.

(xiii) $u_t = k^2(u_{xx} + u_{yy}) + f(x, y, t)$.

(xiv) $x^2 u_{xxy} + y^2 u_{yy} - \log(1 + y^2)u = 0$.

(xv) $u_x + u^3 = 1$.

(xvi) $u_{xxyy} + e^x u_x = y$.

(xvii) $uu_{xx} + u_{yy} - u = 0$.

(xviii) $u_{xx} + u_t = 3u$.

Exercise 5. Rewrite the following PDE in the new coordinates v and w.

(i) $u_x + u_y = 1$ for $v = x + y$ and $w = x - y$.

(ii) $au_t + bu_x = u$ for $v = ax - bt$ and $w = t/a$ where $a, b \neq 0$.

(iii) $au_x + bu_y = 0$ for $v = ax + by$ and $w = bx - ay$, where $a^2 + b^2 > 0$.

(iv) $u_{tt} = c^2 u_{xx}$ for $v = x + ct$ and $w = x - ct$.

(v) $u_{xx} + 2u_{xy} + u_{yy} = 0$ for $v = x$ and $w = x - y$.

(vi) $u_{xx} - 2u_{xy} + 5u_{yy} = 0$ for $v = x + y$ and $w = 2x$.

(vii) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$ for $v = y - 2x$ and $w = x$. (should get $u_{ww} = 0$).

(viii) $u_{xx} + 2u_{xy} - 3u_{yy} = 0$ for $v = y - 3x$ and $w = x + y$.

1.2 Well-Posedness of PDE

We have seen through examples that a given PDE may one, many or no solution(s). The choice of our solution depends on the motivation for the study. One needs to enforce extra side conditions to a *particular solution*. These extra conditions are called boundary conditions (boundary value problem) or whose value at initial time is known(initial value problem). It is usually desirable to solve a *well-posed* problem, in the sense of Hadamard . By well-posedness we mean that the PDE along with the boundary condition (or initial condition)

(a) has a solution (existence)

(b) the solution is unique (uniqueness)

(c) the solution depends continuously on the data given (stability).

Any PDE not meeting the above criteria is said to be *ill-posed*. If too many initial/boundary conditions are specified, then the PDE will have *no solution*. If too few initial/boundary conditions are specified, then the PDE will not have unique solution. However, if the right amount of initial/boundary conditions are specified, but at wrong places, then the solution may exist and be unique except that that it may not depend continuously on the initial or boundary data.

Example 1.10. The initial value problem

$$\begin{cases} u_{tt} & = u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) & = u_t(x,0) = 0 \end{cases}$$

has the trivial solution $u(x,t) = 0$. We consider the same problem with a small change in data, i.e.,

$$\begin{cases} u_{tt} & = u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) & = 0 \\ u_t(x,0) & = \varepsilon \sin\left(\frac{x}{\varepsilon}\right) \end{cases}$$

which has the solution $u_\varepsilon(x,t) = \varepsilon^2 \sin(x/\varepsilon) \sin(t/\varepsilon)$. Since

$$\sup_{(x,t)}\{|u_\varepsilon(x,t) - u(x,t)|\} = \varepsilon^2 \sup_{(x,t)} \{|\sin(x/\varepsilon)\sin(t/\varepsilon)|\} = \varepsilon^2$$

the problem is well-posed.

Example 1.11 (Ill-posed). The initial value problem

$$\begin{cases} u_{tt} & = -u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) & = u_t(x,0) = 0 \end{cases}$$

has the trivial solution $u(x,t) = 0$. We consider the same problem with a small change in data, i.e.,

$$\begin{cases} u_{tt} & = -u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) & = 0 \\ u_t(x,0) & = \varepsilon \sin\left(\frac{x}{\varepsilon}\right) \end{cases}$$

which has the solution $u^\varepsilon(x,t) = \varepsilon^2 \sin(x/\varepsilon) \sinh(t/\varepsilon)$. Since

$$\sup_{(x,t)}\{|u_t^\varepsilon(x,t) - u_t(x,t)|\} = \varepsilon \sup_{(x,t)} \{|\sin(x/\varepsilon)|\} = \varepsilon$$

and

$$\begin{aligned} \lim_{t\to\infty} \sup_{x}\{|u^\varepsilon(x,t) - u(x,t)|\} &= \varepsilon \lim_{t\to\infty} \sup_{x} \{|\sin(x/\varepsilon)|\} \\ &= \lim_{t\to\infty} \varepsilon^2 |\sinh(t/\varepsilon)| = \infty, \end{aligned}$$

the problem is ill-posed. Because a small change in initial data leads to a large change in the solution.

Exercise 6. Consider the initial value problem

$$u_t + u_{xx} = 0 \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

with $u(x,0) = 1$.

(a) Show that $u(x,t) \equiv 1$ is a solution to this problem.

(b) Show that $u_n(x,t) = 1 + \frac{e^{n^2 t}}{n} \sin(nx)$ is a solution to the problem with initial value

$$u(x,0) = 1 + \frac{\sin(nx)}{n}.$$

(c) Find $\sup_x \{|u_n(x,0) - 1|\}$.

(d) Find $\sup_x \{|u_n(x,t) - 1|\}$.

(e) Show that the problem is ill-posed.

The fundamental question is, given a PDE, find these extra conditions that make a PDE well-posedness.

1.3 Three Basic PDE: History

The study of partial differential equations started as a tool to analyse the models of physical science. The PDE's usually arise from the physical laws such as balancing forces (Newton's law), momentum, conservation laws etc. The first PDE was introduced in 1752 by d'Alembert as a model to study vibrating strings. He introduced the one dimensional *wave* equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

This was then generalised to two and three dimensions by Euler (1759) and D. Bernoulli (1762), i.e.,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t),$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$.

In physics, a *field* is a physical quantity associated to each point of space-time. A field can be classified as a scalar field or a vector field according to whether the value of the field at each point is a scalar or a vector, respectively. Some examples of field are Newton's gravitational field, Coulomb's electrostatic field and Maxwell's electromagnetic field.

Given a vector field V, it may be possible to associate a scalar field u, called *potential*, such that $\nabla u = V$. Moreover, the gradient of any function u, ∇u is a vector field. In gravitation theory, the gravity potential is the potential energy per unit mass. Thus, if E is the potential energy of an object with mass m, then $u = E/m$ and the potential associated with a mass distribution is the superposition of potentials of point masses.

The Newtonian gravitation potential can be computed to be

$$u(x) = \frac{1}{4\pi} \int_\Omega \frac{\rho(y)}{|x-y|}\, dy$$

where $\rho(y)$ is the density of the mass distribution, occupying $\Omega \subset \mathbb{R}^3$, at y. In 1782, Laplace discovered that the Newton's gravitational potential satisfies the equation:

$$\Delta u = 0 \quad \text{on } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Thus, the operator $\Delta = \nabla \cdot \nabla$ is called the *Laplacian* and any function whose Laplacian is zero (as above) is said to be a *harmonic* function.

Later, in 1813, Poisson discovered that on Ω the Newtonian potential satisfies the equation:

$$-\Delta u = \rho \quad \text{on } \Omega.$$

Such equations are called the *Poisson* equation. The identity obtained by Laplace was, in fact, a consequence of the conservation laws and can be generalised to any scalar potential. Green (1828) and Gauss (1839) observed that the Laplacian and Poisson equations can be applied to any scalar potential including electric and magnetic potentials. Suppose there is a scalar potential u such that $V = \nabla u$ for a vector field V and V is such that $\int_{\partial\gamma} V \cdot \nu \, d\sigma = 0$ for all closed surfaces $\partial\gamma \subset \Gamma$. Then, by Gauss divergence theorem[1] (cf. Appendix D), we have

$$\int_\gamma \nabla \cdot V \, dx = 0 \quad \forall \gamma \subset \Gamma.$$

[1] a mathematical formulation of conservation laws

Thus, $\nabla \cdot V = \text{div}V = 0$ on Γ and hence $\Delta u = \nabla \cdot (\nabla u) = \nabla \cdot V = 0$ on Γ. Thus, the scalar potential is a harmonic function. The study of potentials in physics is called *Potential Theory* and, in mathematics, it is called Harmonic Analysis. Note that, for any potential u, its vector field $V = \nabla u$ is *irrotational*, i.e., $\text{curl}(V) = \nabla \times V = 0$.

Later, in 1822 J. Fourier on his work on heat flow in *Théorie analytique de la chaleur* introduced the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t),$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. The heat flow model was based on Newton's law of cooling.

Thus, by the beginning of 19th century, the three most important PDE's were identified.

1.4 Continuity Equation

Let us consider an ideal compressible fluid (viz. gas) occupying a bounded region $\Omega \subset \mathbb{R}^n$ (in practice, we take $n = 3$, but the derivation is true for all dimensions). For mathematical precision, we assume Ω to be a bounded open subset of $\mathbb{R}^n$. Let $\rho(x,t)$ denote the density of the fluid for $x \in \Omega$ at time $t \in I \subset \mathbb{R}$, for some open interval I. Mathematically, we presume that $\rho \in C^1(\Omega \times I)$. We cut a region $\Omega_t \subset \Omega$ and follow Ω_t, the position at time t, as t varies in I. For mathematical precision, we will assume that Ω_t have C^1 boundaries (cf. Appendix D). Now, the law of conservation of mass states that during motion the mass is conserved and mass is the product of density and volume. Thus, the mass of the region as a function of t is constant and hence its derivative should vanish. Therefore,

$$\frac{d}{dt}\int_{\Omega_t} \rho(x,t)\,dx = 0.$$

We regard the points of Ω_t, say $x \in \Omega_t$, following the trajectory $x(t)$ with velocity $\mathbf{v}(x,t)$. We also assume that the deformation of Ω_t is smooth, i.e.,

$\mathbf{v}(x,t)$ is continuous in a neighbourhood of $\Omega \times I$. Consider

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}\rho(x,t)\,dx &= \lim_{h\to 0}\frac{1}{h}\left(\int_{\Omega_{t+h}}\rho(x,t+h)\,dx - \int_{\Omega_t}\rho(x,t)\,dx\right)\\ &= \lim_{h\to 0}\int_{\Omega_t}\frac{\rho(x,t+h)-\rho(x,t)}{h}\,dx\\ &\quad + \lim_{h\to 0}\frac{1}{h}\left(\int_{\Omega_{t+h}}\rho(x,t+h)\,dx - \int_{\Omega_t}\rho(x,t+h)\,dx\right)\end{aligned}$$

The first integral becomes

$$\lim_{h\to 0}\int_{\Omega_t}\frac{\rho(x,t+h)-\rho(x,t)}{h}\,dx = \int_{\Omega_t}\frac{\partial\rho}{\partial t}(x,t)\,dx.$$

The second integral reduces as,

$$\begin{aligned}\int_{\Omega_{t+h}}\rho(x,t+h)\,dx - \int_{\Omega_t}\rho(x,t+h)\,dx &= \int_{\Omega}\rho(x,t+h)\left(\chi_{\Omega_{t+h}} - \chi_{\Omega_t}\right)\\ &= \int_{\Omega_{t+h}\setminus\Omega_t}\rho(x,t+h)\,dx\\ &\quad - \int_{\Omega_t\setminus\Omega_{t+h}}\rho(x,t+h)\,dx.\end{aligned}$$

We now evaluate the above integral in the sense of Riemann. We fix t. Our aim is to partition the set $(\Omega_{t+h}\setminus\Omega_t)\cup(\Omega_t\setminus\Omega_{t+h})$ with cylinders and evaluate the integral by letting the cylinders as small as possible. To do so, we choose $0 < s \ll 1$ and a polygon that covers $\partial\Omega_t$ from outside such that the area of each of the face of the polygon is less than s and the faces are tangent to some point $x_i \in \partial\Omega_t$. Let the polygon have m faces. Then, we have $x_1, x_2, \ldots x_m$ at which the faces $F_1, F_2, \ldots, F_m$ are a tangent to $\partial\Omega_t$. Since Ω_{t+h} is the position of Ω_t after time h, any point $x(t)$ moves to $x(t+h) = \mathbf{v}(x,t)h$. Hence, the cylinders with base F_i and height $\mathbf{v}(x_i,t)h$ is expected to cover our annular region depending on whether we move inward or outward. Thus, $\mathbf{v}(x_i,t)\cdot\nu(x_i)$ is positive or negative depending on whether Ω_{t+h} moves outward or inward, where $\nu(x_i)$ is the unit outward normal at $x_i \in \partial\Omega_t$.

$$\int_{\Omega_{t+h}\setminus\Omega_t}\rho(x,t+h)\,dx - \int_{\Omega_t\setminus\Omega_{t+h}}\rho(x,t+h)\,dx = \lim_{s\to 0}\sum_{i=1}^{m}\rho(x_i,t)\mathbf{v}(x_i,t)\cdot\nu(x_i)hs.$$

Thus,

$$\frac{1}{h}\left(\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t+h)\,dx\right) \overset{h\to 0}{\to} \int_{\partial\Omega_t} \rho(x,t)\mathbf{v}(x,t)\cdot\nu(x)\,d\sigma.$$

By Green's theorem (cf. Appendix D), we have

$$\frac{d}{dt}\int_{\Omega_t} \rho(x,t)\,dx = \int_{\Omega_t}\left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v})\right)\,dx.$$

Now, using conservation of mass, we get

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \text{ in } \Omega \times \mathbb{R}. \tag{1.4.1}$$

Equation (1.4.1) is called the equation of continuity. In fact, any quantity that is conserved as it moves in an open set Ω satisfies the equation of continuity (1.4.1).

Chapter 2

First Order PDE

In this chapter, we try to find the general solutions and particular of first order PDE. A general first PDE has the form

$$F(\nabla u(x), u(x), x) = 0.$$

2.1 Family Of Curves

We observed in the previous chapters that solutions of PDE occur as family of curves given by a constant or arbitrary function. In fact, by eliminating the constant or function, via differentiation may lead to the differential equation it solves. We now look at some family of curves which arise as a solution to first order PDE's.

Let $A \subset \mathbb{R}^2$ be an open subset that represents a parameter set and consider

$$u : \mathbb{R}^2 \times A \to \mathbb{R}$$

a two parameter family of "smooth surfaces" in $\mathbb{R}^3$, $u(x, y, a, b)$, where $(a, b) \in A$. For instance, $u(x, y, a, b) = (x - a)^2 + (y - b)^2$ is a family of circles with centre at (a, b). Differentiate w.r.t x and y, we get $u_x(x, y, a, b)$ and $u_y(x, y, a, b)$, respectively. Eliminating a and b from the two equations, we get a first order PDE

$$F(u_x, u_y, u, x, y) = 0$$

whose solutions are the given surfaces u.

Example 2.1. Consider the family of circles

$$u(x, y, a, b) = (x - a)^2 + (y - b)^2.$$

Thus, $u_x = 2(x-a)$ and $u_y = 2(y-b)$ and eliminating a and b, we get

$$u_x^2 + u_y^2 - 4u = 0$$

is a first order PDE.

Example 2.2. Find the first order PDE, by eliminating the arbitrary function f, satisfied by u.

(i) $u(x,y) = xy + f(x^2+y^2)$

(ii) $u(x,y) = f(x/y)$

Proof. (i) Differentiating the given equation w.r.t x and y, we get

$$u_x = y + 2xf', \quad u_y = x + 2yf',$$

respectively. Eliminating f', by multiplying y and x respectively, we get

$$yu_x - xu_y = y^2 - x^2.$$

(ii) Differentiating the given equation w.r.t x and y, we get

$$u_x = \frac{1}{y}f', \quad , u_y = \frac{-x}{y^2}f',$$

respectively. Eliminating f', by multiplying x and y respectively, we get

$$xu_x + yu_y = 0.$$

□

Example 2.3. Find the first order PDE, by eliminating the arbitrary constants a and b, satisfied by u

(i) $u(x,y) = (x+a)(y+b)$

(ii) $u(x,y) = ax + by$

Proof. (i) Differentiating the given equation w.r.t x and y, we get

$$u_x = y + b, \quad u_y = x + a,$$

respectively. Eliminating a and b, we get

$$u_x u_y = u.$$

(ii) Differentiating the given equation w.r.t x and y, we get

$$u_x = a, \quad u_y = b$$

respectively. Eliminating a and b, we get

$$xu_x + yu_y = u.$$

□

Exercise 7. Find the first order PDE, by eliminating the arbitrary function f, satisfied by $u(x, y) = e^x f(2x - y)$. (Answer: $u_x + 2u_y - u = 0$).

Exercise 8. Find the first order PDE, by eliminating the arbitrary function f, satisfied by $u(x, y) = e^{-4x} f(2x - 3y)$. (Answer: $3u_x + 2u_y + 12u = 0$).

2.2 Linear Transport Equation

2.2.1 One Space Dimension

Derivation

Imagine a object (say, a wave) moving on the surface (of a water) with constant speed b. At any time instant t, every point on the wave would have travelled a distance of bt from its initial position. At any given instant t, let $u(x, t)$ denote the shape of the wave in space. Let us fix a point $(x_0, 0)$ on the wave. Now note that the value of $u(x, t)$ is constant along the line $x = bt + x_0$ or $x - bt = x_0$. Therefore the directional derivative of u in the direction of $(b, 1)$ is zero. Therefore

$$0 = \nabla u(x, t) \cdot (b, 1) = u_t + bu_x.$$

This is a simple first order linear equation called the *transport equation*.

Solving

We wish to solve the transport equation $u_t + bu_x = 0$ which describes the motion of an object moving with constant speed b, as seen by a fixed observer A.

Let us imagine that another observer B (say, on a skateboard) moving with speed b observes the same object, moving in the direction of the object.

For B the wave would appear stationary while for A, the fixed observer, the wave would appear to travel with speed b. What is the equation of the motion of the object for the moving observer B? To understand this we need to identify the coordinate system for B relative to A. Let us fix a point x at time $t = 0$. After time t, the point x remains as x for the fixed observer A, while for the moving observer B, the point x is now $x - bt$. Therefore the coordinate system for B is (ξ, s) where $\xi = x - bt$ and $s = t$. Let $v(\xi, s)$ describe the motion of the object from B's view. Then the PDE describing the motion of object, as seen by B, is $v_s(\xi, s) = 0$. Therefore, $v(\xi, s) = f(\xi)$, for some arbitrary function f (sufficiently differentiable), is the solution from B's perspective. To solve the problem from A's perspective, we observe the relations

$$u_t = v_\xi \xi_t + v_s s_t = -bv_\xi + v_s \text{ and}$$

$$u_x = v_\xi \xi_x + v_s s_x = v_\xi.$$

Therefore, $u_t + bu_x = -bv_\xi + v_s + bv_\xi = v_s$ and, hence, $u(x,t) = v(\xi, s) = f(\xi) = f(x - bt)$.

2.2.2 Higher Dimension

We consider the homogeneous initial-value transport problem in higher dimensions. Given a vector $b \in \mathbb{R}^n$, we need to find $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ satisfying

$$u_t(x,t) + b \cdot \nabla u(x,t) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{2.2.1}$$

By setting a new variable $y = (x,t)$ in $\mathbb{R}^n \times (0, \infty)$, (2.2.1) can be rewritten as

$$(b, 1) \cdot \nabla_y u(y) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

This means that the directional derivative of $u(y)$ along the direction $(b, 1)$ is zero. Thus, u must be constant along all lines in the direction of $(b, 1)$. The parametric representation of a line passing through a given point $(x,t) \in \mathbb{R}^n \times [0, \infty)$ and in the direction of $(b, 1)$ is given by $s \mapsto (x + sb, t + s)$, for all $s \geq -t$. Thus, u is constant on the line $(x + sb, t + s)$ for all $s \geq -t$ and, in particular, the value of u at $s = 0$ and $s = -t$ are same. Hence,

$$u(x,t) = u(x - tb, 0).$$

The procedure explained above can be formalised as below.

The equation of a line passing through (x,t) and parallel to $(b,1)$ is $(x,t)+s(b,1)$, for all $s \in (-t,\infty)$, i.e., $(x+sb, t+s)$. Thus, for a fixed $(x,t) \in \mathbb{R}^n \times (0,\infty)$, we set $v(s) := u(x+sb, t+s)$ for all $s \in (-t,\infty)$. Consequently,

$$\begin{aligned}\frac{dv(s)}{ds} &= \nabla u(x+sb,t+s)\cdot\frac{d(x+sb)}{ds} + \frac{\partial u}{\partial t}(x+sb,t+s)\frac{d(t+s)}{ds} \\ &= \nabla u(x+sb,t+s)\cdot b + \frac{\partial u}{\partial t}(x+sb,t+s)\end{aligned}$$

and from (2.2.1) we have $\frac{dv}{ds} = 0$ and hence $v(s) \equiv$ constant for all $s \in \mathbb{R}$. Thus, in particular, $v(0) = v(-t)$ which implies that $u(x,t) = u(x-tb,0)$. If the value of u is known at time $t = 0$, for instance, $u(x,0) = g(x)$ on $\mathbb{R}^n \times \{t = 0\}$ for a function $g : \mathbb{R}^n \to \mathbb{R}$, then

$$u(x,t) = u(x-tb,0) = g(x-tb).$$

Since (x,t) was arbitrary in $\mathbb{R}^n \times (0,\infty)$, we have $u(x,t) = g(x-tb)$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Thus, $g(x-tb)$ is a classical solution to (2.2.1) if $g \in C^1(\mathbb{R}^n)$. If $g \notin C^1(\mathbb{R}^n)$, we shall call $g(x-tb)$ to be a *weak* solution of (2.2.1).

2.2.3 Inhomogeneous Transport Equation

We now consider the inhomogeneous transport problem. Given a vector $b \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$, find $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ satisfying

$$\frac{\partial u}{\partial t}(x,t) + b\cdot\nabla u(x,t) = f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{2.2.2}$$

As before, we set $v(s) := u(x+sb, t+s)$, for all $s \in \mathbb{R}$, and for any given point $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Thus,

$$\frac{dv(s)}{ds} = b\cdot\nabla u(x+sb,t+s) + \frac{\partial u}{\partial t}(x+sb,t+s) = f(x+sb,t+s).$$

Consider,

$$\begin{aligned} u(x,t) - u(x-tb,0) &= v(0) - v(-t) \\ &= \int_{-t}^{0} \frac{dv}{ds}\, ds \\ &= \int_{-t}^{0} f(x+sb, t+s)\, ds \\ &= \int_{0}^{t} f(x+(s-t)b, s)\, ds. \end{aligned}$$

Thus, $u(x,t) = u(x-tb,0) + \int_0^t f(x+(s-t)b,s)\,ds$ solves (2.2.2).

2.3 Integral Surfaces and Monge Cone

2.3.1 Quasi-linear Equations

We begin by describing the method for first order quasi-linear PDE,

$$F(\nabla u, u, x) := b(x,u(x))\cdot\nabla u(x) - c(x,u(x)) = 0 \quad \text{for } x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ open subset, $b(x,u(x)) \in \mathbb{R}^n$ and $c(x,u(x)) \in \mathbb{R}$. Thus, we have $(b(x,u(x)), c(x,u(x)))\cdot(\nabla u(x), -1) = 0$. Finding the solution u is equivalent to finding the surface S in $\mathbb{R}^{n+1}$ which is graph of the solution u of the given quasi-linear PDE, i.e.,

$$S = \{(x,z) \in \Omega\times\mathbb{R} \mid u(x) - z = 0\}.$$

The equation of the surface S is given by $G(x,z) = u(x) - z$. The normal to S, at any point, is given by the gradient of G. Hence, $\nabla G = (\nabla u(x), -1)$ (cf. Appendix B). Therefore, for every point $(x_0, u(x_0)) \in S$, the coefficient vector $(b(x_0,u(x_0)), c(x_0,u(x_0))) \in \mathbb{R}^{n+1}$ is perpendicular to the normal vector $(\nabla u(x_0), -1)$. Thus, the coefficient vector must lie on the tangent plane at $(x_0, u(x_0))$ of S. Define the vector field $V(x,z) = (b(x,z), c(x,z))$ formed by the coefficients of the quasi-linear PDE. Then, we note from the above discussion that S must be such that the coefficient vector field V is tangential to S at every point of S.

Definition 2.3.1. *A curve in $\mathbb{R}^n$ is said to be an* integral curve *for a given vector field, if the curve is tangent to the vector field at each of its point.*

Similarly, a surface in $\mathbb{R}^n$ is said to be an integral surface *for a given vector field, if the surface is tangent to the vector field at each of its point.*

In the spirit of the above definition and arguments, finding a solution to the quasi-linear PDE is equivalent to finding an integral surface S corresponding to the coefficient vector field V. We view an integral surface w.r.t V as an union of integral curves w.r.t V.

2.3.2 Nonlinear Equations

The arguments in previous section can be carried over to a general nonlinear first order PDE. Consider the first order nonlinear PDE, $F : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that (cf. (1.1.1))

$$F(\nabla u(x), u(x), x) = 0 \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$, F is given and u is the unknown to be found. We want find a surface described by the solution u in $\mathbb{R}^{n+1}$. In the quasi-linear case, using the equation, we obtained a unique direction (b, c) which is tangential, at each point, to our desired surface S! In the non-linear case, however, we have no such unique direction cropping out of the equation. For any point $x \in \Omega$, the point (x, z), where $z = u(x)$ is in the solution surface. Further, if $p = \nabla u(x)$ then we have the relation $F(p, z, x) = 0$. Thus, for a fixed $(x, z) \in \Omega \times \mathbb{R}$, we consider the more general equation $F(p, z, x) = 0$ and denote the solution set as

$$V(x, z) := \{p \in \mathbb{R}^n \mid F(p, z, x) = 0\}.$$

Therefore, solving (2.5.1) is equivalent to finding a $u \in C^1(\Omega)$ such that, for all $x \in \Omega$, there is a pair (x, z) for which $z = u(x)$ and $p = \nabla u(x)$. Every choice of p is a possible normal vector candidate $(p, -1)$ at (x_0, z_0) on S. In general, these family of normals envelope a cone with vertex at (x_0, z_0) perpendicular to S. As p varies in $V(x, z)$, we have a possible family of (tangent) planes through (x_0, z_0) given by the equation

$$(z - z_0) = p \cdot (x - x_0),$$

where one of the planes is tangential to the surface S. The envelope of this family is a cone $C(x_0, z_0)$, called *Monge cone*, with vertex at (x_0, z_0). The envelope of the family of planes is that surface which is tangent at each of its point to some plane from the family.

Definition 2.3.2. *A surface S in $\mathbb{R}^{n+1}$ is said to be an* integral surface *if at each point $(x_0, z_0) \in S \subset \mathbb{R}^n \times \mathbb{R}$ it is tangential to the Monge cone with vertex at (x_0, z_0).*

2.4 Method of Characteristics

We have already noted that solving a first order PDE is equivalent to finding an integral surface corresponding to the given PDE. The integral surfaces are usually the union of integral curves, also called the *characteristic* curves. Thus, finding an integral surface boils down to finding a family of characteristic curves. The method of characteristics gives the equation to find these curves in the form of a system of ODE. The method of characteristics is a technique to reduce a given first order PDE into a system of ODE and then solve the ODE using known methods, to obtain the solution of the first order PDE.

Let us consider the first order nonlinear PDE (2.5.1) in new independent variables $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in \Omega$. Consequently,

$$F(p, z, x) = F(p_1, p_2, \ldots, p_n, z, x_1, x_2, \ldots, x_n)$$

is a map of $2n + 1$ variable. We now introduce the derivatives (assume it exists) of F corresponding to each variable,

$$\begin{cases} \nabla_p F &= (F_{p_1}, \ldots, F_{p_n}) \\ \nabla_x F &= (F_{x_1}, \ldots, F_{x_n}). \end{cases}$$

The method of characteristics reduces a given first order PDE to a system of ODE. The present idea is a generalisation of the idea employed in the study of linear transport equation (cf. (2.2.1)). We must choose a curve $x(s)$ in Ω such that we can compute u and ∇u along this curve. In fact, we would want the curve to intersect the boundary.

We begin by differentiating F w.r.t x_i in (2.5.1), we get

$$\sum_{j=1}^{n} F_{p_j} u_{x_j x_i} + F_z u_{x_i} + F_{x_i} = 0.$$

Thus, we seek to find $x(s)$ such that

$$\begin{aligned} \sum_{j=1}^{n} F_{p_j}(p(s), z(s), x(s)) u_{x_j x_i}(x(s)) &+ F_z(p(s), z(s), x(s)) p_i(s) \\ &+ F_{x_i}(p(s), z(s), x(s)) = 0. \end{aligned}$$

To free the above equation of second order derivatives, we differentiate $p_i(s)$ w.r.t s,

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \frac{dx_j(s)}{ds}$$

and set

$$\frac{dx_j(s)}{ds} = F_{p_j}(p(s), z(s), x(s)).$$

Thus,

$$\frac{dx(s)}{ds} = \nabla_p F(p(s), z(s), x(s)). \tag{2.4.1}$$

Now substituting this in the first order equation, we get

$$\frac{dp_i(s)}{ds} = -F_z(p(s), z(s), x(s))p_i(s) - F_{x_i}(p(s), z(s), x(s)).$$

Thus,

$$\frac{dp(s)}{ds} = -F_z(p(s), z(s), x(s))p(s) - \nabla_x F(p(s), z(s), x(s)). \tag{2.4.2}$$

Similarly, we differentiate $z(s)$ w.r.t s,

$$\begin{aligned} \frac{dz(s)}{ds} &= \sum_{j=i}^{n} u_{x_j}(x(s)) \frac{dx_j(s)}{ds} \\ &= \sum_{j=i}^{n} u_{x_j}(x(s)) F_{p_j}(p(s), z(s), x(s)) \end{aligned}$$

Thus,

$$\frac{dz(s)}{ds} = p(s) \cdot \nabla_p F(p(s), z(s), x(s)). \tag{2.4.3}$$

We have $2n+1$ first order ODE called the *characteristic equations* of (2.5.1). The steps derived above can be summarised in the following theorem:

Theorem 2.4.1. *Let $u \in C^2(\Omega)$ solve (2.5.1) and $x(s)$ solve (2.4.1), where $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$. Then $p(s)$ and $z(s)$ solve (2.4.2) and (2.4.3), respectively, for all $x(s) \in \Omega$.*

We end this section with remark that for linear, semi-linear and quasi-linear PDE one can do away with (2.4.2), the ODE corresponding to p, because for these problems (2.4.3) and (2.4.1) form a determined system. However, for a fully nonlinear PDE one needs to solve all the 3 ODE's to compute the characteristic curve. The method of characteristics may be generalised to higher order hyperbolic PDE's.

Remark 2.4.2. If the PDE is linear, i.e., a and b are independent of u, then the characteristic curves are lying in the xy-plane. If the a and b are constants (independent of both x and u) then the characteristic curves are straight lines. In the linear case the characteristics curves will not intersect. This is easily seen from the fact that if they intersect then, at the point of intersection, they have the same tangent. This is not possible.

(a) In first order linear problem, the ODE reduces to, as follows: Let

$$F(\nabla u, u, x) := b(x) \cdot \nabla u(x) + c(x)u(x) = 0 \quad x \in \Omega.$$

Then, in the new variable, $F(p, z, x) = b(x) \cdot p + c(x)z$. Therefore, by (2.4.1), we have

$$\frac{dx(s)}{ds} = \nabla_p F = b(x(s)).$$

Also,

$$\frac{dz(s)}{ds} = b(x(s)) \cdot p(s) = b(x(s)) \cdot \nabla u(x(s)) = -c(x(s))z(s).$$

(b) For a semi-linear PDE

$$F(\nabla u, u, x) := b(x) \cdot \nabla u(x) + c(x, u(x)) = 0 \quad \text{for } x \in \Omega,$$

we have

$$\frac{dx(s)}{ds} = \nabla_p F = b(x(s)).$$

Also,

$$\frac{dz(s)}{ds} = b(x(s), z(s)) \cdot p(s) = b(x(s), z(s)) \cdot \nabla u(x(s)) = -c(x(s), z(s)).$$

(c) A quasi-linear PDE has the form

$$F(\nabla u, u, x) := b(x, u(x)) \cdot \nabla u(x) + c(x, u(x)) = 0 \quad \text{for } x \in \Omega.$$

Then, in the new variable, $F(p, z, x) = b(x, z) \cdot p + c(x, z)$. Therefore, by (2.4.1), we have

$$\frac{dx(s)}{ds} = \nabla_p F = b(x(s), z(s)).$$

Also,

$$\frac{dz(s)}{ds} = b(x(s), z(s)) \cdot p(s) = b(x(s), z(s)) \cdot \nabla u(x(s)) = -c(x(s), z(s)).$$

Example 2.4. To understand the derivation of the characteristic equation and the geometry involved, explained in this section, let us see what is happening two varibale first order quasi-linear equation:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{2.4.4}$$

Solving for $u(x, y)$ in the above equation is equivalent to finding the surface $S \equiv \{(x, y, u(x, y))\}$ generated by u in $\mathbb{R}^3$. If u is a solution of (2.4.4), at each (x, y) in the domain of u,

$$\begin{aligned} a(x, y, u)u_x + b(x, y, u)u_y &= c(x, y, u) \\ a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) &= 0 \\ (a(x, y, u), b(x, y, u), c(x, y, u)) \cdot (u_x, u_y, -1) &= 0 \\ (a(x, y, u), b(x, y, u), c(x, y, u)) \cdot (\nabla u(x, y), -1) &= 0. \end{aligned}$$

But $(\nabla u(x, y), -1)$ is normal to S at the point (x, y) (cf. Appendix B). Hence, the coefficients $(a(x, y, u), b(x, y, u), c(x, y, u))$ are perpendicular to the normal. Thus, the coefficients $(a(x, y, u), b(x, y, u), c(x, y, u))$ lie on the tangent plane to S at $(x, y, u(x, y))$. Hence, finding u is equivalent to finding the integral surface corresponding to the coefficient vector field $V = (a(x, y, u), b(x, y, u), c(x, y, u))$.

The surface is the union of curves which satisfy the property of S. Thus, for any curve $\Gamma \subset S$ such that at each point of Γ, the vector $V(x, y) = (a(x, y, u), b(x, y, u), c(x, y, u)$ is tangent to the curve. Parametrizing the curve Γ by the variable s, we see that we are looking for the curve $\Gamma = \{x(s), y(s), z(s)\} \subset \mathbb{R}^3$ such that

$$\frac{dx}{ds} = a(x(s), y(s), u(x(s), y(s))), \quad \frac{dy}{ds} = b(x(s), y(s), u(x(s), y(s))),$$

$$\text{and } \frac{dz}{ds} = c(x(s), y(s), u(x(s), y(s))).$$

The three ODE's obtained are called *characteristic equations*. The union of these characteristic (integral) curves give us the integral surface.

We shall now illustrate the method of characteristics for various examples of first order PDE.

Example 2.5 (Linear Transport Equation). We have already solved the linear transport equation by elementary method. We solve the same now using method of characteristics. Consider the linear transport equation in two variable,

$$u_t + bu_x = 0, \quad x \in \mathbb{R} \text{ and } t \in (0, \infty),$$

where the constant $b \in \mathbb{R}$ is given. Thus, the given vector field $V(x,t) = (b, 1, 0)$. The characteristic equations are

$$\frac{dx}{ds} = b, \quad \frac{dt}{ds} = 1, \text{ and } \frac{dz}{ds} = 0.$$

Solving the 3 ODE's, we get

$$x(s) = bs + c_1, \quad t(s) = s + c_2, \text{ and } z(s) = c_3.$$

Eliminating the parameter s, we get the curves (lines) $x - bt =$ a constant and $z =$ a constant. Therefore, $u(x,t) = f(x - bt)$ is the general solution, for an arbitrary function f.

In the inhomogeneous equation case, where the ODE corresponding to z is not zero, we intend to seek a function $\Phi(x, u)$ such that the solution u is defined implicitly by $\Phi(x, u) =$ a constant. Suppose there is such a function Φ then, by setting $z := u$,

$$\Phi_{x_i} + \Phi_z u_{x_i} = 0 \quad \forall i = 1, 2, \ldots, n.$$

Assuming $\Phi_z \neq 0$, we get

$$u_{x_i} = \frac{-\Phi_{x_i}}{\Phi_z} \quad \forall i = 1, 2, \ldots, n.$$

If u solves $b(x, u) \cdot \nabla u(x) = c(x, u)$ then $(b(w), c(w)) \cdot \nabla_w \Phi = 0$ is a homogeneous first order equation of Φ in $n + 1$ variables. In this case,

$$\frac{dw(s)}{ds} = (b, c) \quad \text{and} \quad \frac{d\Phi(w(s))}{ds} = 0.$$

Example 2.6 (Inhomogeneous Transport Equation). Given a constant $b \in \mathbb{R}$ and a function $f(x,t)$, we wish to solve the *inhomogeneous* linear transport equation,

$$u_t(x,t) + bu_x(x,t) = f(x,t), \quad x \in \mathbb{R} \text{ and } t \in (0,\infty).$$

As before, the first two ODE will give the projection of characteristic curve in the xt plane, $x - bt =$ a constant, and the third ODE becomes

$$\frac{dz(s)}{ds} = f(x(s), t(s)).$$

Let's say we need to find the value of u at the point (x,t). The parametrisation of the line passing through (x,t) and $(x - bt, 0)$ in the first variable is $x(s) = x - bt + bs$ and in the second variable is $t(s) = s$, as s varies from 0 to t. If z has to be on the integral curve, then $z(s) = u(x - bt + bs, s)$. Therefore, the third ODE becomes,

$$\frac{dz(s)}{ds} = f(x(s), s) = f(x - bt + bs, s).$$

Integrating both sides from 0 to t, we get

$$\begin{aligned}\int_0^t f(x - b(t-s), s)\, ds &= z(t) - z(0) \\ &= u(x,t) - u(x - bt, 0).\end{aligned}$$

Thus,

$$u(x,t) = u(x - bt, 0) + \int_0^t f(x - b(t-s), s)\, ds$$

is the general solution.

Example 2.7. Let us compute the general solution (in terms of arbitrary functions) of the first order PDE $xu_x(x,y) + yu_y(x,y) = u(x,y)$. The characteristic equations (ODE's) are

$$\frac{dx}{ds} = x(s) \quad \frac{dy}{ds} = y(s) \text{ and } \frac{dz}{ds} = z(s).$$

Thus, $x(s) = c_1 e^s$, $y(s) = c_2 e^s$ and $z(s) = c_3 e^s$. Eliminating the parameter s, we get $y/x = c_4$ and $z/x = c_5$. Thus, the general solution is $F(y/x, z/x) = 0$ for an arbitrary function F. Explicitly,

$$u(x,y) = xg(y/x) \text{ or } u(x,y) = yf(x/y),$$

for some arbitrary smooth functions f and g. Compare this answer with Example 2.3(ii).

Example 2.8. Let us compute the general solution (in terms of arbitrary functions) of the first order PDE $yu(x,y)u_x(x,y) + xu(x,y)u_y(x,y) = xy$. The characteristic equations are

$$\frac{dx}{ds} = yz, \quad \frac{dy}{ds} = xz \text{ and } \frac{dz}{ds} = xy.$$

Hence,

$$\begin{aligned} 0 &= x\frac{dx}{ds} - y\frac{dy}{ds} \\ &= \frac{d(x^2)}{ds} - \frac{d(y^2)}{ds} \\ &= \frac{d(x^2 - y^2)}{ds}. \end{aligned}$$

Thus, $x^2 - y^2 = c_1$ and, similarly, $x^2 - z^2 = c_2$. Hence, the general solution is $F(x^2 - y^2, x^2 - z^2) = 0$ for some arbitrary function F. Explicitly, for some f and g,

$$u^2(x,y) = x^2 + f(x^2 - y^2) \text{ or } u^2(x,y) = y^2 + g(x^2 - y^2).$$

Example 2.9. Let us compute the general solution (in terms of arbitrary functions) of the first order PDE $2yu_x + uu_y = 2yu^2$. The characteristic equations (ODE's) are

$$\frac{dx}{ds} = 2y(s) \quad \frac{dy}{ds} = z(s) \text{ and } \frac{dz}{ds} = 2y(s)z^2(s).$$

Solving in the parametric form is quite cumbersome, because we will have a second order nonlinear ODE of y, $y''(s) = 2y(y')^2$. However, for $u \neq 0$, we get $\frac{dz}{dy} = 2y(s)z(s)$ solving which we get $\ln|z| = y^2 + c_1$ or $z = c_2 e^{y^2}$. Similarly, $\frac{dy}{dx} = \frac{z}{2y} = \frac{c_2 e^{y^2}}{2y}$ solving which we get $c_2 x + e^{-y^2} = c_3$ or, equivalently, $xze^{-y^2} + e^{-y^2} = c_3$. Thus, the general solution is

$$F(e^{-y^2}(1 + xz), e^{-y^2} z) = 0.$$

or

$$u(x,y) = f\left(e^{-y^2}(1 + xu)\right) e^{y^2}$$

for some arbitrary smooth functions f. Note that $u \equiv 0$ is a solution if we choose $f = 0$. The characteristic curves are $(1 + xu)e^{-y^2}$ = a constant and along these curves ue^{-y^2} is constant.

Example 2.10. Let us compute the general solution (in terms of arbitrary functions) of the first order PDE $u_x + 2xu_y = u^2$. The characteristic equations (ODE's) are

$$\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = 2x(s) \text{ and } \frac{dz}{ds} = z^2(s).$$

Solving which we get $x(s) = s + c_1$, $y(s) = s^2 + 2c_1 s + c_2$ and $z(s) = -1/(s + c_3)$. Eliminating s between x and y, we get the characteristic curves to be $y - x^2$ = a constant and $x + 1/z$ = a constant. Thus, the general solution is $F(y - x^2, x + 1/z) = 0$. Explicitly,

$$u(x, y) = \frac{-1}{x + f(y - x^2)}$$

for some arbitrary smooth functions f.

Example 2.11. Let us compute the general solution (in terms of arbitrary functions) of the first order PDE $yu_x - xu_y = 2xyu$. The characteristic equations (ODE's) are

$$\frac{dx}{ds} = y(s) \quad \frac{dy}{ds} = -x(s) \text{ and } \frac{dz}{ds} = 2xyz.$$

To avoid cumbersome ODE, let us begin by assuming $y \neq 0$, then dividing the entire equation by y, we get

$$\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = -x(s)/y(s) \text{ and } \frac{dz}{ds} = 2xz.$$

Solving which we get $x(s) = s + c_1$, $y(s) = -s^2 - 2c_1 s + 2c_2$ and $|z(s)| = c_3 e^{s^2 + 2c_1 s}$. Eliminating s between x and y, we get the characteristic curves to be $y^2 + x^2$ = a constant and $z = c_4 e^{x^2}$. Thus, the general solution is $F(y^2 + x^2, e^{-x^2} z) = 0$. Explicitly,

$$u(x, y) = f(y^2 + x^2)e^{x^2}$$

for some arbitrary smooth functions f.

Example 2.12. Let us compute the general solution (in terms of arbitrary functions) of the first order PDE $u_{x_1} + e^{x_1} u_{x_2} + e^{x_3} u_{x_3} = (2x_1 + e^{x_1})e^u$. The characteristic equations (ODE's) are

$$\frac{dx_1}{ds} = 1 \quad \frac{dx_2}{ds} = e^{x_1} \quad \frac{dx_3}{ds} = e^{x_3} \text{ and } \frac{dz}{ds} = (2x_1 + e^{x_1})e^{z(s)}.$$

Solving which we get $x_1(s) = s + c_1$, $x_2(s) = e^{c_1}e^s + c_2$, $e^{-x_3(s)} = -s + c_3$ and

$$e^{-z(s)} = -s^2 - 2c_1 s - e^{s+c_1} + c_4.$$

Eliminating s between x_1 and x_2, we get $x_2 - e^{x_1} =$ a constant, $e^{-x_3} + x_1 =$ a constant and the general solution is $F(x_2 - e^{x_1}, e^{-x_3} + x_1, e^{-u} + x_1^2 + e^{x_1}) = 0$. Explcitly,

$$e^{-u} = f(x_2 - e^{x_1}, e^{-x_3} + x_1) - x_1^2 - e^{x_1}$$

for some arbitrary smooth functions f.

2.5 Complete Integrals

In this section, we study the form of general solutions of a first order PDE, i.e.,

$$F(\nabla u(x), u(x), x) = 0 \quad \text{in } \Omega. \tag{2.5.1}$$

Let $A \subset \mathbb{R}^n$ be an open set which is the parameter set. Let us introduce the $n \times (n+1)$ matrix

$$(D_a u, D^2_{xa} u) := \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \dots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \dots & u_{x_n a_n} \end{pmatrix}.$$

Definition 2.5.1. *A C^2 function $u = u(x; a)$ is said to be a* complete integral *in $\Omega \times A$ if $u(x; a)$ solves (2.5.1) for each $a \in A$ and the rank of the matrix $(D_a u, D^2_{xa} u)$ is n.*

The condition on the rank of the matrix means that $u(x; a)$ strictly depends on all the n components of a.

Example 2.13. The nonlinear equation $u_x u_y = u$ has a complete integral $u(x, y; a, b) = xy + ab + (a, b) \cdot (x, y)$. We shall derive this solution in Example 2.22.

Example 2.14. The nonlinear equation $u_x^2 + u_y = 0$ has a complete integral $u(x, y; a, b) = ax - ya^2 + b$.

Example 2.15. For any given $f : \mathbb{R}^n \to \mathbb{R}$, the complete integral of the *Clairaut's* equation

$$x \cdot \nabla u + f(\nabla u) = u$$

is

$$u(x; a) = a \cdot x + f(a) \quad \forall a \in \mathbb{R}^n.$$

Example 2.16. The complete integral of the *eikonal* equation $|\nabla u| = 1$ is $u(x; a, b) = a \cdot x + b$ for all $a \in S(0, 1)$ and $b \in \mathbb{R}$.

Example 2.17. The Hamilton-Jacobi is a special case of the nonlinear equation where $F(x, z, p) = p_n + H(x, p_1, \ldots, p_{n-1})$ where H is independent of z and p_n. For any given $H : \mathbb{R}^n \to \mathbb{R}$, the complete integral of the *Hamilton-Jacobi* equation

$$u_t + H(\nabla u) = 0$$

is

$$u(x, t; a, b) = a \cdot x - tH(a) + b \quad \forall a \in \mathbb{R}^n, b \in \mathbb{R}.$$

2.5.1 Envelopes and General Integrals

Definition 2.5.2. *Let $\Omega \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be open subsets and let $u = u(x; a)$ be a C^1 function of both x and a. Suppose the equation $D_a u(x; a) = 0$ is solvable for a, as a C^1 function of x, say $a = \phi(x)$, i.e., $D_a u(x; \phi(x)) = 0$, then $v(x) := u(x; \phi(x))$ is the* envelope *of the functions $\{u(\cdot; a)\}_{a \in A}$.*

The idea is that for each $x \in \Omega$, the graph of v is tangent to the graph of $u(\cdot; a)$ for $a = \phi(x)$.

Example 2.18. The complete integral of the nonlinear PDE $u^2(1+|\nabla u|^2) = 1$ is $u(x; a) = \pm(1 - |x - a|^2)^{1/2}$ with $|x - a| < 1$. Now, solving $D_a u = \pm\frac{x-a}{(1-|x-a|^2)^{1/2}} = 0$ for a, we get $a = \phi(x) := x$. Thus, the envelope is $v(x) = \pm 1$.

Theorem 2.5.3. *Suppose for each $a \in A$, $u(\cdot; a)$ is a solution to (2.5.1) and the envelope v of u, given as $v(x) = u(x, \phi(x))$, exists then v also solves (2.5.1).*

Proof. Since $v(x) = u(x; \phi(x))$, for each $i = 1, 2, \ldots, n$,

$$v_{x_i}(x) = u_{x_i} + \sum_{j=1}^{m} u_{a_j} \phi^j_{x_i}(x) = u_{x_i}$$

because $D_a u(x; \phi(x)) = 0$. Therefore,

$$F(\nabla v(x), v(x), x) = F(\nabla u(x; \phi(x)), u(x; \phi(x)), x) = 0.$$

□

Definition 2.5.4. *The general integral is the envelope w of the functions $u(x; a', h(a'))$ where $a' := (a_1, a_2, \ldots, a_{n-1})$ and $h : A' \subset \mathbb{R}^{n-1} \to \mathbb{R}$, provided it exists.*

Example 2.19. The envelope of the nonlinear equation $u_x u_y = u$, considered in Example 2.13, is $u(x, y; a, b) = xy + ab + (a, b) \cdot (x, y)$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(a) = a$, then $u(x, y; a, h(a)) = u(x, y; a, a) = xy + a^2 + a(x + y)$. Solve $D_a u = 2a + x + y = 0$ which yields $a = \phi(x) := \frac{-(x+y)}{2}$. Therefore, the envelope $w(x) = u(x, y; \phi(x), h(\phi(x))) = -(x - y)^2/4$.

Example 2.20. Consider the eikonal equation $|\nabla u(x, y)| = 1$ in two dimension, i.e., $\sqrt{u_x^2 + u_y^2} = 1$ whose complete integral is $u(x, y; a_1, a_2) = (x, y) \cdot (\cos a_1, \sin a_1) + a_2$. Consider $h \equiv 0$, then $u(x, y; a_1, h(a_1)) = (x, y) \cdot (\cos a_1, \sin a_1)$. Thus, solving $D_{a_1} u = -x \sin a_1 + y \cos a_1 = 0$, we get $a_1 = \arctan(y/x)$. Since

$$\cos(\arctan(z)) = \frac{1}{\sqrt{1 + z^2}} \text{ and } \sin(\arctan(z)) = \frac{z}{\sqrt{1 + z^2}},$$

we have the envelope $w(x) = \pm\sqrt{x^2 + y^2}$ for non-zero vectors.

Example 2.21. Consider the Hamilton-Jacobi Equation $u_t + |\nabla u|^2 = 0$ whose complete integral is $u(x, t; a, b) = x \cdot a - t|a|^2 + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Consider $h \equiv 0$, then $u(x, t; a, h(a)) = x \cdot a - t|a|^2$. Thus, solving $D_a u = x - 2ta = 0$, we get $a = \frac{x}{2t}$. We get the envelope $w(x) = \frac{|x|^2}{4t}$.

2.5.2 Method Of Characteristics

We have already derived the characteristic equation for a first order PDE. We illustrate through examples its application for nonlinear PDE.

Example 2.22. Let us compute a complete integral of the first order PDE $u_x u_y = u(x, y)$. The equation is of the form $F(p, z, x) = p_1 p_2 - z$. The characteristic equations are (using (2.4.1))

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (p_2(s), p_1(s)),$$

(using (2.4.2))

$$\frac{dp(s)}{ds} = p(s)$$

and (using (2.4.3))

$$\frac{dz(s)}{ds} = (p_1(s), p_2(s)) \cdot (p_2(s), p_1(s)) = 2p_1(s)p_2(s)$$

Thus, on integrating, we get $p_1(s) = c_1 e^s$ and $p_2(s) = c_2 e^s$. Solving for z, we get

$$z(s) = c_1 c_2 e^{2s} + c_3.$$

Using p, we solve for x to get $x(s) = c_2 e^s + b$ and $y(s) = c_1 e^s + a$.Therefore,

$$u(x, y) = (y - a)(x - b) + c_3$$

is a complete integral for arbitrary constants a and b, if we choose $c_3 = 0$. We have already seen in Example 2.13 that $u(x, y) = xy + ab + (a, b) \cdot (x, y)$ is a complete integral.

Example 2.23. Let us find a different complete integral of the nonlinear PDE $u_x u_y = u$. Note that $F(p, z, x) = p_1 p_2 - z$. Then the ODE

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (p_2(s), p_1(s)),$$

$$\frac{dp(s)}{ds} = p(s)$$

Thus, on integrating, we get $p_1(s) = c_1 e^s$ and $p_2(s) = c_2 e^s$. Therefore, $p_1/p_2 = a$. Using this equation with $p_1 p_2 = z$, we get $p_1 = \pm\sqrt{az}$ and

$p_2 = \pm\sqrt{z/a}$. Now,

$$
\begin{aligned}
\frac{dz(s)}{ds} &= (p_1(s), p_2(s)) \cdot \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) \\
&= \pm\sqrt{az}\frac{dx(s)}{ds} \pm \sqrt{z/a}\frac{dy(s)}{ds} \\
\frac{1}{\sqrt{z}}\frac{dz(s)}{ds} &= \pm\left(\sqrt{a}\frac{dx(s)}{ds} + 1/\sqrt{a}\frac{dy(s)}{ds}\right) \\
2\sqrt{z} &= \pm\left(\sqrt{a}x + y/\sqrt{a}\right) + c_3.
\end{aligned}
$$

Thus,

$$
u(x, y) = \left[b + \frac{1}{2}\left(\sqrt{a}x + y/\sqrt{a}\right)\right]^2
$$

is a complete integral, if we had chosen $a > 0$.

Note that previous two examples compute two different complete integral for same equation. However, in both examples, no choice of a and b will give the zero solution $u \equiv 0$. Thus, $u \equiv 0$ is called *singular solution*.

Example 2.24. Let us find the complete integral, general solution and singular solution of the fully non-linear PDE $u_x^2 + u_y^2 = 1 + 2u$. Since $F(p, z, x) = p_1^2 + p_2^2 - 1 - 2z$, the ODEs are

$$
\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (2p_1(s), 2p_2(s)),
$$

$$
\frac{dp(s)}{ds} = (2p_1, 2p_2)
$$

Thus, on dividing and integrating, we get $p_1/p_2 = a$. Using the PDE, we get $(1 + a^2)p_2^2 = 1 + 2z$. Thus,

$$
p_2 = \pm\sqrt{\frac{1 + 2z}{1 + a^2}} \quad p_1 = \pm a\sqrt{\frac{1 + 2z}{1 + a^2}}.
$$

Now,

$$\begin{aligned}
\frac{dz(s)}{ds} &= (p_1(s), p_2(s)) \cdot \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) \\
&= \pm\sqrt{\frac{1+2z}{1+a^2}}\left(a\frac{dx(s)}{ds} + \frac{dy(s)}{ds}\right) \\
\frac{1}{\sqrt{1+2z}}\frac{dz(s)}{ds} &= \pm\frac{1}{\sqrt{1+a^2}}\left(a\frac{dx(s)}{ds} + \frac{dy(s)}{ds}\right) \\
\sqrt{1+2z} &= \pm\frac{ax+y}{\sqrt{1+a^2}} \pm b.
\end{aligned}$$

Thus,

$$u(x,y) = \frac{1}{2}\left(\frac{ax+y}{\sqrt{1+a^2}} + b\right)^2 - \frac{1}{2}$$

is a complete integral. Note that no choice of a and b will give the constant solution $u = -1/2$. Thus, $u \equiv -1/2$ is called *singular solution*.

Exercise 9. Find the general solution (in terms of arbitrary functions) of the given first order PDE

(i) $xu_x + yu_y = xe^{-u}$ with $x > 0$. (Answer: $u(x,y) = f(y/x)$ for some arbitrary f).

(ii) $u_x + u_y = y + u$. (Answer: $u(x,y) = -(1+y) + f(y-x)e^x$).

(iii) $x^2u_x + y^2u_y = (x+y)u$. (Answer: $u(x,y) = f((1/x) - (1/y))(x-y)$).

(iv) $x(y^2 - u^2)u_x - y(u^2 + x^2)u_y = (x^2 + y^2)u$. (Answer: $u(x,y) = \frac{x}{y}f(x^2 + y^2 + u^2)$).

(v) $(\ln(y+u))u_x + u_y = -1$.

(vi) $x(y-u)u_x + y(u-x)u_y = (x-y)u$.

(vii) $u(u^2 + xy)(xu_x - yu_y) = x^4$.

(viii) $(y + xu)u_x - (x + yu)u_y = x^2 - y^2$.

(ix) $(y^2 + u^2)u_x - xyu_y + xu = 0$.

(x) $(y-u)u_x + (u-x)u_y = x - y$.

(xi) $x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$.

(xii) $\sqrt{1 - x^2}u_x + u_y = 0$.

(xiii) $(x + y)uu_x + (x - y)uu_y = x^2 + y^2$.

(xiv) Find a complete integral of $uu_xu_y = u_x + u_y$.

(xv) Find a complete integral of $u_x^2 + u_y^2 = xu$.

2.6 Cauchy Problem

Recall that the general solution of the transport equation depends on the value of u at time $t = 0$, i.e., the value of u on the curve $(x, 0)$ in the xt-plane. Thus, the problem of finding a function u satisfying the first order PDE (2.4.4) such that u is known on a curve Γ in the xy-plane is called the *Cauchy problem*.

Definition 2.6.1. *A Cauchy problem states that: given a hypersurface $\Gamma \subset \mathbb{R}^n$, can we find a solution u of $F(x, u(x), \nabla u(x)) = 0$ whose graph contains Γ?*

The question that arises at this moment is that: Does the knowledge of u on *any* hypersurface $\Gamma \subset \mathbb{R}^n$ lead to solving the first order PDE? The answer is a "no". For instance, in the transport problem, if we choose the curve $\Gamma = \{(x, t) \mid x - bt = 0\}$, then we had no information to conclude u off the line $x - bt = 0$.

2.6.1 Quasilinear

Consider the general first order qausilinear PDE with n independent variable

$$F(x, u, Du, D^2u) := b(x, u(x)) \cdot Du(x) - c(u(x), x), \tag{2.6.1}$$

where $b = b_i$ is a vector with n components. Let $\Gamma \subset \Omega$ be an hypersurface defined implicitly as $S(x) = 0$ and $\nabla S \neq 0$. Given u on Γ, can we calculate all first derivatives of u on Γ? To answer this, we map Γ to a hyperplane Γ_0 by the map $\phi : \Gamma \to \Gamma_0$ with its components $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ such that $\phi_n(x) = S(x)$ and $J_\phi(x) \neq 0$, non-zero Jacobian of ϕ, for all $x \in \Omega$. For sufficiently regular hypersurfaces such a mapping exists. Note that Γ_0 is a subset of the hyperplane whose final coordinate is zero. Let $y = \phi(x)$ and $v(y) := u(x)$. Then

$$u_{x_i} = \sum_{k=1}^{n} v_{y_k} \frac{\partial \phi_k}{\partial x_i}.$$

Using the above equation, the first order linear PDE becomes

$$\sum_{i,k=1}^{n} b_i \frac{\partial \phi_k}{\partial x_i} v_{y_k} = \tilde{c}(v, y), \tag{2.6.2}$$

where RHS is all known on Γ_0. To understand LHS on Γ_0, note that

$$v_{y_k}(y) = \lim_{h \to 0} \frac{1}{h}[v(y_1, \ldots, y_k + h, \ldots, y_{n-1}, 0) - v(y_1, \ldots, y_k, \ldots, y_{n-1}, 0)].$$

Therefore, we know v_{y_k}, for all $k = 1, 2, \ldots, n-1$, on Γ_0. We only do not know v_{y_n} on Γ_0. Thus, (2.6.2) can be rewritten as

$$\sum_{i=1}^{n} b_i \frac{\partial \phi_n}{\partial x_i} v_{y_n} = \text{ terms known on } \Gamma_0.$$

Since $\phi_n(x) = S(x)$, we can compute v_{y_n} if

$$\sum_{i=1}^{n} b_i(x) S_{x_i} \neq 0$$

on Γ. Note that ∇S is the normal to the hypersurface $S(x) = 0$.

Definition 2.6.2. *We say a hypersurface* $\Gamma \subset \Omega \subset \mathbb{R}^n$ *is* non-characteristic *w.r.t* (2.6.1) *if*

$$\sum_{i=1}^{n} b_i(u(x), x)\nu_i(x) \neq 0 \quad \forall x \in \Gamma,$$

where $\nu(x)$ *is the normal vector of* Γ *at* x.

For instance, in the two dimension case, $\Gamma = \{\gamma_1(r), \gamma_2(r)\} \subset \Omega \subset \mathbb{R}^2$ is *non-characteristic* for the quasilinear Cauchy problem

$$\left\{ \begin{array}{rll} a(x,y,u)u_x + b(x,y,u)u_y & = c(x,y,u) & (x,y) \in \Omega \\ u & = g & \text{on } \Gamma \end{array} \right. \tag{2.6.3}$$

if Γ is nowhere tangent to $(a(\gamma_1, \gamma_2, g), b(\gamma_1, \gamma_2, g))$, i.e.,

$$(a(\gamma_1, \gamma_2, g), b(\gamma_1, \gamma_2, g)) \cdot (-\gamma_2', \gamma_1') \neq 0.$$

Example 2.25. Consider the equation

$$2u_x(x,y) + 3u_y(x,y) = 1 \quad \text{in } \mathbb{R}^2.$$

Let Γ be a straight line $y = mx + c$ in $\mathbb{R}^2$. The parametrisation of the line is $\Gamma(r) := (r, mr + c)$ for $r \in \mathbb{R}$. Therefore,

$$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (2,3) \cdot (-m, 1) = 3 - 2m.$$

Thus, the line is not a non-characteristic for $m = 3/2$, i.e., all lines with slope $3/2$ is not a non-characteristic.

Theorem 2.6.3. *Let a, b and c, the coefficients of (2.6.3), have continuous partial derivatives w.r.t x, y, u. Let $\Gamma(r) := (\gamma_1(r), \gamma_2(r))$ be the parametrization of an initial curve on which $u(\gamma_1(r), \gamma_2(r)) = \phi(r)$ such that γ_1, γ_2 and ϕ are continuously differentiable and the initial curve is non-characteristic, i.e.,*

$$(a(\gamma_1, \gamma_2, \phi), b(\gamma_1, \gamma_2, \phi)) \cdot (-\gamma_2', \gamma_1') \neq 0.$$

Then there exists a unique solution $u(x,y)$ in some neighbourhood of Γ which satisfies (2.6.3).

Proof. The characteristic curves are solution to the ODE's

$$\frac{dx}{ds}(r,s) = a(x,y,u); \frac{dy}{ds}(r,s) = b(x,y,u) \quad \text{and } \frac{du}{ds}(r,s) = c(x,y,u)$$

such that on Γ, $x(r,s) = \gamma_1(r)$, $y(r,s) = \gamma_2(r)$ and $z(r,s) = \phi(r)$. The non-characteristic property of Γ implies that the following Jacobian on Γ:

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{pmatrix} = (a(\gamma_1, \gamma_2, \phi), b(\gamma_1, \gamma_2, \phi)) \cdot (-\gamma_2', \gamma_1') \neq 0.$$

By implicit function theorem (cf. Theorem C.0.2), one can solve for r and s in terms of x and y, locally, in the neighbourhood of Γ. Set $v(x,y) = u(r(x,y), s(x,y))$. We will show that v is a unique solution to (2.6.3). Note that, by implicit function theorem, the value of v on Γ is

$$v(x,y) = u(\gamma_1(r), \gamma_2(r)) = \phi(r).$$

Moreover,

$$\begin{aligned} av_x + bv_y &= a(u_r r_x + u_s s_x) + b(u_r r_y + u_s s_y) \\ &= u_r(ar_x + br_y) + u_s(as_x + bs_y) \\ &= u_r(r_x x_s + r_y y_s) + u_s(s_x x_s + s_y y_s) \\ &= u_r \frac{dr}{ds} + u_s \frac{ds}{ds} = \frac{du}{ds} = c \end{aligned}$$

□

Theorem 2.6.4. *If the initial* Γ *is non-characteristic and there exists functions* α *and* β*, as above, then there exists a unique (depending on* α *and* β*) solution* $u(x, y)$ *in some neighbourhood of* Γ *which satisfies* (2.6.5).

Recall that we already introduced the notion of well-posedness of a PDE in Chapter 1. We see the existence issues in a example which highlights the importance of well-posedness of Cauchy problem. In particular, if Γ is not non-characteristic, then the Cauchy problem is not well-posed.

Example 2.26. (i) Find the general solution (in terms of arbitrary functions) of the first order PDE $2u_x(x, y) + 3u_y(x, y) + 8u(x, y) = 0$.

(ii) For the PDE given above, check for the characteristic property of the following curves

(a) $y = x$ in the xy-plane

(b) $y = \frac{3x-1}{2}$.

(iii) Discuss the particular solutions of the above PDE, corresponding to

(a) $u(x, x) = x^4$ on $y = x$

(b) $u(x, (3x - 1)/2) = x^2$ on $y = (3x - 1)/2$

(c) $u(x, (3x - 1)/2) = e^{-4x}$.

Observe the nature of solutions for the same PDE on a characteristic curve and on non-characteristic curve.

(i) The characteristic equations are

$$\frac{dx}{ds} = 2, \quad \frac{dy}{ds} = 3 \text{ and } \frac{dz}{ds} = -8z.$$

Hence,

$$x(s) = 2s + c_1 \quad y(s) = 3s + c_2 \text{ and } z(s) = c_3 e^{-8s}.$$

Thus, $3x - 2y = c_4$ and $z = c_4 e^{-4x}$ or $z = c_5 e^{-8y/3}$. Hence, the general solution is $F(3x - 2y, e^{4x} z) = 0$. Explicitly, for some f and g,

$$u(x, y) = f(3x - 2y)e^{-4x} \text{ or } u(x, y) = g(3x - 2y)e^{-8y/3}.$$

(ii) (a) Parametrise the curve $y = x$ as $\Gamma(r) : r \mapsto (r, r)$. Thus $\gamma_1(r) = \gamma_2(r) = r$. Since the coefficients of the PDE are $a(r) = 2$ and $b(r) = 3$, we have

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (2, 3) \cdot (-1, 1) = -2 + 3 = 1 \neq 0.$$

Hence Γ is non-characteristic.

(b) Parametrise the curve $y = (3x - 1)/2$ as $\Gamma(r) : r \mapsto (r, (3r - 1)/2)$. Hence $\gamma_1(r) = r$ and $\gamma_2(r) = (3r - 1)/2$ and

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (2, 3) \cdot (-3/2, 1) = -3 + 3 = 0.$$

Hence Γ is a characteristic curve.

(iii) Recall that the general solution is $F(3x - 2y, e^{4x} z) = 0$ or

$$u(x, y) = f(3x - 2y)e^{-4x} \text{ or } u(x, y) = g(3x - 2y)e^{-8y/3}.$$

(a) Now, $u(x, x) = x^4$ implies $F(x, e^{4x} x^4) = 0$. Thus,

$$e^{4x} z = e^{12x - 8y}(3x - 2y)^4$$

and

$$u(x, y) = (3x - 2y)^4 e^{8(x-y)}.$$

Thus, we have a unique solution u.

(b) Using the given condition, we have $F(1, x^2 e^{4x}) = 0$. Either $f(1) = x^2 e^{4x}$ or $f(x^2 e^{4x}) = 1$. The first case is not valid (multi-valued function). The second case corresponds to $z = e^{-4x}$ which will not satisfy the Cauchy data. Hence there is no solution u solving the given PDE with the given data.

(c) Once again using the given condition, we have $F(1, x^2e^{4x}) = 0$. Either $f(1) = x^2e^{4x}$ or $f(x^2e^{4x}) = 1$. The first case is not valid (multi-valued function). The second case corresponds to $z = e^{-4x}$ which will satisfy the Cauchy data. Since there many choices of f that satisfies $f(x^2e^{4x}) = 1$, we have infinite number of solutions (or choices for) u that solves the PDE.

Example 2.27. (i) Find the general solution (in terms of arbitrary functions) of the first order PDE $u_x(x, y) + u_y(x, y) = 1$.

(ii) For the PDE given above, check for the characteristic property of the following curves

(a) the x-axis, $\{(x, 0)\}$, in the xy-plane

(b) $y = x$.

(iii) Discuss the particular solutions of the above PDE, corresponding to

(a) $u(x, 0) = \phi(x)$ on x-axis.

(b) $u(x, x) = x$ on $y = x$.

(c) $u(x, x) = 1$ on $y = x$.

Observe the nature of solutions for the same PDE on a characteristic curve and on non-characteristic curve.

(i) The characteristic equations are

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 1 \text{ and } \frac{dz}{ds} = 1.$$

Hence,

$$x(s) = s + c_1 \quad y(s) = s + c_2 \text{ and } z(s) = s + c_3.$$

Thus, $y - x = c_4$ and $z - x = c_5$ or $x - y = c_4$ and $z - y = c_5$. Hence, for some f and g,

$$u(x, y) = x + f(y - x) \text{ or } u(x, y) = y + g(x - y).$$

(ii) (a) Parametrise the curve x-axis as $\Gamma(r) : r \mapsto (r, 0)$. Thus $\gamma_1(r) = r$ and $\gamma_2(r) = 0$. Since the coefficients of the PDE are $a(r) = 1$ and $b(r) = 1$, we have

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (1, 1) \cdot (0, 1) = 1 \neq 0.$$

Hence Γ is non-characteristic.

(b) Parametrise the curve $y = x$ as $\Gamma(r) : r \mapsto (r, r)$. Hence $\gamma_1(r) = r = \gamma_2(r)$ and

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (1, 1) \cdot (-1, 1) = -1 + 1 = 0.$$

Hence Γ is a characteristic curve.

(iii) Recall that the general solution is

$$u(x, y) = x + f(y - x) \text{ or } u(x, y) = y + g(x - y).$$

(a) Now, $u(x, 0) = \phi(x)$ implies $f(x) = \phi(-x) + x$ or $g(x) = \phi(x)$, and

$$u(x, y) = y + \phi(x - y).$$

Thus, we have a unique solution u.

(b) Using the given condition, we have $f(0) = 0$ or $g(0) = 0$. One has many choices of function satisying these conditions. Thus, we have infinite number of solutions (or choices for) u that solves the PDE.

(c) Once again using the given condition, we have $f(0) = 1 - x$ or $g(0) = 1 - x$ for all $x \in \mathbb{R}$. This implies f and g are not well defined. We have no function f and g, hence there is no solution u solving the given PDE with the given data.

For any given (smooth enough) function $\phi : \mathbb{R} \to \mathbb{R}$, consider the linear transport equation

$$\begin{cases} u_t + bu_x &= 0 \quad x \in \mathbb{R} \text{ and } t \in (0, \infty) \\ u(x, 0) &= \phi(x) \quad x \in \mathbb{R}. \end{cases} \tag{2.6.4}$$

We know that the general solution of the transport equation is $u(x, t) = u(x - bt, 0)$. Thus, $u(x, t) = \phi(x - bt)$ is the unique solution of (2.6.4). We derive the particular solution of the Cauchy problem (2.6.4) using parametrisation of the data curve Γ. The example also shows how the information on data curve Γ is reduced as initial condition for the characteristic ODE's. Note that in the example below the data curve Γ is parametrised using the variable r and the characteristic curves is parametrised using the variable s.

Example 2.28. We shall compute the solution of the Cauchy problem (2.6.4). We first check for non-characteristic property of Γ. Note that $\Gamma \equiv \{(x, 0)\}$, the x-axis of xt-plane, is the (boundary) curve on which the value of u is given. Thus, $(\Gamma, \phi) = \{(x, 0, \phi(x))\}$ is the known curve on the solution surface of u.

We parametrize the curve Γ with r-variable, i.e., $\Gamma = \{\gamma_1(r), \gamma_2(r)\} = \{(r, 0)\}$. Γ is non-characteristic, because $(b, 1) \cdot (0, 1) = 1 \neq 0$. The characteristic equations are:

$$\frac{dx(r,s)}{ds} = b, \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dz(r,s)}{ds} = 0$$

with initial conditions,

$$x(r, 0) = r, \quad t(r, 0) = 0, \text{ and } z(r, 0) = \phi(r).$$

Solving the ODE's, we get

$$x(r, s) = bs + c_1(r), \quad t(r, s) = s + c_2(r)$$

and $z(r, s) = c_3(r)$ with initial conditions

$$x(r, 0) = c_1(r) = r$$

$$t(r, 0) = c_2(r) = 0, \text{ and } z(r, 0) = c_3(r) = \phi(r).$$

Therefore,

$$x(r, s) = bs + r, \quad t(r, s) = s, \text{ and } z(r, s) = \phi(r).$$

We solve for r, s in terms of x, t and set $u(x, t) = z(r(x, t), s(x, t))$.

$$r(x, t) = x - bt \text{ and } s(x, t) = t.$$

Therefore, $u(x, t) = z(r, s) = \phi(r) = \phi(x - bt)$.

Example 2.29. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$. Let $\Gamma := \{(x,0) \mid x > 0\}$. Consider the linear PDE

$$\begin{cases} xu_y(x,y) - yu_x(x,y) &= u(x,y) \quad \text{in } \Omega \\ u(x,0) &= \phi(x) \quad \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r,0)$ for $r > 0$. Therefore,

$$(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (0,r)\cdot(0,1) = r \neq 0.$$

Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = -y; \frac{dy(r,s)}{ds} = x \text{ and } \frac{dz(r,s)}{ds} = z(s)$$

with initial conditions

$$x(r,0) = r, \quad y(r,0) = 0, \text{ and } z(r,0) = \phi(r).$$

Note that

$$\frac{d^2x(r,s)}{ds} = -x(r,s) \text{ and } \frac{d^2y(r,s)}{ds} = -y(r,s).$$

Then, $x(r,s) = c_1(r)\cos s + c_2(r)\sin s$ and $y(r,s) = c_3(r)\cos s + c_4(r)\sin s$. Using the initial condition, we get $c_1(r) = r$ and $c_3(r) = 0$. Also,

$$0 = -y(r,0) = \frac{dx(r,s)}{ds}\,|_{s=0} = -c_1(r)\sin 0 + c_2(r)\cos 0 = c_2(r).$$

and, similarly, $c_4(r) = r$. Also, $z(r,s) = c_5(r)e^s$ where $c_5(r) = \phi(r)$. Thus, we have $(x(r,s), y(r,s)) = (r\cos s, r\sin s)$, where $r > 0$ and $0 \leq s \leq \pi/2$. Hence, $r = (x^2+y^2)^{1/2}$ and $s = \arctan(y/x)$. Therefore, for any given (x,y), we have

$$u(x,y) = z(r,s) = \phi(r)e^s = \phi(\sqrt{x^2+y^2})e^{\arctan(y/x)}.$$

Example 2.30. Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. Let $\Gamma := \{(x,0) \mid x \in \mathbb{R}\}$. Consider the semi-linear PDE

$$\begin{cases} u_x(x,y) + u_y(x,y) &= u^2(x,y) \quad \text{in } \Omega \\ u(x,0) &= \phi(x) \quad \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r,0)$ for all $r \in \mathbb{R}$. Therefore,

$$(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (1,1)\cdot(0,1) = 1 \neq 0.$$

Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = 1, \frac{dy(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = z^2(r,s)$$

with initial conditions

$$x(r,0) = r, \quad y(r,0) = 0, \text{ and } z(r,0) = \phi(r).$$

Therefore, $x(r,s) = s + c_1(r)$, $y(r,s) = s + c_2(r)$ and $z(r,s) = \frac{-1}{s+c_3(r)}$. Using the initial conditions, we get $c_1(r) = r$, $c_2(r) = 0$ and $c_3(r) = -(\phi(r))^{-1}$. Note that this makes sense only if $\phi(r) \neq 0$ for all r. To overcome this situation, we write $z(r,s) = \frac{\phi(r)}{1-\phi(r)s}$. Also, we have $(x(r,s), y(r,s)) = (s+r, s)$, where $s \geq 0$ and $r \in \mathbb{R}$. Moreover, $r = x - y$ and $s = y$. Therefore, $u(x,y) = z(r,s) = \frac{\phi(r)}{1-\phi(r)s} = \frac{\phi(x-y)}{1-\phi(x-y)y}$. Note that the non-linearity in the z-variable, even though the equation is linear, may cause a possible blow-up (or singularity) in the solution. For instance, even if we assume ϕ is bounded, a very large value of y may induce a singularity.

Example 2.31. Consider the quasi-linear PDE

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) & = x \quad \text{in } \Omega \\ u(x,0) & = 1 \quad \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r, 0)$ for all $r \in \mathbb{R}$. Therefore,

$$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (u, 1) \cdot (0, 1) = 1 \neq 0.$$

Hence, the given initial curve is non-characteristic. The characteristic equations are

$$\frac{dx(r,s)}{ds} = z(r,s), \frac{dt(r,s)}{ds} = 1 \text{ and } \frac{dz(r,s)}{ds} = x(r,s)$$

with initial conditions

$$x(r,0) = r, \quad t(r,0) = 0, \text{ and } z(r,0) = 1.$$

Therefore, $t(r,s) = s + c_1(r)$, $x(r,s) = c_2(r)e^s + c_3(r)e^{-s}$ and $z(r,s) = c_4(r)e^s + c_5(r)e^{-s}$. Using the initial conditions, we get $c_1(r) = 0$, $c_2(r) = (r+1)/2 = c_4(r)$, $c_3(r) = (r-1)/2$ and $c_5(r) = (1-r)/2$. Solving for r and s, in terms of x, t and z, we get $s = t$ and

$$r = \frac{2x - e^t + e^{-t}}{e^t + e^{-t}}.$$

Therefore, $u(x,y) = x\frac{e^{2t}-1}{e^{2t}+1} + \frac{2e^t}{e^{2t+1}}$.

2.6.2 Nonlinear

Definition 2.6.5. *We say* $\Gamma \subset \Omega \subset \mathbb{R}^n$ *is* non-characteristic *for the nonlinear Cauchy problem*

$$\begin{cases} F(\nabla u(x), u(x), x) &= 0 \quad x \in \Omega \\ u &= g \quad on\ \Gamma \end{cases} \tag{2.6.5}$$

if Γ *is nowhere tangent to the Monge cone, i.e., there exists function* $\mathbf{v}$ *such that* $F(\mathbf{v}(r), g(r), \gamma_1(r), \ldots, \gamma_n(r)) = 0$ *and* $\mathbf{v}_i = g_{x_i}$*, for* $i = 1, 2, \ldots, n-1$ *on* Γ*, and satisfies*

$$\sum_{i=1}^{n} F_{p_i}(\mathbf{v}, g, \gamma_1, \ldots, \gamma_n) \cdot \nu(x) \neq 0$$

where $\nu(x)$ *is normal to* Γ *at* x*.*

In particular, in the two dimension case, $\Gamma = \{\gamma_1(r), \gamma_2(r)\} \subset \Omega \subset \mathbb{R}^2$ is *non-characteristic* for the nonlinear Cauchy problem (2.6.5) if Γ is nowhere tangent to the Monge cone, i.e., there exists function $\alpha(r)$ and $\beta(r)$ such that $F(\alpha(r), \beta(r), g(r), \gamma_1(r), \gamma_2(r)) = 0$ and $g'(r) = \alpha(r)\gamma_1'(r) + \beta(r)\gamma_2'(r)$ and Γ satisfies

$$(F_{p_2}(\gamma_1, \gamma_2, g, \alpha, \beta), F_{p_1}(\gamma_1, \gamma_2, g, \alpha, \beta)) \cdot (-\gamma_2', \gamma_1') \neq 0.$$

Example 2.32. For a nonlinear PDE the concept of characteristics also depend on initial values. Consider the nonlinear PDE $\sum_{i=1}^n u_{x_i}^2 = 1$ with $u(x', 0) = \phi(x')$. Note that $u = x_i$ is a solution with $\phi(x') = x_i$. However, any choice of ϕ such that $|\nabla_{x'}\phi(x')|^2 > 1$ has no solution.

Suppose v is a solution of $F(x, u, \nabla u) = 0$ in a neighbourhood of some point $x_0 \in \Gamma$.

Example 2.33. (i) Find the general solution (in terms of arbitrary functions) of the first order PDE $xu_x(x, y) + 2xuu_y(x, y) = u(x, y)$.

(ii) For the PDE given above, check if the following curves in xy-plane are non-characteristic and discuss the particular solutions of the PDE

(a) $y = 2x^2 + 1$ and $u(x, 2x^2 + 1) = 2x$.

(b) $y = 3x^3$ and $u(x, 3x^3) = 2x^2$.

(c) $y = x^3 - 1$ and $u(x, x^3 - 1) = x^2$.

Observe the nature of solutions for the same PDE on a characteristic curve and on non-characteristic curve.

(i) The characteristic equations are

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = 2xz \text{ and } \frac{dz}{ds} = z.$$

Hence,

$$x(s) = c_1 e^s \quad z(s) = c_2 e^s \text{ and } y(s) = c_1 c_2 e^{2s} + c_3.$$

Thus, $y = c_2/c_1 x^2 + c_3$ and $z = c_2/c_1 x$. Therefore, $y - zx = c_3$ and, for some f,

$$u(x, y) = xf(y - xu).$$

The characteristic curves are $y - xu =$ a constant which depends on u.

(ii) (a) Parametrise the curve $y = 2x^2 + 1$ as $\Gamma(r) : r \mapsto (r, 2r^2 + 1)$. Thus $\gamma_1(r) = r$ and $\gamma_2(r) = 2r^2 + 1$. Since the coefficients of the PDE are $a(r) = r$ and $b(r, u) = 4r^2$, we have

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (r, 4r^2) \cdot (-4r, 1) = -4r^2 + 4r^2 = 0.$$

Hence Γ is not non-characteristic. But on the characteristic curves $y - 2x^2 = 1$ the function $u = 2x$ solves the PDE. Elsewhere the solution is non-unique and there are many choices because $u(x, 2x^2 + 1) = 2x$ implies $f(1) = 2$. Thus, we have infinite number of solutions (or choices for) u that solves the PDE on other characterisitc curves.

(b) Parametrise the curve $y = 3x^3$ as $\Gamma(r) : r \mapsto (r, 3r^3)$. Hence $\gamma_1(r) = r$ and $\gamma_2(r) = 3r^3$ and

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (r, 4r^3) \cdot (-9r^2, 1) = -9r^3 + 4r^3 = -5r^3 \neq 0$$

for $r \neq 0$. Hence Γ is a non-characteristic curve. Using $(u(x, 3x^3) = 2x^2$ we get $2x^2 = xf(3x^3 - 2x^3)$ which implies $2x = f(x^3)$. Thus, $f(x) = 2x^{1/3}$ and $u(x, y) = 2x(y - xu)^{1/3}$ or $u^3(x, y) + 8x^4 u = 8x^3 y$. Thus, we have a unique solution. The characteristic curves are $y - 2x^3 =$ a constant and the data $u = 2x^2$ is given on $y - 3x^3 = 0$.

(c) Parametrise the curve $y = x^3 - 1$ as $\Gamma(r) : r \mapsto (r, r^3 - 1)$. Hence $\gamma_1(r) = r$ and $\gamma_2(r) = r^3 - 1$ and

$$(a, b) \cdot (-\gamma_2'(r), \gamma_1'(r)) = (r, 2r^3) \cdot (-3r^2, 1) = -3r^3 + 2r^3 = -r^3 \neq 0$$

for $r \neq 0$. Hence Γ is a non-characteristic curve. Using $(u(x, x^3 - 1) = x^2$ we get $x = f(-1)$. Thus f is not well defined and, hence there is no solution. The characteristic curves are $y - x^3 =$ a constant and $u = x^2$ given on $y - x^3 = -1$ is not a solution.

Example 2.34. We now give an example of a fully non linear PDE. Let $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. Let $\Gamma := \{(0, y) \mid y \in \mathbb{R}\}$. Consider the fully non-linear PDE

$$\begin{cases} u_x u_y &= u(x, y) \quad \text{in } \Omega \\ u(0, y) &= y^2 \quad \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (0, r)$ for all $r \in \mathbb{R}$. We first look for the functions α and β such that

$$\alpha(r)\beta(r) = r^2 \text{ and } 2r = \beta(r).$$

Solving them, we get $\beta(r) = 2r$ and $\alpha(r) = r/2$. Since $F(p, z, x) = p_1 p_2 - z$, we have

$$F_{p_2}\gamma_2'(r) = p_1 = \alpha(r) = r/2 \neq 0, \quad \text{for } r \neq 0.$$

Hence, the given initial curve is non-characteristic. The characteristic equations are (using (2.4.1))

$$\left(\frac{dx(r, s)}{ds}, \frac{dy(r, s)}{ds}\right) = (p_2(r, s), p_1(r, s)),$$

(using (2.4.2))

$$\frac{dp(r, s)}{ds} = p(r, s)$$

and (using (2.4.3))

$$\frac{dz(r, s)}{ds} = (p_1(r, s), p_2(r, s)) \cdot (p_2(r, s), p_1(r, s)) = 2p_1(r, s)p_2(r, s)$$

with initial conditions

$$x(r, 0) = 0, y(r, 0) = r, z(r, 0) = r^2, p_1(r, 0) = \alpha(r) = \frac{r}{2}$$

and

$$p_2(r,0) = \beta(r) = 2r.$$

Thus, on integrating, we get $p_1(r,s) = (r/2)e^s$ and $p_2(r,s) = 2re^s$, for all $s \in \mathbb{R}$. Using p, we solve for x to get $x(r,s) = 2r(e^s - 1)$ and $y(r,s) = (r/2)(e^s + 1)$. Solving for z, we get

$$z(r,s) = \frac{r^2}{2}(e^{2s} + 1).$$

Solving r and s in terms of x and y, we get

$$r = \frac{4y - x}{4} \text{ and } e^s = \frac{x + 4y}{4y - x}.$$

Hence $u(x,y) = z(r(x,y), s(x,y)) = \frac{(x+4y)^2}{16}$.

Example 2.35. Consider the fully non-linear PDE

$$\begin{cases} u_x u_y &= u(x,y) \quad \text{in } \mathbb{R}^2 \\ u(x, 1+x) &= x^2 \quad \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r, 1+r)$ for all $r \in \mathbb{R}$. We first look for the functions α and β such that

$$\alpha(r)\beta(r) = r^2 \text{ and } 2r = \alpha(r) + \beta(r).$$

Solving them, we get $\beta(r) = r = \alpha(r)$. Since $F(p,z,x) = p_1p_2 - z$, we have

$$F_{p_1}\gamma_1' - F_{p_2}\gamma_2'(r) = p_2 - p_1 = \beta(r) - \alpha(r) = r - r = 0, \quad \forall r.$$

Hence, the given initial curve is not non-characteristic. The characteristic equations are (using (2.4.1))

$$\left(\frac{dx(r,s)}{ds}, \frac{dy(r,s)}{ds}\right) = (p_2(r,s), p_1(r,s)),$$

(using (2.4.2))

$$\frac{dp(r,s)}{ds} = p(r,s)$$

and (using (2.4.3))

$$\frac{dz(r,s)}{ds} = (p_1(r,s), p_2(r,s)) \cdot (p_2(r,s), p_1(r,s)) = 2p_1(r,s)p_2(r,s)$$

with initial conditions

$$x(r,0) = r, \quad y(r,0) = 1+r, z(r,0) = r^2, p_1(r,0) = \alpha(r) = r$$

and

$$p_2(r,0) = \beta(r) = r.$$

Thus, on integrating, we get $p_1(r,s) = re^s$ and $p_2(r,s) = re^s$, for all $s \in \mathbb{R}$. Using p, we solve for x to get $x(r,s) = re^s$ and $y(r,s) = re^s + 1$. Solving for z, we get $z(r,s) = r^2e^{2s}$. Note that there is no unique way of solving r and s in terms of x and y. In fact, we have three possible represenation of u, viz., $u = x^2$, $u = (y-1)^2$ and $u = x(y-1)$. This is because the Jacobian is zero. Of these three possibilities, only $u = x(y-1)$ satisfies the equation.

Example 2.36. For any given $\lambda \in \mathbb{R}$, consider the fully non-linear PDE

$$\begin{cases} u_x^2 + u_y^2 &= 1 \quad \text{in } \mathbb{R}^2 \\ u(x,x) &= \lambda x \quad \text{on } \Gamma. \end{cases}$$

The parametrisation of the initial curve is $\Gamma(r) := (r,r)$ for all $r \in \mathbb{R}$. We first look for the functions α and β such that

$$\alpha^2(r) + \beta^2(r) = 1 \text{ and } \lambda = \alpha(r) + \beta(r).$$

We can view $\alpha(r) = \cos\theta$ and $\beta(r) = \sin\theta$ where θ is such that $\cos\theta + \sin\theta = \lambda$. Since $F(p,z,x) = p_1^2 + p_2^2 - 1$, we have

$$F_{p_1}\gamma_1' - F_{p_2}\gamma_2'(r) = 2(p_1 - p_2) = 2(\cos\theta - \sin\theta) = 0, \quad \forall \theta = \pi/4 + k\pi,$$

where $k = 1, 2, \ldots$. Hence, the given initial curve is non-characteristic for $\theta \neq \pi/2 + k\pi$ for all k and $\cos\theta + \sin\theta = \lambda$. The characteristic equations are (using (2.4.1))

$$\left(\frac{dx(r,s)}{ds}, \frac{dy(r,s)}{ds}\right) = (2p_1(r,s), 2p_2(r,s)),$$

(using (2.4.2))

$$\frac{dp(r,s)}{ds} = 0$$

and (using (2.4.3))

$$\frac{dz(r,s)}{ds} = (p_1(r,s), p_2(r,s)) \cdot (2p_1(r,s), 2p_2(r,s)) = 2(p_1^2 + p_2^2) = 2$$

with initial conditions

$$x(r,0)=r,\quad y(r,0)=r, z(r,0)=\lambda r, p_1(r,0)=\alpha(r)=\cos\theta$$

and

$$p_2(r,0)=\beta(r)=\sin\theta.$$

Thus, on integrating, we get $p_1(r,s)=\cos\theta$ and $p_2(r,s)=\sin\theta$. Using p, we solve for x to get $x(r,s)=2s\cos\theta+r$ and $y(r,s)=2s\sin\theta+r$. Solving for z, we get $z(r,s)=2s+\lambda r$. Solving r and s in terms of x and y, we get

$$s=\frac{x-y}{2(\cos\theta-\sin\theta)}\quad\text{and } r=\frac{x\sin\theta-y\cos\theta}{\sin\theta-\cos\theta}.$$

Therefore, the general solution is

$$\begin{aligned}u(x,y) &= \frac{x-y+\lambda(y\cos\theta-x\sin\theta)}{\cos\theta-\sin\theta}\\ &= \frac{x-y+(\cos\theta+\sin\theta)(y\cos\theta-x\sin\theta)}{\cos\theta-\sin\theta}\\ &= \frac{x(1-\sin\theta\cos\theta-\sin^2\theta)+y(\cos^2\theta+\sin\theta\cos\theta-1)}{\cos\theta-\sin\theta}\\ &= x\cos\theta+y\sin\theta.\end{aligned}$$

Example 2.37. Consider the equation $xu_xu_y+yu_y^2=1$ with $u(2r,0)=r$. Note that $F(p,z,x,y)=xp_1p_2+yp_2^2-1=0$. Thus the ODE

$$\left(\frac{dx(s)}{ds},\frac{dy(s)}{ds}\right)=(xp_2,xp_1+2yp_2),$$

$$\frac{dp(s)}{ds}=-(p_1p_2,p_2^2)$$

Thus, on integrating, we get $p_1/p_2=a$. Using the PDE, we get $(xa+y)p_2^2=1$. Thus,

$$p_2=\pm\frac{1}{\sqrt{xa+y}}\quad p_1=\pm\frac{a}{\sqrt{xa+y}}.$$

Now,

$$\begin{aligned}\frac{dz(s)}{ds} &= (p_1(s),p_2(s))\cdot\left(\frac{dx(s)}{ds},\frac{dy(s)}{ds}\right)\\ &= \pm\frac{1}{\sqrt{xa+y}}\left(a\frac{dx(s)}{ds}+\frac{dy(s)}{ds}\right)\\ u(x,y) &= \pm2\sqrt{ax+y}+b.\end{aligned}$$

Thus, $(u(x,y)-b)^2 = 4(ax+y)$ is a complete integral. Using the initial data, we get $(r-b)^2 = 8ar$. Differentiating w.r.t r, we get $r = 4a+b$ and eliminating r, we get $b = h(a) := -2a$. Hence, $(u+2a)^2 = 4(ax+y)$. Now solving for a in $D_a u = 0$, we get $4(u+2a) = 4x$. Then $a = \phi(x) := (x-u)/2$ and the solution is

$$x^2 = 4\left(\frac{x-u}{2}\right)x + 4y$$

which yields $u(x,y) = \frac{x^2+4y}{2x}$.

Example 2.38. Consider the equation $y(u_x^2 - u_y^2) + uu_y = 0$ with $u(2y,y) = 3y$. Note that $F(p,z,x,y) = y(p_1^2 - p_2^2) + zp_2 = 0$. Thus the ODEs become

$$\left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) = (2yp_1, z - 2yp_2),$$

$$\frac{dp(s)}{ds} = -(p_1 p_2, p_1^2)$$

Thus, on integrating, we get $p_1^2 - p_2^2 = a$. Using the PDE, we get $ya + up_2 = 0$. Thus,

$$p_2 = -\frac{ay}{z} \quad p_1 = \pm\sqrt{a + \frac{a^2y^2}{z^2}}.$$

Now,

$$\begin{aligned}
\frac{dz(s)}{ds} &= (p_1(s), p_2(s)) \cdot \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}\right) \\
\mp\sqrt{a}\frac{dx(s)}{ds} &= -\frac{1}{\sqrt{z^2+ay^2}}\left(z\frac{dz(s)}{ds} + ay\frac{dy(s)}{ds}\right) \\
\mp\sqrt{a}x \mp b &= -\sqrt{z^2+ay^2} \\
u^2(x,y) &= (b + x\sqrt{a})^2 - ay^2.
\end{aligned}$$

Using the initial data, we get $9y^2 = (b + 2y\sqrt{a})^2 - ay^2$ which is satisfied with $b = 0$ and $a = 3$. Thus, $u(x,y)^2 = 3(x^2 - y^2)$.

Example 2.39. Consider the equation $y(u_x^2 - u_y^2) + uu_y = 0$ with $u(r^2, 0) = 2r$. Note that $F(p,z,x,y) = y(p_1^2 - p_2^2) + zp_2 = 0$. As in above example, we get $u^2(x,y) = (b + x\sqrt{a})^2 - ay^2$. Using the initial conditions, we get $4r^2 = (b + r^2\sqrt{a})^2$. Differentiating this w.r.t to r and solving for r, we get

$$r^2 = \frac{1}{\sqrt{a}}\left(\frac{2}{\sqrt{a}} - b\right).$$

Substituting this in the equation of r, we get $b = 1/\sqrt{a}$. This gives

$$u^2 = \left(\frac{1}{\sqrt{a}} + x\sqrt{a}\right)^2 - ay^2 = \frac{1}{a}(1+xa)^2 - ay^2.$$

Now solving for a in $D_a u = 0$, we get

$$\begin{aligned} 0 &= 2\left(\frac{1}{\sqrt{a}} + x\sqrt{a}\right)\left(-\frac{1}{2a\sqrt{a}} + \frac{x}{2\sqrt{a}}\right) - y^2 \\ y^2 &= \left(x^2 - \frac{1}{a^2}\right) \\ a &= \phi(x) := \frac{1}{\sqrt{x^2 - y^2}}. \end{aligned}$$

We choose the positive root above to keep $a > 0$ so that all roots above made sense. Therefore,

$$u^2(x,y) = \sqrt{x^2 - y^2}\left(1 + \frac{x}{\sqrt{x^2-y^2}}\right)^2 - \frac{y^2}{\sqrt{x^2-y^2}}.$$

Exercise 10. Find the general solution of the following PDE. Check if the given data curve is non-characteristic or not. Also find the solution(s) (if it exists) given the value of u on the prescribed curves.

(i) $2u_t + 3u_x = 0$ with $u(x,0) = \sin x$.

(ii) $u_x - u_y = 1$ with $u(x,0) = x^2$.

(iii) $u_x + u_y = u$ with $u(x,0) = \cos x$.

(iv) $u_x - u_y = u$ with $u(x,-x) = \sin x$.

(v) $4u_x + u_y = u^2$ with $u(x,0) = \frac{1}{1+x^2}$.

(vi) $au_x + u_y = u^2$ with $u(x,0) = \cos x$.

(vii) $u_x + 4u_y = x(u+1)$ with $u(x,5x) = 1$.

(viii) $(1 - xu)u_x + y(2x^2 + u)u_y = 2x(1 - xu)$. Also, when $u(0,y) = e^y$ on $x = 0$.

(ix) $e^{2y}u_x + xu_y = xu^2$. Also, when $u(x,0) = e^{x^2}$ on $y = 0$.

(x) $u_x - 2xuu_y = 0$. Also, when $u(x,2x) = x^{-1}$ on $y = 2x$ and when $u(x,x^3) = x$ on $y = x^3$.

(xi) $-3u_x + u_y = 0$ with $u(x,0) = e^{-x^2}$. (Answer: $u(x,y) = e^{-(x+3y)^2}$).

(xii) $yu_x + xu_y = x^2 + y^2$ with $u(x,0) = 1 + x^2$ and $u(0,y) = 1 + y^2$. (Answer: $u(x,y) = xy + |x^2 - y^2|$).

(xiii) $yu_x + xu_y = 4xy^3$ with $u(x,0) = -x^4$ and $u(0,y) = 0$.

(xiv) $yu_x + xu_y = u$ with $u(x,0) = x^3$.

(xv) $u_x + yu_y = y^2$ with $u(0,y) = \sin y$.

(xvi) $u_x + yu_y = u^2$ with $u(0,y) = \sin y$.

(xvii) $u_x + yu_y = u$ with $u(x, 3e^x) = 2$.

(xviii) $u_x + yu_y = u$ with $u(x, e^x) = e^x$.

(xix) $u_x + xu_y = u$ with $u(1,y) = \phi(y)$.

(xx) $xu_x + u_y = 3x - u$ with $u(x,0) = \arctan x$.

(xxi) $xu_x + u_y = 0$ with $u(x,0) = \phi(x)$.

(xxii) $xu_x + yu_y = u$ with $u(x,1) = 2 + e^{-|x|}$.

(xxiii) $xu_x + yu_y = xe^{-u}$ with $u(x,x^2) = 0$.

(xxiv) $xu_x - yu_y = 0$ with $u(x,x) = x^4$.

(xxv) $e^{2y}u_x + xu_y = xu^2$ with $u(x,0) = e^{x^2}$.

(xxvi) $uu_x + u_y = 1$ with $u(2r^2, 2r) = 0$ for $r > 0$. (Answer: No solution for $y^2 > 4x$).

(xxvii) $(y-u)u_x + (u-x)u_y = x - y$ with $u(x, 1/x) = 0$.

(xxviii) $x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$ with $u(x,-x) = 1$.

(xxix) $\sqrt{1-x^2}u_x + u_y = 0$ with $u(0,y) = y$.

Exercise 11. Solve the equation $xu_x + 2yu_y = 0$ with $u(1, y) = e^y$. Does a solutions exist with data on $u(0, y) = g(y)$ or $u(x, 0) = h(x)$? What happens to characteristic curves at $(0, 0)$?

Exercise 12. Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?

Exercise 13. Solve the equation $u_x + yu_y = 0$ with $u(x, 0) = 1$. Also, solve the equation with $u(x, 0) = x$. If there is no solution, give reasons for non-existence.

Exercise 14. (i) Find a complete integral of $x(u_x^2 + u_y^2) = uu_x$ with u given on the curve

(a) $u(2y, y) = 5y$

(b) $u(0, r^2) = 2r$

(ii) Find a complete integral of $4uu_x - u_y^3 = 0$ with u given on the curve $u(0, r) = 4r$.

2.7 Non-smooth Solutions and Shocks

Observe that in the Cauchy problem of Transport equation (2.6.4), suppose we choose a discontinuous ϕ, for instance,

$$\phi(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

then $u(x, t)$ inherits this jump continuity. In applications it is often necessary to consider such solutions which, by our definition, is not even a differentiable, hence, not a solution. This demand motivates us to redefine the notion of solution to accommodate non-smooth solutions. Further, in some quasilinear situations, eventhough we start with a smooth initial data ϕ, jump discontinuity might occur at some positive time t. For instance, consider the *Burgers' equation* given as

$$\begin{cases} u_t + uu_x &= 0 & x \in \mathbb{R} \text{ and } t \in (0, \infty) \\ u(x, 0) &= \phi(x) & x \in \mathbb{R}. \end{cases}$$

We parametrize the curve Γ with r-variable, i.e., $\Gamma = \{\gamma_1(r), \gamma_2(r)\} = \{(r, 0)\}$. Γ is non-characteristic, because $(u, 1) \cdot (0, 1) = 1 \neq 0$. The characteristic equations are:

$$\frac{dx(r,s)}{ds} = z, \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dz(r,s)}{ds} = 0$$

with initial conditions,

$$x(r, 0) = r, \quad t(r, 0) = 0, \text{ and } z(r, 0) = \phi(r).$$

Solving the ODE corresponding to z, we get $z(r, s) = c_3(r)$ with initial conditions $z(r, 0) = c_3(r) = \phi(r)$. Thus, $z(r, s) = \phi(r)$. Using this in the ODE of x, we get

$$\frac{dx(r,s)}{ds} = \phi(r).$$

Solving the ODE's, we get

$$x(r, s) = \phi(r)s + c_1(r), \quad t(r, s) = s + c_2(r)$$

with initial conditions

$$x(r, 0) = c_1(r) = r \text{ and } t(r, 0) = c_2(r) = 0.$$

Therefore,

$$x(r, s) = \phi(r)s + r, \text{ and } t(r, s) = s.$$

Solving r and s, in terms of x, t and z, we get $s = t$ and $r = x - zt$. Therefore, $u(x, t) = \phi(x - tu)$ is the solution in the implicit form.

Example 2.40. If the data $\phi(x) = c$, some constant, then $u(x, t) = c$ and the characteristic curves are $t = x/c$.

Example 2.41. In the Burgers' equation, suppose we choose the data $\phi(x) = x$, then

$$u(x, t) = \frac{x}{1 + t}.$$

Note that u has a singularity at $(0, -1)$. This can be observed in the following way: u takes the constant value c along the line $t = x/c - 1$ and all these curves intersect at $(0, -1)$ which means u is multiply defined at $(0, -1)$ or, rather undefined at $(0, -1)$.

Example 2.42. In the Burgers' equation, suppose we choose the data ϕ to be the function

$$\phi(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & x \geq 1. \end{cases}$$

Then the characteristic curves are

$$x = \begin{cases} t+c & c \leq 0 \\ (1-c)t & 0 \leq c \leq 1 \\ c & c \geq 1. \end{cases}$$

Therefore,

$$u(x,t) = \begin{cases} 1 & x \leq t \\ \frac{1-x}{1-t} & t \leq x \leq 1 \\ 0 & x \geq 1. \end{cases}$$

Note that for $t \leq 1$ the solution behaves well, but for $t \geq 1$, the characteristics start *crossing* each other on the line $t = x$ and $u(x,t)$ takes both 0 and 1 on the line $t = x$, for $t \geq 1$. This situation is called the *shock*.

Example 2.43. In the Burgers' equation, suppose we choose ϕ to be the function

$$\phi(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 1 \\ 1 & 1 < x. \end{cases}$$

Then the characteristic curves are

$$x = \begin{cases} -t+c & c < -1 \\ c(t+1) & -1 \leq c \leq 1 \\ t+c & 1 < c. \end{cases}$$

Therefore,

$$u(x,t) = \begin{cases} -1 & x+t < -1 \\ \frac{x}{t+1} & -(t+1) \leq x \leq (t+1) \\ 1 & 1 < x-t. \end{cases}$$

Example 2.44. In the Burgers' equation, suppose we choose ϕ to be the function

$$\phi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

Then note that

$$u(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > t, \end{cases}$$

but there is no information of u on the wedge $\{0 < x < t\}$.

Chapter 3

Classification by Characteristics

A general second order PDE is of the form $F\left(D^2u(x), Du(x), u(x), x\right) = 0$, for each $x \in \Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ is the unknown. A *Cauchy problem* poses the following: Given the knowledge of u on a smooth hypersurface $\Gamma \subset \Omega$ can one find the solution u of the PDE? The knowledge of u on Γ is said to be the *Cauchy data*.

What should be the minimum required Cauchy data for the Cauchy problem to be solved? Viewing the Cauchy problem as an initial value problem corresponding to ODE, we know that a unique solution exists to the second order ODE

$$\left\{\begin{array}{rll} y''(x) + P(x)y'(x) + Q(x)y(x) & = 0 & x \in I \\ y(x_0) & = y_0 & \\ y'(x_0) & = y'_0. & \end{array}\right.$$

where P and Q are continuous on I (assume I closed interval of $\mathbb{R}$) and for any point $x_0 \in I$. This motivates us to define the Cauchy problem for second order PDE as:

$$\left\{\begin{array}{rll} F\left(D^2u(x), Du(x), u(x), x\right) & = 0 & x \in \Omega \\ u(x) & = g(x) & x \in \Gamma \\ Du(x) \cdot \nu(x) & = h(x) & x \in \Gamma \end{array}\right. \tag{3.0.1}$$

where ν is the outward unit normal vector on the hypersurface Γ and g, h are known functions on Γ.

3.1 Semilinear

Consider the general second order qausilinear PDE with n independent variable

$$F(x, u, Du, D^2u) := A(x) \cdot D^2u - D(\nabla u, u, x), \tag{3.1.1}$$

where $A = A_{ij}$ is an $n \times n$ matrix with entries $A_{ij}(x, u, \nabla u)$, D^2u is the Hessian matrix. The dot product in LHS is in $\mathbb{R}^{n^2}$. Since we demand the solution to be in C^2, the mixed derivatives are equal and we can assume, without loss generality that, A is symmetric. In fact if A is not symmetric, we can replace A with $A^s := \frac{1}{2}(A_0 + A_0^t)$, which is symmetric since $A \cdot D^2u = A^s \cdot D^2u$.

Let $\Gamma \subset \Omega$ be an hypersurface defined implicitly as $S(x) = 0$ and $\nabla S \neq 0$. Given u and $\nabla u \cdot \nu$ on Γ, can we calculate all other derivatives of u on Γ? To answer this, we map Γ to a hyperplane Γ_0 by the map $\phi : \Gamma \to \Gamma_0$ with its components $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ such that $\phi_n(x) = S(x)$ and $J_\phi(x) \neq 0$, non-zero Jacobian of ϕ, for all $x \in \Omega$. For sufficiently regular hypersurfaces such a mapping exists. Note that Γ_0 is a subset of the hyperplane whose final coordinate is zero. Let $y = \phi(x)$ and $v(y) := u(x)$. Then

$$u_{x_i} = \sum_{k=1}^n v_{y_k} \frac{\partial \phi_k}{\partial x_i} \quad u_{x_i x_j} = \sum_{k,l=1}^n v_{y_k y_l} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_l}{\partial x_j} + \sum_{k=1}^n v_{y_k} \frac{\partial^2 \phi_k}{\partial x_i \partial x_j}.$$

Using the second equation the second order linear PDE becomes

$$\sum_{i,j,k,l=1}^n A_{ij} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_l}{\partial x_j} v_{y_k y_l} = \tilde{D}(\nabla v, v, y) - \sum_{i,k=1}^n A_{ij} v_{y_k} \frac{\partial^2 \phi_k}{\partial x_i \partial x_j}, \tag{3.1.2}$$

where RHS is all known on Γ_0. To understand LHS on Γ_0, note that using the first equation, $v_{y_k}(y_1, \ldots, y_{n-1}, 0)$, for all $k = 1, 2, \ldots, n$, are known in Γ_0. Therefore, we know $v_{y_k y_l}$, for all $l = 1, 2, \ldots, n-1$, on Γ_0 because

$$v_{y_k y_l}(y) = \lim_{h \to 0} \frac{1}{h}[v_{y_k}(y_1, \ldots, y_l + h, \ldots, y_{n-1}, 0) - v_{y_k}(y_1, \ldots, y_l, \ldots, y_{n-1}, 0)].$$

We only do not know $v_{y_n y_n}$ on Γ_0. Therefore the (3.1.2) can be rewritten as

$$\sum_{i,j=1}^n A_{ij} \frac{\partial \phi_n}{\partial x_i} \frac{\partial \phi_n}{\partial x_j} v_{y_n y_n} = \text{ terms known on } \Gamma_0.$$

Since $\phi_n(x) = S(x)$, we can compute $v_{y_n y_n}$ if

$$\sum_{i,j=1}^{n} A_{ij}(x) S_{x_i} S_{x_j} \neq 0$$

on Γ. Note that ∇S is the normal to the hypersurface $S(x) = 0$.

Definition 3.1.1. *We say a hypersurface* $\Gamma \subset \Omega \subset \mathbb{R}^n$ *is* non-characteristic *w.r.t* (3.1.1) *if*

$$\sum_{i,j=1}^{n} A_{ij}(x) \nu_i(x) \nu_j(x) \neq 0.$$

where $\nu(x)$ *is the normal vector of* Γ *at* x*.*

Since any real symmetric matrix can always be diagonalised, there is a coordinate transformation $T(x)$ such that the matrix $T(x)A(x)T^t(x)$ is diagonal with diagonal entries, say $\lambda_1, \lambda_2, \ldots, \lambda_n$, for each $x \in \Omega$. Since $A(x)$ is real symmetric all $\lambda_i \in \mathbb{R}$, for all i. Thus, we classify PDE at a point $x \in \Omega$ based on the eigenvalues of the matrix $A(x)$. Let p denote the number of eigenvalues that are strictly positive and z denote the number of zero eigenvalues.

Definition 3.1.2. *We say a PDE is* hyperbolic *at a point* $x \in \Omega$*, if* $z = 0$ *and either* $p = 1$ *or* $p = n - 1$*. We say it is* parabolic *if* $z > 0$*. We say it is* elliptic*, if* $z = 0$ *and either* $p = n$ *or* $p = 0$*. If* $z = 0$ *and* $1 < p < n - 1$ *then the PDE is said to be* ultra hyperbolic*.*

Note that the elliptic case corresponds to the situation that every hypersurface $S(x) = 0$ with $\nabla S \neq 0$ is non-characteristic corresponding the elliptic operator, i.e., there are no real characteristics curves. Thus, one can equivalently define a linear second order PDE to be *elliptic* at x if

$$\sum_{i,j=1}^{n} A_{ij}(x) \xi_i \xi_j \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

3.1.1 Semilinear: Two Dimension

Consider the Cauchy problem (3.0.1) in two variables and set $x = (x, y)$. Let τ denote the unit tangent vector on Γ. Then, the directional derivative along the tangent vector, $Du(x,y)\cdot\tau(x,y) = g'(x,y)$ is known because g is known. Thus, we may compute directional derivative of u in any direction, along Γ, as a linear combination of $Du\cdot\tau$ and $Du\cdot\nu$. Using this we may reformulate (3.0.1) as

$$\begin{cases} F\left(D^2u, Du, u, x, y\right) &= 0 \quad (x,y)\in\Omega \\ u(x,y) &= g(x,y) \quad (x,y)\in\Gamma \\ u_x(x,y) &= h_1(x,y) \quad (x,y)\in\Gamma \\ u_y(x,y) &= h_2(x,y) \quad (x,y)\in\Gamma \end{cases}$$

with the *compatibility condition* that $\dot{g}(s) = h_1\dot{\gamma}_1(s) + h_2\dot{\gamma}_2(s)$, where $s\mapsto(\gamma_1(s),\gamma_2(s))$ is the parametrisation[1] of the hypersurface Γ. The compatibility condition is an outcome of the fact that

$$\dot{u}(s) = u_x\dot{\gamma}_1(s) + u_y\dot{\gamma}_2(s).$$

The above condition implies that among g, h_1, h_2 only two can be assigned independently.

Consider the Cauchy problem for the second order semi-linear PDE in two variables $(x,y)\in\Omega\subset\mathbb{R}^2$,

$$\begin{cases} A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} &= D \quad (x,y)\in\Omega \\ u(x,y) &= g(x,y) \quad (x,y)\in\Gamma \\ u_x(x,y) &= h_1(x,y) \quad (x,y)\in\Gamma \\ u_y(x,y) &= h_2(x,y) \quad (x,y)\in\Gamma. \end{cases} \tag{3.1.3}$$

where $D(x,y,u,u_x,u_y)$ may be non-linear and Γ is a smooth[2] curve in Ω. Also, one of the coefficients A, B or C is identically non-zero (else the PDE is not of second order). Let $s\mapsto(\gamma_1(s),\gamma_2(s))$ be a parametrisation of the curve Γ. Then we have the compatibility condition that

$$\dot{g}(s) = h_1\dot{\gamma}_1(s) + h_2\dot{\gamma}_2(s).$$

By computing the second derivatives of u on Γ and considering u_{xx}, u_{yy} and u_{xy} as unknowns, we have the linear system of three equations in three unknowns on Γ,

$$\begin{array}{rrrl} Au_{xx} & +2Bu_{xy} & +Cu_{yy} & = D \\ \dot{\gamma}_1(s)u_{xx} & +\dot{\gamma}_2(s)u_{xy} & & = \dot{h}_1(s) \\ & \dot{\gamma}_1(s)u_{xy} & +\dot{\gamma}_2(s)u_{yy} & = \dot{h}_2(s). \end{array}$$

[1]($\prime$) denotes the derivative with respect to space variable and ($\cdot$) denotes the derivative with respect to parameter

[2]twice differentiable

This system of equation is solvable if the determinant of the coefficients are non-zero, i.e.,

$$\begin{vmatrix} A & 2B & C \\ \dot{\gamma}_1 & \dot{\gamma}_2 & 0 \\ 0 & \dot{\gamma}_1 & \dot{\gamma}_2 \end{vmatrix} \neq 0.$$

Definition 3.1.3. *We say a curve* $\Gamma \subset \Omega \subset \mathbb{R}^2$ *is* characteristic *(w.r.t (3.1.3)) if*

$$A\dot{\gamma}_2^2 - 2B\dot{\gamma}_1\dot{\gamma}_2 + C\dot{\gamma}_1^2 = 0.$$

where $(\gamma_1(s), \gamma_2(s))$ *is a parametrisation of* Γ.

Note that the geometry hidden in the above definition is very similar to that we encountered in first order equation. Since $\nu = (-\dot{\gamma}_2, \dot{\gamma}_1)$ is the normal to Γ at each point, the above definition says that the curve is non-characteristic if

$$\sum_{i,j=1}^{2} A_{ij}\nu_i\nu_j = A\dot{\gamma}_2^2 - 2B\dot{\gamma}_1\dot{\gamma}_2 + C\dot{\gamma}_1^2 \neq 0$$

where $A_{11} = A$, $A_{12} = A_{21} = B$ and $A_{22} = C$. If $y = y(x)$ is a representation of the curve Γ (locally, if necessary), we have $\gamma_1(s) = s$ and $\gamma_2(s) = y(s)$. Then the characteristic equation reduces as

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0.$$

Therefore, the characteristic curves of (3.1.3) are given by the graphs whose equation is

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

Thus, we have three situations arising depending on the sign of the *discriminant*, $B^2 - AC$. This classifies the given second order PDE based on the sign of its discriminant $d = B^2 - AC$.

Definition 3.1.4. *We say a second order PDE is of*

(a) hyperbolic *type if* $d > 0$,

(b) parabolic *type if* $d = 0$ *and*

(c) elliptic *type if* $d < 0$.

The hyperbolic PDE have two families of characteristics, parabolic PDE has one family of characteristic and elliptic PDE have no characteristic. We caution here that these names are no indication of the shape of the graph of the solution of the PDE. The classification tells us the right amount of initial/boundary condition to be imposed for a PDE to be well-posed. For hyperbolic, which has two real characteristics, requires as many initial condition as the number of characteristics emanating from initial time and as many boundary conditions as the number of characteristics that pass into the spatial boundary. For parabolic, which has exactly one real characteristic, we need one boundary condition at each point of the spatial boundary and one initial condition at initial time. For elliptic, which has no real characteristic curves, we need one boundary condition at each point of the spatial boundary.

Note that the classification depends on the determinant of the coefficient matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

For every $(x, y) \in \Omega$, the matrix is symmetric and hence diagonalisable. If λ_1, λ_2 are the diagonal entries, then $d = -\lambda_1\lambda_2$. Thus, a equation is hyperbolic at a point (x, y) if the eigen values have opposite sign. It is ellipic if the eigenvalues have same sign and is parabolic if, at least, one of the eigenvalue is zero.

3.2 Quasilinear

All the arguments of semilinear PDE can be carried over to a quasilinear PDE $A(x, u(x), Du(x))$. For each specific point $x_0 \in \Omega$, $u(x_0) = u_0$ and $A_0 = A(x_0, u_0, \nabla u(x_0))$. The solutions to characteristic equation for a quasilinear equation depends on the solution considered. Set

$$U := \{(x, z, p) \mid x \in \Omega, z \in \mathbb{R}, p \in \mathbb{R}^n\}.$$

Definition 3.2.1. *A quasilinear equation*

$$A(x, u(x), Du(x)) \cdot D^2u(x) = D(x, u(x), Du(x))$$

is said to be elliptic *if the matrix* $A_{ij}(x,z,p)$ *is positive definite for each* $(x,z,p) \in U$. *Further,*

$$0 < \alpha(x,z,p)|\xi|^2 \leq \sum_{i,j=1}^{n} A_{ij}(x,z,p)\xi_i\xi_j \leq \beta(x,z,p)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

The bounds $\alpha(x,z,p)$ and $\beta(x,z,p)$ are minimum and maximum eigenvalues, respectively. If β/α is uniformly bounded in U then PDE is *uniformly elliptic*. The interesting thing about uniformly elliptic equation is that they behave very similar to linear elliptic equations.

Example 3.1. Consider the minimal surface equation

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = f(x)$$

where the second order coefficients are

$$A_{ij}(x,z,p) = (1+|p|^2)^{-1/2} \left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2} \right)$$

and

$$\alpha(x,z,p) = \frac{1}{(1+|p|^2)^{3/2}} \quad \beta(x,z,p) = \frac{1}{(1+|p|^2)^{1/2}}.$$

Thus, the equation is not uniformly elliptic.

The minimal surface equation and the capillary equation are not uniformly elliptic.

3.3 Examples

Example 3.2 (Wave Equation). For a given $c \in \mathbb{R}$, $u_{yy} - c^2 u_{xx} = 0$ is hyperbolic. Since $A = -c^2$, $B = 0$ and $C = 1$, we have $d = B^2 - AC = c^2 > 0$. The eigen values of the coefficient matrix are $1, -c^2$ which have opposite sign.

Example 3.3 (Heat Equation). For a given $c \in \mathbb{R}$, $u_y - cu_{xx} = 0$ is parabolic. Since $A = -c$, $B = 0$ and $C = 0$, thus $d = B^2 - AC = 0$. The eigen values of the coefficient matrix are $0, -c$ has a zero eigenvalue.

Example 3.4 (Laplace equation). $u_{xx} + u_{yy} = 0$ is elliptic. Since $A = 1$, $B = 0$ and $C = 1$, thus $d = B^2 - AC = -1 < 0$. The eigen values of the coefficient matrix are $1, 1$ which have same sign.

Example 3.5 (Velocity Potential Equation). In the equation $(1 - M^2)u_{xx} + u_{yy} = 0$, $A = (1 - M^2)$, $B = 0$ and $C = 1$. Then $d = B^2 - AC = -(1 - M^2)$. The eigen values of the coefficient matrix are $1 - M^2, 1$. Thus, for $M > 1$ (opposite sign), the equation is hyperbolic (supersonic flow), for $M = 1$ (zero eigenvalue) it is parabolic (sonic flow) and for $M < 1$ (same sign) it is elliptic (subsonic flow).

Note that the classification of PDE is dependent on its coefficients. Thus, for constant coefficients the type of PDE remains unchanged throughout the region Ω. However, for variable coefficients, the PDE may change its classification from region to region.

Example 3.6. An example is the *Tricomi* equation , $u_{xx} + xu_{yy} = 0$. The discriminant of the Tricomi equation is $d = -x$. The eigenvalues are $1, x$. Thus, tricomi equation is hyperbolic when $x < 0$, elliptic when $x > 0$ and degenerately parabolic when $x = 0$, i.e., y-axis. Such equations are called *mixed type*.

The notion of classification of second order semi-linear PDE, discussed in this section, could be generalised to quasi-linear, non-linear PDE and system of ODE. However, in these cases the classification may also depend on the solution u, as seen in the examples below.

Example 3.7. Consider the quasi-linear PDE $u_{xx} - uu_{yy} = 0$. The discriminant is $d = u$. The eigenvalues are $1, -u(x)$. It is hyperbolic for $\{u > 0\}$[3], elliptic when $\{u < 0\}$ and parabolic when $\{u = 0\}$.

Example 3.8. Consider the quasi-linear PDE

$$(c^2 - u_x^2)u_{xx} - 2u_x u_y u_{xy} + (c^2 - u_y^2)u_{yy} = 0$$

where $c > 0$. Then $d = B^2 - AC = c^2(u_x^2 + u_y^2 - c^2) = c^2(|\nabla u|^2 - c^2)$. It is hyperbolic if $|\nabla u| > c$, parabolic if $|\nabla u| = c$ and elliptic if $|\nabla u| < c$.

Example 3.9. Find the family of characteristic curves for the following second order PDE, whenever they exist.

(i) For a given $c \in \mathbb{R}$, $u_{yy} - c^2 u_{xx} = 0$.

(ii) For a given $c \in \mathbb{R}$, $u_y - cu_{xx} = 0$.

(iii) $u_{xx} + u_{yy} = 0$.

[3] The notation $\{u > 0\}$ means $\{x \in \Omega \mid u(x) > 0\}$

(iv) $u_{xx} + xu_{yy} = 0$.

Solution. (i) We have already seen the equation is hyperbolic and hence it should have two characteristic curves. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{\pm\sqrt{c^2}}{-c^2} = \frac{\mp 1}{c}.$$

Thus, $cy \pm x =$ a constant is the equation of the two characteristic curves. Note that the characteristic curves $y = \mp x/c + y_0$ are boundary of two cones in $\mathbb{R}^2$ with vertex at $(0, y_0)$.

(ii) We have already seen the equation is parabolic and hence it should have one characteristic curve. The characteristic curve are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = 0.$$

Thus, $y =$ a constant is the equation of the characteristic curve. i.e., any horizontal line in $\mathbb{R}^2$ is a charateristic curve.

(iii) We have already seen the equation is elliptic and hence has no real characteristics.

(iv) The equation is of mixed type. In the region $x > 0$, the characteristic curves are $y \mp 2x^{3/2}/3 =$ a constant.

□

Exercise 15. Classify the following second order PDE in terms of the number of characteristics:

(a) $3u_{xx} + u_{xy} + 2u_{yy} = 0$.

(b) $u_{zz} + u_z + u_{rr} + \frac{1}{r}u_\theta + c = 0$.

(c) $u_t + \beta u_x + \alpha u_{xx} = 0$.

(d) $4u_{xx} + y^2 u_x + xu_x + u_{yy} + 4u_{xy} - 4xy = 0$.

(e) $xu_{xx} + xu_{xy} + yu_{yy} = 0$.

(f) $xu_{xx} + yu_{xy} + c = 0$.

(g) $x^2 y u_{xx} + xy u_{xy} - y^2 u_{yy} = 0.$

(h) $\sin x u_{xx} + 2\cos x u_{xy} + \sin x u_{yy} = 0.$

(i) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0.$

(j) $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x + 4u = 0.$

(k) $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0.$

(l) $(1+x)u_{xx} + 2xy u_{xy} - y^2 u_{yy} = 0.$

(m) $2u_{xx} - 4u_{xy} + 7u_{yy} - u = 0.$

(n) $u_{xx} - 2\cos x u_{xy} - \sin^2 x u_{yy} = 0.$

(o) $y u_{xx} + 2(x-1)u_{xy} - (y+2)u_{yy} = 0.$

(p) $y u_{xx} + u_{xy} - x^2 u_{yy} - u_x - u = 0.$

Exercise 16. Classify the following second order PDE, in terms of the number of characteristics, and find their characteristics, when it exists:

(a) $u_{xx} + (5+2y^2)u_{xy} + (1+y^2)(4+y^2)u_{yy} = 0.$

(b) $y u_{xx} + u_{yy} = 0.$

(c) $y u_{xx} = x u_{yy}.$

(d) $u_{yy} - x u_{xy} + y u_x + x u_y = 0.$

(e) $y^2 u_{xx} + 2xy u_{xy} + x^2 u_{yy} = 0.$

(f) $u_{xx} + 2x u_{xy} + (1-y^2)u_{yy} = 0.$

3.4 System of First Order PDE

Recall that any second order ODE $y'' = f(x, y, y')$ can be equivalently written as a system of ODE, by using $u_1 = y$ and $u_2 = y'$, as follows:

$$\begin{cases} u_1' &= u_2 \\ u_2' &= f(x, u_1, u_2). \end{cases}$$

A similar procedure also makes a second order PDE into a system of first order PDEs. Thus, we expect that our classification of second order PDE to induce a classification for a system of first order PDE.

A general system of m first order linear PDE in n variables will be of the form

$$\sum_{j=1}^{n} \mathbf{A}_j(x, \mathbf{u})\mathbf{u}_{x_j} = \mathbf{f}(x, \mathbf{u}) \quad \text{in } \Omega, \tag{3.4.1}$$

where each $\mathbf{A}_j(x, \mathbf{u})$ is a $m \times m$ matrix, $\mathbf{u}(x) = (u^1, \ldots, u^m)$ and $\mathbf{f} = (f_1, \ldots, f_m)$ has m components. Following the arguments for semilinear PDE and using the map $\phi(x)$ and $v(y)$ between Γ and Γ_0, we get

$$\sum_{j=1}^{n} \mathbf{A}_j(x, \mathbf{u}) v_{y_n} S_{x_j} = \text{ terms known on } \Gamma_0.$$

This system is solvable for v_{y_n} if

$$\det \left(\sum_{j=1}^{n} \mathbf{A}_j(x, \mathbf{u}) S_{x_j} \right) \neq 0$$

on Γ.

Definition 3.4.1. *We say a hyperspace Γ in $\mathbb{R}^n$ is* non-characteristic *if*

$$det \left(\sum_{j=1}^{n} \mathbf{A}_j(x, \mathbf{u}) \nu_j(x) \right) \neq 0$$

on Γ, where $\nu(x) = (\nu_1, \ldots, \nu_n)$ is the normal at x of Γ.

Definition 3.4.2. *The system of first order PDE (3.4.1) is called* hyperbolic *if the $m \times m$ matrix*

$$\sum_{j=1}^{n} \xi_j A_j(x, \mathbf{u})$$

is diagonalisable for every $\xi = (\xi_j) \in \mathbb{R}^n$ and every $x \in \Omega$. If the eigenvalues are all distinct, for non-zero $\xi \in \mathbb{R}^n$, then the PDE is called strictly hyperbolic*.*

Definition 3.4.3. *The system of first order PDE* (3.4.1) *is called* elliptic *if*

$$det\left(\sum_{j=1}^{n}\mathbf{A}_j(x,\mathbf{u})\xi_j\right)=0$$

only for $\xi = 0$.

Example 3.10 (Beltrami Equations). Consider the system of first order equations

$$\begin{cases} W(x,y)u_x(x,y) - b(x,y)v_x(x,y) - c(x,y)v_y(x,y) &= 0 \\ W(x,y)u_y(x,y) + a(x,y)v_x(x,y) + b(x,y)v_y(x,y) &= 0 \end{cases}$$

where W, a, b, c are given such that $W \neq 0$ and the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite. Set

$$\mathbf{A}_1 = \begin{pmatrix} W & -b \\ 0 & a \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & -c \\ W & b \end{pmatrix}.$$

Then the system can be rewritten as

$$\mathbf{A}_1\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \mathbf{A}_2\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$\det\left(\sum_{j=1}^{n}\mathbf{A}_j(x,\mathbf{u})\xi_j\right) = \begin{vmatrix} W\xi_1 & -b\xi_1 - c\xi_2 \\ W\xi_2 & a\xi_1 + b\xi_2 \end{vmatrix} = W(a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2) \neq 0$$

if $\xi \neq 0$. Therefore, the Beltrami equation is elliptic. The Beltrami system is a generalization of the Cauchy-Riemann equations.

3.5 System of Second Order PDE

A general system of m second order linear PDE in n variables will be of the form

$$\sum_{i,j=1}^{n}\mathbf{A}_{ij}(x,\mathbf{u},\nabla\mathbf{u})\mathbf{u}_{x_ix_j} + \text{ lower order terms} = 0 \quad \text{in } \Omega, \tag{3.5.1}$$

where each $\mathbf{A}_{ij}$ is a $m \times m$ matrix and $\mathbf{u}(x) = (u^1, \ldots, u^m)$. Following the arguments for semilinear PDE and using the map $\phi(x)$ and $v(y)$ between Γ and Γ_0, we get

$$\sum_{i,j=1}^{n} \mathbf{A}_{ij}(x, \mathbf{u}, \nabla \mathbf{u}) v_{y_n y_n} S_{x_i} S_{x_j} = \text{ terms known on } \Gamma_0.$$

Definition 3.5.1. *The system is called* elliptic *if*

$$det \left(\sum_{i,j=1}^{n} \mathbf{A}_{ij} \xi_i \xi_j \right) = 0$$

only for $\xi = 0$.

3.6 Invariance of Discriminant

The classification of second order semi-linear PDE is based on the discriminant $B^2 - AC$. In this section, we note that the classification is independent of the choice of coordinate system (to represent a PDE). Consider the two-variable semilinear PDE

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = D(x,y,u,u_x,u_y) \quad (x,y) \in \Omega \tag{3.6.1}$$

where the variables (x, y, u, u_x, u_y) may appear non-linearly in D and $\Omega \subset \mathbb{R}^2$. Also, one of the coefficients A, B or C is identically non-zero (else the PDE is not of second order). We shall observe how (3.6.1) changes under coordinate transformation.

Definition 3.6.1. *For any PDE of the form* (3.6.1) *we define its* discriminant *as* $B^2 - AC$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the coordinate transformation with components $T = (w, z)$, where $w, z : \mathbb{R}^2 \to \mathbb{R}$. We assume that $w(x, y)$, $z(x, y)$ are such that w, z are both continuous and twice differentiable w.r.t (x, y), and the Jacobian J of T is non-zero,

$$J = \begin{vmatrix} w_x & w_y \\ z_x & z_y \end{vmatrix} \neq 0.$$

We compute the derivatives of u in the new variable,

$$\begin{aligned}
u_x &= u_w w_x + u_z z_x, \\
u_y &= u_w w_y + u_z z_y, \\
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx} \\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy} \\
u_{xy} &= u_{ww} w_x w_y + u_{wz}(w_x z_y + w_y z_x) + u_{zz} z_x z_y + u_w w_{xy} + u_z z_{xy}
\end{aligned}$$

Substituting above equations in (3.6.1), we get

$$a(w,z)u_{ww} + 2b(w,z)u_{wz} + c(w,z)u_{zz} = d(w,z,u,u_w,u_z).$$

where D transforms in to d and

$$a(w,z) = Aw_x^2 + 2Bw_x w_y + Cw_y^2 \tag{3.6.2}$$

$$b(w,z) = Aw_x z_x + B(w_x z_y + w_y z_x) + Cw_y z_y \tag{3.6.3}$$

$$c(w,z) = Az_x^2 + 2Bz_x z_y + Cz_y^2. \tag{3.6.4}$$

Note that the coefficients in the new coordinate system satisfy

$$b^2 - ac = (B^2 - AC)J^2.$$

Since $J \neq 0$, we have $J^2 > 0$. Thus, both $b^2 - ac$ and $B^2 - AC$ have the same sign. Thus, the sign of the *discriminant* is invariant under coordinate transformation. All the above arguments can be carried over to quasi-linear and non-linear PDE.

3.7 Standard or Canonical Forms

The advantage of above classification helps us in reducing a given PDE into simple forms. Given a PDE, one can compute the sign of the discriminant and depending on its clasification we can choose a coordinate transformation (w,z) such that

(i) For hyperbolic, $a = c = 0$ or $b = 0$ and $a = -c$.

(ii) For parabolic, $c = b = 0$ or $a = b = 0$. We conveniently choose $c = b = 0$ situation so that $a \neq 0$ (so that division by zero is avoided in the equation for characteristic curves).

(iii) For elliptic, $b = 0$ and $a = c$.

If the given second order PDE (3.6.1) is such that $A = C = 0$, then (3.6.1) is of hyperbolic type and a division by $2B$ (since $B \neq 0$) gives

$$u_{xy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/2B$. The above form is the *first standard form* of second order hyperbolic equation. If we introduce the linear change of variable $X = x + y$ and $Y = x - y$ in the first standard form, we get the *second standard form* of hyperbolic PDE

$$u_{XX} - u_{YY} = \hat{D}(X, Y, u, u_X, u_Y).$$

If the given second order PDE (3.6.1) is such that $A = B = 0$, then (3.6.1) is of parabolic type and a division by C (since $C \neq 0$) gives

$$u_{yy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/C$. The above form is the *standard form* of second order parabolic equation.

If the given second order PDE (3.6.1) is such that $A = C$ and $B = 0$, then (3.6.1) is of elliptic type and a division by A (since $A \neq 0$) gives

$$u_{xx} + u_{yy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/A$. The above form is the *standard form* of second order elliptic equation.

Note that the standard forms of the PDE is an expression with no mixed derivatives.

3.8 Reduction to Standard Form

Consider the second order semi-linear PDE (3.6.1) not in standard form. We look for transformation $w = w(x, y)$ and $z = z(x, y)$, with non-vanishing Jacobian, such that the reduced form is the standard form.

If $B^2 - AC > 0$, we have two characteristics. We are looking for the coordinate system w and z such that $a = c = 0$. This implies from equation (3.6.2) and (3.6.4) that we need to find w and z such that

$$\frac{w_x}{w_y} = \frac{-B \pm \sqrt{B^2 - AC}}{A} = \frac{z_x}{z_y}.$$

Therefore, we need to find w and z such that along the slopes of the characteristic curves,

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{-w_x}{w_y}.$$

This means that, using the parametrisation of the characteristic curves, $w_x\dot{\gamma}_1(s) + w_y\dot{\gamma}_2(s) = 0$ and $\dot{w}(s) = 0$. Similarly for z. Thus, w and z are chosen such that they are constant on the characteristic curves.

The characteristic curves are found by solving

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

and the coordinates are then chosen such that along the characteristic curve $w(x, y) =$ a constant and $z(x, y) =$ a constant. Note that $w_x z_y - w_y z_x = w_y z_y \left(\frac{2}{A}\sqrt{B^2 - AC}\right) \neq 0$.

Example 3.11. Let us reduce the PDE $u_{xx} - c^2 u_{yy} = 0$ to its canonical form. Note that $A = 1$, $B = 0$, $C = -c^2$ and $B^2 - AC = c^2$ and the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \pm c.$$

Solving we get $y \mp cx =$ a constant. Thus, $w = y + cx$ and $z = y - cx$. Now writing

$$\begin{aligned}
u_{xx} &= u_{ww}w_x^2 + 2u_{wz}w_x z_x + u_{zz}z_x^2 + u_w w_{xx} + u_z z_{xx} \\
&= c^2(u_{ww} - 2u_{wz} + u_{zz}) \\
u_{yy} &= u_{ww}w_y^2 + 2u_{wz}w_y z_y + u_{zz}z_y^2 + u_w w_{yy} + u_z z_{yy} \\
&= u_{ww} + 2u_{wz} + u_{zz} \\
-c^2 u_{yy} &= -c^2(u_{ww} + 2u_{wz} + u_{zz})
\end{aligned}$$

Substituting into the given PDE, we get

$$\begin{aligned}
0 &= 4c^2 u_{wz} \\
&= u_{wz}.
\end{aligned}$$

Example 3.12. Let us reduce the PDE $u_{xx} - x^2 y u_{yy} = 0$ given in the region $\{(x, y) \mid x \in \mathbb{R}, x \neq 0, y > 0\}$ to its canonical form. Note that $A = 1$, $B = 0$,

$C = -x^2y$ and $B^2 - AC = x^2y$. In the given region $x^2y > 0$, hence the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \pm x\sqrt{y}.$$

Solving we get $x^2/2 \mp 2\sqrt{y}$ = a constant. Thus, $w = x^2/2 + 2\sqrt{y}$ and $z = x^2/2 - 2\sqrt{y}$. Now writing

$$\begin{aligned}
u_x &= u_w w_x + u_z z_x = x(u_w + u_z)\\
u_y &= u_w w_y + u_z z_y = \frac{1}{\sqrt{y}}(u_w - u_z)\\
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx}\\
&= x^2(u_{ww} + 2u_{wz} + u_{zz}) + u_w + u_z\\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy}\\
&= \frac{1}{y}(u_{ww} - 2u_{wz} + u_{zz}) - \frac{1}{2y\sqrt{y}}(u_w - u_z)\\
-x^2 y u_{yy} &= -x^2(u_{ww} - 2u_{wz} + u_{zz}) + \frac{x^2}{2\sqrt{y}}(u_w - u_z)
\end{aligned}$$

Substituting into the given PDE, we get

$$\begin{aligned}
0 &= 4x^2 u_{wz} + \frac{2\sqrt{y} + x^2}{2\sqrt{y}} u_w + \frac{2\sqrt{y} - x^2}{2\sqrt{y}} u_z\\
&= 8x^2\sqrt{y} u_{wz} + (x^2 + 2\sqrt{y}) u_w + (2\sqrt{y} - x^2) u_z.
\end{aligned}$$

Note that $w + z = x^2$ and $w - z = 4\sqrt{y}$. Now, substituting x, y in terms of w, z, we get

$$\begin{aligned}
0 &= 2(w^2 - z^2) u_{wz} + \left(w + z + \frac{w - z}{2}\right) u_w + \left(\frac{w - z}{2} - w - z\right) u_z\\
&= u_{wz} + \left(\frac{3w + z}{4(w^2 - z^2)}\right) u_w - \left(\frac{w + 3z}{4(w^2 - z^2)}\right) u_z. \quad \square
\end{aligned}$$

Example 3.13. Let us reduce the PDE $u_{xx} + u_{xy} - 2u_{yy} + 1 = 0$ given in the region $\{(x, y) \mid 0 \le x \le 1, y > 0\}$ to its canonical form. Note that $A = 1$, $B = 1/2$, $C = -2$ and $B^2 - AC = 9/4 > 0$. Hence the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{1}{2} \pm \frac{3}{2} = 2 \text{ or } -1.$$

Solving we get $y-2x =$ a constant and $y+x =$ a constant. Thus, $w = y-2x$ and $z = y + x$.

In the parabolic case, $B^2 - AC = 0$, we have a single characteristic. We are looking for a coordinate system such that either $b = c = 0$.

Example 3.14. Let us reduce the PDE $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$ to its canonical form. Note that $A = e^{2x}$, $B = e^{x+y}$, $C = e^{2y}$ and $B^2 - AC = 0$. The PDE is parabolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B}{A} = \frac{e^y}{e^x}.$$

Solving, we get $e^{-y} - e^{-x} =$ a constant. Thus, $w = e^{-y} - e^{-x}$. Now, we choose z such that $w_x z_y - w_y z_x \neq 0$. For instance, $z = x$ is one such choice. Then

$$\begin{aligned}
u_x &= e^{-x}u_w + u_z \\
u_y &= -e^{-y}u_w \\
u_{xx} &= e^{-2x}u_{ww} + 2e^{-x}u_{wz} + u_{zz} - e^{-x}u_w \\
u_{yy} &= e^{-2y}u_{ww} + e^{-y}u_w \\
u_{xy} &= -e^{-y}(e^{-x}u_{ww} - u_{wz})
\end{aligned}$$

Substituting into the given PDE, we get

$$e^x e^{-y} u_{zz} = (e^{-y} - e^{-x})u_w$$

Replacing x, y in terms of w, z gives

$$u_{zz} = \frac{w}{1 + we^z}u_w.$$

Example 3.15. Let us reduce the PDE $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{1}{xy}(y^3u_x + x^3u_y)$ to its canonical form. Note that $A = y^2$, $B = -xy$, $C = x^2$ and $B^2 - AC = 0$. The PDE is parabolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{B}{A} = \frac{-x}{y}.$$

Solving, we get $x^2 + y^2 =$ a constant. Thus, $w = x^2 + y^2$. Now, we choose z such that $w_x z_y - w_y z_x \neq 0$. For instance, $z = x$ is one such choice. Then

$$\begin{aligned}
u_x &= 2xu_w + u_z \\
u_y &= 2yu_w
\end{aligned}$$

In the elliptic case, $B^2 - AC < 0$, we have no real characteristics. Thus, we choose w, z to be the real and imaginary part of the solution of the characteristic equation.

Example 3.16. Let us reduce the PDE $x^2 u_{xx} + y^2 u_{yy} = 0$ given in the region $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ to its canonical form. Note that $A = x^2$, $B = 0$, $C = y^2$ and $B^2 - AC = -x^2y^2 < 0$. The PDE is elliptic. Solving the characteristic equation

$$\frac{dy}{dx} = \pm \frac{iy}{x}$$

we get $\ln x \pm i \ln y = c$. Let $w = \ln x$ and $z = \ln y$. Then

$$\begin{aligned} u_x &= u_w/x \\ u_y &= u_z/y \\ u_{xx} &= -u_w/x^2 + u_{ww}/x^2 \\ u_{yy} &= -u_z/y^2 + u_{zz}/y^2 \end{aligned}$$

Substituting into the PDE, we get

$$u_{ww} + u_{zz} = u_w + u_z.$$

Example 3.17. Let us reduce the PDE $u_{xx} + 2u_{xy} + 5u_{yy} = xu_x$ to its canonical form. Note that $A = 1$, $B = 1$, $C = 5$ and $B^2 - AC = -4 < 0$. The PDE is elliptic. The characteristic equation is

$$\frac{dy}{dx} = 1 \pm 2i.$$

Solving we get $x - y \pm i2x = c$. Let $w = x - y$ and $z = 2x$. Then

$$\begin{aligned} u_x &= u_w + 2u_z \\ u_y &= -u_w \\ u_{xx} &= u_{ww} + 4u_{wz} + 4u_{zz} \\ u_{yy} &= u_{ww} \\ u_{xy} &= -(u_{ww} + 2u_{wz}) \end{aligned}$$

Substituting into the PDE, we get

$$u_{ww} + u_{zz} = x(u_w + 2u_z)/4.$$

Replacing x, y in terms of w, z gives

$$u_{ww} + u_{zz} = \frac{z}{8}(u_w + 2u_z).$$

Example 3.18. Let us reduce the PDE $u_{xx} + u_{xy} + u_{yy} = 0$ to its canonical form. Note that $A = 1$, $B = 1/2$, $C = 1$ and $B^2 - AC = -3/4 < 0$. The PDE is elliptic. Solving the characteristic equation

$$\frac{dy}{dx} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

we get $2y = x \pm i\sqrt{3}x + c$. Let $w = 2y - x$ and $z = \sqrt{3}x$.

Exercise 17. Rewrite the PDE in their canonical forms and solve them.

(a) $u_{xx} + 2\sqrt{3}u_{xy} + u_{yy} = 0$

(b) $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$

(c) $u_{xx} - (2\sin x)u_{xy} - (\cos^2 x)u_{yy} - (\cos x)u_y = 0$

(d) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

Chapter 4

Wave Equation

4.1 One Dimension

The one dimensional wave equation is the first ever partial differential equation (PDE) to be studied by mankind, introduced in 1752 by d'Alembert as a model to study vibrating strings. He introduced the one dimensional *wave* equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

d'Alembert used the travelling wave technique to solve the wave equation. In this chapter we shall explain this technique of d'Alembert and also give the standing wave technique which motivates the idea of separation of variable and in turn the evolution of Fourier series. The wave equations was generalised to two and three dimensions by Euler (1759) and D. Bernoulli (1762), respectively. Note the invariance of the wave equation under the transformation $t \mapsto -t$, i.e., if $u(x,t)$ is a solution to the wave equation for $t \geq 0$, then $\hat{u}(x,t) := u(x,\tau)$ is a solution of the wave equation for $t < 0$ and $\tau := -t > 0$, because $d\tau/dt = -1$, $u_t(x,t) = -u_\tau(x,\tau)$. This means that wave equation is reversible in time and do not distinguish between past and future.

4.1.1 The Vibrating String: Derivation

Let us consider a homogeneous string of length L, stretched along the x-axis, with one end fixed at $x = 0$ and the other end fixed at $x = L$. We assume that the string is free to move only in the vertical direction. Let $\rho > 0$ denote the density of the string and $T > 0$ denote the coefficient of tension of the string. Let $u(x,t)$ denote the vertical displacement of the string at the point x and time t.

We shall imagine the string of length L as system of N objects, for N sufficiently large. Think of N objects sitting on the string L at equidistant (uniformly distributed). The position of the n-th object on the string is given by $x_n = nL/N$. One can think of the vibrating string as the harmonic oscillator of N objects governed by the tension on the string (which behaves like the spring). Let $y_n(t) = u(x_n,t)$ denote the displacement of the object x_n at time t. The distance between any two successive objects is $h = x_{n+1} - x_n = L/N$. Then mass of each of the N object is mass of the string divided by N. Since mass of the string is $\rho \times L$, mass of each of the object x_n, $n = 1, 2, \ldots, N$, is ρh. Thus, by Newton's second law, $\rho h y_n''(t)$ is same as the force acting on the n-th object. The force on x_n is coming both from left (x_{n-1}) and right (x_{n+1}) side. The force from left and right is given as $T(y_{n-1} - y_n)/h$ and $T(y_{n+1} - y_n)/h$, respectively. Therefore,

$$\begin{aligned}
\rho h y_n''(t) &= \frac{T}{h}\left\{y_{n+1}(t) + y_{n-1}(t) - 2y_n(t)\right\} \\
&= \frac{T}{h}\left\{u(x_n + h, t) + u(x_n - h, t) - 2u(x_n, t)\right\} \\
y_n''(t) &= \frac{T}{\rho}\left(\frac{u(x_n + h, t) + u(x_n - h, t) - 2u(x_n, t)}{h^2}\right)
\end{aligned}$$

Note that assuming u is twice differentiable w.r.t the x variable, the term on RHS is same as

$$\frac{T}{\rho}\frac{1}{h}\left(\frac{u(x_n + h, t) - u(x_n, t)}{h} + \frac{u(x_n - h, t) - u(x_n, t)}{h}\right)$$

which converges to the second partial derivative of u w.r.t x as $h \to 0$. The $h \to 0$ is the limit case of the N objects we started with. Therefore the vibrating string system is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

where T is the *tension* and ρ is the *density* of the string. Equivalently,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.1.1}$$

where $c^2 = T/\rho$, $c > 0$, on $x \in (0, L)$ and $t > 0$.

Exercise 18. Show that the wave equation (4.1.1), derived above can be written as

$$u_{zz} = u_{ww} \text{ in } (w, z) \in (0, L) \times (0, \infty).$$

under a new coordinate system (w, z). One may, in fact choose coordinate such that the string is fixed between $(0, \pi)$.

Proof. Set $w = x/a$ and $z = t/b$, where a and b will be chosen appropriately. Then, $w_x = 1/a$ and $z_t = 1/b$. Therefore, $u_x = u_w/a$, $u_t = u_z/b$, $a^2 u_{xx} = u_{ww}$ and $b^2 u_{tt} = u_{zz}$. Choosing $a = 1$ and $b = 1/c$. Choosing $a = L/\pi$ and $b = L/c\pi$ makes the domain $(0, \pi)$. □

4.1.2 Travelling Waves

Consider the wave equation $u_{tt} = c^2 u_{xx}$ on $\mathbb{R} \times (0, \infty)$, describing the vibration of an infinite string. We have already seen in Chapter 3 that the equation is hyperbolic and has the two characteristics $x \pm ct=$ a constant. Introduce the new coordinates $w = x + ct$, $z = x - ct$ and set $u(w, z) = u(x, t)$. Thus, we have the following relations, using chain rule:

$$\begin{aligned} u_x &= u_w w_x + u_z z_x = u_w + u_z \\ u_t &= u_w w_t + u_z z_t = c(u_w - u_z) \\ u_{xx} &= u_{ww} + 2u_{zw} + u_{zz} \\ u_{tt} &= c^2(u_{ww} - 2u_{zw} + u_{zz}) \end{aligned}$$

In the new coordinates, the wave equation satisfies $u_{wz} = 0$. Integrating[1] this twice, we have $u(w, z) = F(w) + G(z)$, for some arbitrary functions F and G. Thus, $u(x, t) = F(x + ct) + G(x - ct)$ is a *general solution* of the wave equation.

Consider the case where G is chosen to be zero function. Then $u(x, t) = F(x+ct)$ solves the wave equation. At $t = 0$, the solution is simply the graph of F and at $t = t_0$ the solution is the graph of F with origin translated to the left by ct_0. Similarly, choosing $F = 0$ and $G = F$, we have $u(x, t) = F(x-ct)$ also solves wave equation and at time t is the translation to the right of the graph of F by ct. This motivates the name "travelling waves" and "wave equation". The graph of F is shifted to right or left with a speed of c.

[1]We are assuming the function is integrable, which may be false

Now that we have derived the general form of the solution of wave equation, we return to understand the physical system of a vibrating infinite string. The initial shape (position at initial time $t = 0$) of the string is given as $u(x, 0) = g(x)$, where the graph of g on $\mathbb{R}^2$ describes the shape of the string. Since we need one more data to identify the arbitrary functions, we also prescribe the initial velocity of the string, $u_t(x, 0) = h(x)$.

Another interesting property that follows from the general solution is that for any four points A, B, C and D that form a rectangle bounded by characteristic curves in $\mathbb{R} \times \mathbb{R}^+$ then $u(A) + u(C) = u(B) + u(D)$ because $u(A) = F(\alpha) + G(\beta)$, $u(C) = F(\gamma) + G(\delta)$, $u(B) = F(\alpha) + G(\delta)$ and $u(D) = F(\gamma) + G(\beta)$.

Theorem 4.1.1. *Given $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, there is a unique C^2 solution u of the Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) &= g(x) \quad \text{in } \mathbb{R} \\ u_t(x,0) &= h(x) \quad \text{in } \mathbb{R}, \end{cases} \tag{4.1.2}$$

which is given by the d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy. \tag{4.1.3}$$

Proof. The general solution is $u(x, t) = F(x + ct) + G(x - ct)$ with $F, G \in C^2(\mathbb{R})$. Using the initial position we get

$$F(x) + G(x) = g(x).$$

Thus, g should be $C^2(\mathbb{R})$. Now, $u_t(x, t) = c\,(F'(w) - G'(z))$ and putting $t = 0$, we get

$$F'(x) - G'(x) = \frac{1}{c}h(x).$$

Thus, h should be $C^1(\mathbb{R})$. Now solving for F' and G', we get $2F'(x) = g'(x) + h(x)/c$. Similarly, $2G'(x) = g'(x) - h(x)/c$. Integrating[2] both these equations, we get

$$F(x) = \frac{1}{2}\left(g(x) + \frac{1}{c}\int_0^x h(y)\, dy\right) + c_1$$

[2]assuming they are integrable and the integral of their derivatives is itself

and

$$G(x) = \frac{1}{2}\left(g(x) - \frac{1}{c}\int_0^x h(y)\,dy\right) + c_2.$$

Since $F(x) + G(x) = g(x)$, we get $c_1 + c_2 = 0$. Therefore, the solution to the wave equation is given by (4.1.3). □

Aliter. Let us derive the d'Alembert's formula in an alternate way. Note that the wave equation can be factored as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = u_{tt} - c^2 u_{xx} = 0.$$

We set $v(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t)$ and hence

$$v_t(x,t) + cv_x(x,t) = 0 \quad \text{in } \mathbb{R}\times(0,\infty).$$

Notice that the above first order PDE obtained is in the form of homogeneous linear transport equation (cf. (2.2.1)), which we have already solved. Hence, for some smooth function ϕ,

$$v(x,t) = \phi(x - ct)$$

and $\phi(x) := v(x,0)$. Using v in the original equation, we get the inhomogeneous transport equation,

$$u_t(x,t) - cu_x(x,t) = \phi(x - ct).$$

Recall the formula for inhomogenoeus transport equation (cf. (2.2.2))

$$u(x,t) = g(x - at) + \int_0^t \phi(x - a(t-s), s)\,ds.$$

Since $u(x,0) = g(x)$ and $a = -c$, in our case the solution reduces to,

$$\begin{aligned}
u(x,t) &= g(x+ct) + \int_0^t \phi(x + c(t-s) - cs)\,ds \\
&= g(x+ct) + \int_0^t \phi(x + ct - 2cs)\,ds \\
&= g(x+ct) + \frac{-1}{2c}\int_{x+ct}^{x-ct} \phi(y)\,dy \\
&= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(y)\,dy.
\end{aligned}$$

But $\phi(x) = v(x,0) = u_t(x,0) - cu_x(x,0) = h(x) - cg'(x)$ and substituting this in the formula for u, we get

$$\begin{aligned} u(x,t) &= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} (h(y) - cg'(y))\, dy \\ &= g(x+ct) + \frac{1}{2}(g(x-ct) - g(x+ct)) \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy \\ &= \frac{1}{2}(g(x-ct) + g(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy \end{aligned}$$

□

For $c = 1$, the *d'Alembert's formula* takes the form

$$u(x,t) = \frac{1}{2}(g(x-t) + g(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.$$

A useful observation from the d'Alembert's formula is that the regularity of u is same as the regularity of its initial value g.

Theorem 4.1.2 (Dirichlet Condition)**.** *Given $g \in C^2[0,\infty)$, $h \in C^1[0,\infty)$ and $\phi \in C^2(0,\infty)$, there is a unique C^2 solution u of the homogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= 0 \quad \text{in } (0,\infty)\times(0,\infty) \\ u(x,0) &= g(x) \quad \text{in } [0,\infty) \\ u_t(x,0) &= h(x) \quad \text{in } [0,\infty) \\ u(0,t) &= \phi(t) \quad \text{in } (0,\infty), \end{cases} \tag{4.1.4}$$

where ϕ, g, h satisfies the compatibility condition

$$g(0) = \phi(0), g''(0) = \phi''(0), h(0) = \phi'(0).$$

Proof. We first consider case $\phi \equiv 0$, homogeneous Dirichlet conditions. We extend g as an odd function on $(-\infty,\infty)$ by setting $g(-x) = -g(x)$ for $x \in [0,\infty)$. Therefore, g is an odd function in $\mathbb{R}$. Then, we have a unique solution $u \in C^2$ solving the Cauchy problem in $\mathbb{R}\times(0,\infty)$. But note that $v(x,t) = -u(-x,t)$ is also a solution to the same Cauchy problem. Thus, $u(x,t) = -u(-x,t)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ and $u(0,t) = 0$.

Now, we consider the case when ϕ is a non-zero function. The line $ct = x$ divides the quadrant in to two regions Ω_1 and Ω_2 defined as

$$\Omega_1 = \{(x,t) \mid x > ct\}$$

and

$$\Omega_2 = \{(x,t) \mid x < ct\}.$$

For $(x,t) \in \Omega_1$, the solution

$$u_1(x,t) = \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy.$$

On the line $x = ct$, we get

$$\chi(x) := u_1(x, x/c) = \frac{1}{2}\left(g(0) + g(2x)\right) + \frac{1}{2c}\int_{0}^{2x} h(y)\,dy.$$

Let u_2 be the solution in Ω_2 of

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= 0 \quad \text{in } \Omega_2 \\ u(x,x/c) &= \chi(x) \quad \text{in } \{x = ct\} \\ u(0,t) &= \phi(t) \quad \text{in } (0,\infty). \end{cases}$$

Fix $A := (x,t) \in \Omega_2$. One of the characteristic curve through A intersects t-axis at $B := (0, t - x/c)$. The other characteristic curve intersects the line $ct = x$ at $C := (1/2(ct + x, t + x/c)$. The characteristic curve through B intersects $ct = x$ at $D := 1/2(ct - x, t - x/c)$. The four points form a parallelogram in Ω_2. Therefore, we know that

$$u_2(x,t) + u_2(1/2(ct - x, t - x/c)) = u_2(0, t - x/c) + u_2(1/2(ct + x, t + x/c))$$

and, hence,

$$\begin{aligned} u_2(x,t) &= \phi(t - x/c) + \chi(1/2(ct + x)) - \chi(1/2(ct - x)) \\ &= \phi(t - x/c) + \frac{1}{2}\left(g(0) + g(ct + x)\right) + \frac{1}{2c}\int_0^{ct+x} h(y)\,dy \\ &\quad -\frac{1}{2}\left(g(0) + g(ct - x)\right) - \frac{1}{2c}\int_0^{ct-x} h(y)\,dy \\ &= \phi(t - x/c) + \frac{1}{2}\left(g(ct + x) - g(ct - x)\right) + \frac{1}{2c}\int_{ct-x}^{ct+x} h(y)\,dy. \end{aligned}$$

By setting

$$u(x,t) = \begin{cases} u_1(x,t) & (x,t) \in \Omega_1 \\ u_2(x,t) & (x,t) \in \Omega_2 \end{cases}$$

and the fact that alll derivatives of u are continuous along $ct = x$ line, due to the compatibility condition implies that u is a solution (4.1.4). □

Corollary 4.1.3 (Dirichlet Condition)**.** *Given $g \in C^2[0,L]$, $h \in C^1[0,L]$ and $\phi, \psi \in C^2(0,\infty)$, there is a unique C^2 solution u of the homogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= 0 \quad \text{in } (0,L) \times (0,\infty) \\ u(x,0) &= g(x) \quad \text{in } [0,L] \\ u_t(x,0) &= h(x) \quad \text{in } [0,L] \\ u(0,t) &= \phi(t) \quad \text{in } (0,\infty) \\ u(L,t) &= \psi(t) \quad \text{in } (0,\infty), \end{cases} \tag{4.1.5}$$

where ϕ, ψ, g, h satisfies the compatibility condition

$$g(0) = \phi(0), g''(0) = \phi''(0), h(0) = \phi'(0)$$

and

$$g(L) = \psi(0), g''(L) = \psi''(0), h(L) = \psi'(0).$$

Proof. We first consider case $\phi = \psi \equiv 0$, homogeneous Dirichlet conditions. We extend g as an odd function on $[-L,L]$ by setting $g(-x) = -g(x)$ for $x \in [0,L]$. Therefore, g is an odd $2L$-periodic function in $\mathbb{R}$. Similarly, we extend h and we have $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then, we have a unique solution $u \in C^2$ solving the Cauchy problem in $\mathbb{R} \times (0,\infty)$. But note that $v(x,t) = -u(-x,t)$ is also a solution to the same Cauchy problem. Thus, $u(x,t) = -u(-x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ and $u(0,t) = 0$. Similarly, $w(x,t) = -u(2L-x,t)$ is also a solution and, hence, $u(x,t) = -u(2L-x,t)$ which implies $u(L,t) = 0$.

Consider the lines $ct = x$ and $ct = -x + cL$, then we will obtain u in the four regions as u_1, u_2, u_3 and u_4. Then follow the idea similar to proof in above theorem. □

4.1.3 Domain of Dependence and Influence

Note that the solution $u(x,t)$ depends only on the interval $[x - ct, x + ct]$ because g takes values only on the end-points of this interval and h takes values between this interval. The interval $[x-ct, x+ct]$ is called the *domain of dependence*. Thus, the region of $\mathbb{R} \times (0,\infty)$ on which the value of $u(x,t)$ depends forms a triangle with base $[x-ct, x+ct]$ and vertex at (x,t). The domain of dependence of (x,t) is marked in x-axis by the characteristic curves passing through (x,t).

Given a point p on the x-axis what values of u on (x,t) will depend on the value of $g(p)$ and $h(p)$. This region turns out to be a cone with vertex at p and is called the *domain of influence*. The domain of influence is the region bounded by the characteristic curves passing through p.

If the initial data g and h are supported in the interval $B_{x_0}(R)$ then the solution u at (x,t) is supported in the region $B_{x_0}(R+ct)$. Consequently, if g and h have compact support then the solution u has compact support in $\mathbb{R}$ for all time $t > 0$. This phenomenon is called the *finite speed of propagation*.

Theorem 4.1.4 (Inhomogeneous Wave Equation)**.** *Given $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R} \times [0,\infty)$, there is a unique C^2 solution u of the inhomogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= f(x,t) \quad \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) &= g(x) \quad \text{in } \mathbb{R} \\ u_t(x,0) &= h(x) \quad \text{in } \mathbb{R}, \end{cases} \tag{4.1.6}$$

given by the formula

$$\frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} h(y)\,dy + \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds\right].$$

Proof. Fix $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Consider the open triangle in $\mathbb{R} \times \mathbb{R}^+$ with vertices $(x,t), (x-ct,0)$ and $(x+ct,0)$, and denote it by $T(x,t)$. Thus,

$$T(x,t) := \{(y,s) \in \mathbb{R} \times \mathbb{R}^+ \mid |y-x| < c(t-s)\}.$$

The boundary of the triangle $\partial T(x,t)$ consists of three parts

$$T_0 := \{(y,0) \mid x-ct < y < x+ct\},$$

$$T_+ := \{(y,s) \in \mathbb{R} \times (0,t) \mid cs + y = x + ct\}$$

and

$$T_- := \{(y,s) \in \mathbb{R} \times (0,t) \mid cs - y = -x + ct\}.$$

The unit outward normal at each point of the boundary $\partial T(x,t)$ is given by $\nu = (\nu_1, \nu_2)$ defined as

$$\nu(y,s) = \begin{cases} (0,-1) & (y,s) \in T_0 \\ \frac{1}{\sqrt{1+c^2}}(1,c) & (y,s) \in T_+ \\ \frac{1}{\sqrt{1+c^2}}(-1,c) & (y,s) \in T_-. \end{cases}$$

Therefore, by Gauss divergence theorem (cf. (D.0.1))

$$\begin{aligned}
\int_{T(x,t)} f((y,s)\,dy\,ds &= \int_{T(x,t)} \left[u_{tt}(y,s) - c^2 u_{xx}(y,s)\right]\,dy\,ds \\
&= \int_{\partial T(x,t)} \left[u_t\nu_2 - c^2 u_x \nu_1\right]\,d\sigma \\
&= \int_{T_0} \left[u_t\nu_2 - c^2 u_x \nu_1\right]\,d\sigma + \int_{T_+} \left[u_t\nu_2 - c^2 u_x \nu_1\right]\,d\sigma \\
&\quad + \int_{T_-} \left[u_t\nu_2 - c^2 u_x \nu_1\right]\,d\sigma \\
&= -\int_{x-ct}^{x+ct} u_t(y,0)\,dy + \frac{c}{\sqrt{1+c^2}} \int_{T_+} [u_t - cu_x]\,d\sigma \\
&\quad + \frac{c}{\sqrt{1+c^2}} \int_{T_-} [u_t + cu_x]\,d\sigma
\end{aligned}$$

Note that the second and third integral are just the direction derivatives of u along the direction $(1,c)$ and $(-1,c)$ in the line T_+ and T_-, respectively. Therefore, we have

$$\begin{aligned}
\int_{T(x,t)} f((y,s)\,dy\,ds &= -\int_{x-ct}^{x+ct} h(y)\,dy + cu(x,t) - cu(x+ct,0) \\
&\quad + c[u(x,t) - u(x-ct,0)] \\
u(x,t) &= \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] \\
&\quad + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} h(y)\,dy + \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds\right].
\end{aligned}$$

□

Aliter. We introduce a new function v defined as $v(x,t) = u_t(x,t)$ and rewrite (4.1.6) as

$$U'(x,t) + AU(x,t) = F(x,t)$$

where $U = (u, v)$, $F = (0, f)$ and

$$A = \begin{pmatrix} 0 & -1 \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

with the initial condition $G(x) := U(x,0) = (g(x), h(x))$. The solution $U(x,t)$ is given as (cf. Appendix G)

$$U(x,t) = S(t)G(x) + \int_0^t S(t-s)F(s)\,ds$$

where $S(t)$ is a solution operator of the homoegeneous system of first order PDE. Therefore, by d'Alembert's formula,

$$S(t)(g,h) = \begin{pmatrix} \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy \\ \frac{c}{2}\left[g'(x+ct) - g'(x-ct)\right] + \frac{1}{2}\left[h(x+ct) + h(x-ct)\right] \end{pmatrix}$$

and, hence, $u(x,t)$ has the required represenation. □

4.1.4 Standing Waves: Separation of Variable

Recall the set-up of the vibrating string given by the equation $u_{tt} = u_{xx}$, we have normalised the constant c. Initially at time t, let us say the string has the shape of the graph of v, i.e., $u(x,0) = v(x)$. The snapshot of the vibrating string at each time are called the "standing waves". The shape of the string at time t_0 can be thought of as some factor (depending on time) of v. This observation motivates the idea of "separation of variable", i.e., $u(x,t) = v(x)w(t)$, where $w(t)$ is the factor depending on time, which scales v at time t to fit with the shape of $u(x,t)$.

The fact that endpoints are fixed is given by the boundary condition

$$u(0,t) = u(L,t) = 0.$$

We are also given the initial position $u(x,0) = g(x)$ (at time $t = 0$) and initial velocity of the string at time $t = 0$, $u_t(x,0) = h(x)$. Given $g, h : [0,L] \to \mathbb{R}$ such that $g(0) = g(L) = 0$ and $h(0) = h(L)$, we need to solve the initial value problem (4.1.5) with $\phi = \psi \equiv 0$.

Let us seek for solutions $u(x,t)$ whose variables can be separated. Let $u(x,t) = v(x)w(t)$. Differentiating and substituting in the wave equation, we get

$$v(x)w''(t) = c^2 v''(x)w(t)$$

Hence

$$\frac{w''(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)}.$$

Since RHS is a function of x and LHS is a function t, they must equal a constant, say λ. Thus,

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{c^2 w(t)} = \lambda.$$

Using the boundary condition $u(0,t) = u(L,t) = 0$, we get

$$v(0)w(t) = v(L)w(t) = 0.$$

If $w \equiv 0$, then $u \equiv 0$ and this cannot be a solution to (4.1.5). Hence, $w \not\equiv 0$ and $v(0) = v(L) = 0$. Thus, we need to solve the *eigen value problem* for the second order differential operator.

$$\begin{cases} v''(x) &= \lambda v(x), \quad x \in (0,L) \\ v(0) &= v(L) \quad = 0, \end{cases}$$

Note that the λ can be either zero, positive or negative. If $\lambda = 0$, then $v'' = 0$ and the general solution is $v(x) = \alpha x + \beta$, for some constants α and β. Since $v(0) = 0$, we get $\beta = 0$, and $v(L) = 0$ and $L \neq 0$ implies that $\alpha = 0$. Thus, $v \equiv 0$ and hence $u \equiv 0$. But, this cannot be a solution to (4.1.5).

If $\lambda > 0$, then $v(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$. Equivalently,

$$v(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

such that $\alpha = (c_1 + c_2)/2$ and $\beta = (c_1 - c_2)/2$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$ and hence

$$v(x) = c_2 \sinh(\sqrt{\lambda}x).$$

Now using $v(L) = 0$, we have $c_2 \sinh \sqrt{\lambda}L = 0$. Thus, $c_2 = 0$ and $v(x) = 0$. We have seen this cannot be a solution.

Finally, if $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$. We need to solve the simple harmonic oscillator problem

$$\begin{cases} v''(x) + \omega^2 v(x) &= 0 \qquad x \in (0, L) \\ v(0) &= v(L) = 0. \end{cases}$$

The general solution is

$$v(x) = \alpha \cos(\omega x) + \beta \sin(\omega x).$$

Using $v(0) = 0$, we get $\alpha = 0$ and hence $v(x) = \beta \sin(\omega x)$. Now using $v(L) = 0$, we have $\beta \sin \omega L = 0$. Thus, either $\beta = 0$ or $\sin \omega L = 0$. But $\beta = 0$ does not yield a solution. Hence $\omega L = k\pi$ or $\omega = k\pi/L$, for all non-zero $k \in \mathbb{Z}$. Since $\omega > 0$, we can consider only $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there is a solution (v_k, λ_k) for the eigen value problem with

$$v_k(x) = \beta_k \sin\left(\frac{k\pi x}{L}\right),$$

for some constant b_k and $\lambda_k = -(k\pi/L)^2$. It now remains to solve w for each of these λ_k. For each $k \in \mathbb{N}$, we solve for w_k in the ODE

$$w_k''(t) + (ck\pi/L)^2 w_k(t) = 0.$$

The general solution is

$$w_k(t) = a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right).$$

For each $k \in \mathbb{N}$, we have

$$u_k(x,t) = \left[a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right)$$

for some constants a_k and b_k. The situation corresponding to $k = 1$ is called the *fundamental mode* and the *frequency* of the fundamental mode is

$$\frac{1}{2\pi}\frac{c\pi}{L} = \frac{c}{2L} = \frac{\sqrt{T/\rho}}{2L}.$$

The frequency of higher modes are integer multiples of the fundamental frequency.

The general solution of (4.1.5), by principle of superposition, is

$$u(x,t) = \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right).$$

Note that the solution is expressed as series, which raises the question of convergence of the series. Another concern is whether all solutions of (4.1.5) have this form. We ignore these two concerns at this moment.

Since we know the initial position of the string as the graph of g, we get

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right).$$

This expression is again troubling and rises the question: Can any arbitrary function g be expressed as an infinite sum of trigonometric functions? Answering this question led to the study of "Fourier series". Let us also, as usual, ignore this concern for time being. Then, can we find the the constants a_k with knowledge of g. By multiplying $\sin\left(\frac{l\pi x}{L}\right)$ both sides of the expression of g and integrating from 0 to L, we get

$$\begin{aligned}\int_0^L g(x)\sin\left(\frac{l\pi x}{L}\right)\,dx &= \int_0^L \left[\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right)\right] \sin\left(\frac{l\pi x}{L}\right)\,dx \\ &= \sum_{k=1}^{\infty} a_k \int_0^L \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi x}{L}\right)\,dx\end{aligned}$$

Therefore, the constants a_k are given as

$$a_k = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right).$$

Finally, by differentiating u w.r.t t, we get

$$u_t(x,t) = \sum_{k=1}^{\infty} \frac{ck\pi}{L}\left[b_k \cos\frac{ck\pi t}{L} - a_k \sin\frac{ck\pi t}{L}\right] \sin\left(\frac{k\pi x}{L}\right).$$

Employing similar arguments and using $u_t(x,0) = h(x)$, we get

$$h(x) = u_t(x,0) = \sum_{k=1}^{\infty} \frac{b_k kc\pi}{L} \sin\left(\frac{k\pi x}{L}\right)$$

and hence

$$b_k = \frac{2}{kc\pi}\int_0^L h(x)\sin\left(\frac{k\pi x}{L}\right).$$

4.2 Higher Dimensions

We shall denote Δ as the Laplacian w.r.t. the space variable and the wave equation is denoted as $\Box u := u_{tt} - c^2\Delta u$. The Cauchy initial value problem in higher dimensions is

$$\begin{cases} \Box u &= f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= g(x) \quad \text{in } \mathbb{R}^n \\ u_t(x,0) &= h(x) \quad \text{in } \mathbb{R}^n. \end{cases} \tag{4.2.1}$$

Due to the linearity of the wave operator $\Box$, any solution $u = v+w+z$ where v, w and z are, respectively, solutions of

$$\begin{cases} \Box v &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) &= g(x) \quad \text{in } \mathbb{R}^n \\ v_t(x,0) &= 0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{4.2.2}$$

$$\begin{cases} \Box w &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ w(x,0) &= 0 \quad \text{in } \mathbb{R}^n \\ w_t(x,0) &= h(x) \quad \text{in } \mathbb{R}^n \end{cases} \tag{4.2.3}$$

and

$$\begin{cases} \Box z &= f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ z(x,0) &= 0 \quad \text{in } \mathbb{R}^n \\ z_t(x,0) &= 0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{4.2.4}$$

Theorem 4.2.1 (Duhamel's Principle)**.** *Let w^h be a solution to (4.2.3). Then*

$$v(x,t) = w^g_t(x,t)$$

and

$$z(x,t) = \int_0^t w^{f_s}(x,t-s)\,ds,$$

where $f_s = f(\cdot, s)$, are solutions to (4.2.2) and (4.2.4).

Proof. Since w^g solves (4.2.3) with $h = g$, we have

$$\Box v = \Box w^g_t(x,t) = \frac{\partial}{\partial t}(\Box w^g) = 0$$

on $\mathbb{R}^n \times (0,\infty)$. Further, $v(x,0) = w^g_t(x,0) = g(x)$ in $\mathbb{R}^n$ and $v_t(x,0) = w^g_{tt}(x,0) = c^2\Delta w^g(x,0) = 0$. Thus, v solves (4.2.2). Now, let us denote

$w(x,t) = w^{f_s}(x, t-s)$, for $0 < s < t$, which satisfies (4.2.3) with $h(\cdot) = f(c;s)$. Then,

$$z_t(x,t) = w^{f_s}(x,0) + \int_0^t w_t^{f_s}(x,t-s)\,ds = \int_0^t w_t^{f_s}(x,t-s)\,ds$$

and

$$\begin{aligned} z_{tt}(x,t) &= w_t^{f_s}(x,0) + \int_0^t w_{tt}^{f_s}(x,t-s)\,ds \\ &= f(x,t) + c^2\Delta \int_0^t w^{f_s}(x,t-s)\,ds \\ &= f(x,t) + c^2\Delta z. \end{aligned}$$

Therefore, z solves (4.2.4). □

The Duhamel's principle can be viewed as a generalization of the method of variations of constants in ODE (cf. Appendix G). Owing to the above theorem it is enough solve for w in (4.2.3). To do so, we shall employ the method of spherical means which reduces such problem to one dimensional framework.

4.2.1 Spherical Means

More generally, in this section we solve for $v + w$ which is the solution to the wave equation

$$\begin{cases} \Box u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= g(x) \quad \text{in } \mathbb{R}^n \\ u_t(x,0) &= h(x) \quad \text{in } \mathbb{R}^n. \end{cases} \tag{4.2.5}$$

For a fixed $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, the spherical mean of a C^2 function $u(x,t)$ is given as

$$M(x;r,t) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y,t)\,d\sigma_y$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Equivalently, after setting $z = (y-x)/r$,

$$M(x;r,t) := \frac{1}{\omega_n} \int_{S_1(0)} u(x+rz,t)\,d\sigma_z.$$

If we take the above form of $M(x;r,t)$ as a definition, we observe that it is valid for all $r \in \mathbb{R}$ and $M(x;0,t) = u(x,t)$. Then

$$\begin{aligned} M_r(x;r,t) &= \frac{1}{\omega_n}\int_{S_1(0)} \sum_{i=1}^{n} u_{y_i}(x+rz,t) z_i \, d\sigma_z \\ &= \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} \nabla_y u(y,t)\cdot z \, d\sigma_y \\ &= \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta_y u(y,t)\, dy. \end{aligned}$$

If $u(x,t)$ is a solution to (4.2.5) then

$$\begin{aligned} r^{n-1} M_r(x;r,t) &= \frac{1}{c^2\omega_n}\int_{B_r(x)} u_{tt}(y,t)\, dy \\ &= \frac{1}{c^2\omega_n}\int_0^r \int_{S_s(x)} u_{tt}(y,t)\, d\sigma_y \, ds \\ \frac{d}{dr}(r^{n-1}M_r(x;r,t)) &= \frac{1}{c^2\omega_n}\int_{S_r(x)} u_{tt}(y,t)\, d\sigma_y \\ &= \frac{r^{n-1}}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\omega_n r^{n-1}}\int_{S_r(x)} \nabla_y u(y,t)\, d\sigma_y\right) = \frac{r^{n-1}}{c^2} M_{tt}. \end{aligned}$$

Thus, the spherical means satisfies the one space variable PDE

$$r^{1-n}\frac{d}{dr}(r^{n-1}M_r) = c^{-2}M_{tt}$$

or

$$M_{rr} + \frac{n-1}{r}M_r = c^{-2}M_{tt},$$

which is called the Euler-Poisson-Darboux equation. Also, if $u(x,t)$ is a solution to (4.2.5) then using the initial condition, we get

$$G(x;r) := M(x;r,0) = \frac{1}{\omega_n}\int_{S_1(0)} g(x+rz)\, d\sigma_z$$

and

$$H(x;r) := M_t(x;r,0) = \frac{1}{\omega_n}\int_{S_1(0)} h(x+rz)\, d\sigma_z.$$

In the following section we shall illustrate solving the Euler-Poisson-Darboux equation in three and two dimensions. For higher dimensions solving E-P-D equation is tedious.

Theorem 4.2.2 (Three dimensions)**.** *Given $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$ there exists a unique solution $u \in C^2(\mathbb{R}^3 \times [0,\infty)$ of (4.2.5) given by the Poisson's formula*

$$u(x,t) = \frac{1}{4\pi c^2}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S_{ct}(x)} g(y)\,d\sigma_y\right) + \frac{1}{t}\int_{S_{ct}(x)} h(y)\,d\sigma_y\right].$$

Proof. For the Euclidean dimension three, the Euler-Poisson-Darboux equation becomes

$$c^2(rM)_{rr} = (rM)_{tt}.$$

Thus, rM is a solution to the one dimensional Cauchy problem

$$\begin{cases} c^2(rM)_{rr}(x;r,t) &= (rM)_{tt}(x;r,t) \quad \text{in } (0,\infty)\times(0,\infty)\\ rM(x;r,0) &= rG(x;r) \quad \text{in } (0,\infty)\\ rM_t(x;r,0) &= rH(x;r) \quad \text{in } (0,\infty)\end{cases}$$

and is given by the d'Alembert formula

$$M(x;r,t) = \frac{1}{2r}\left((r-ct)G(r-ct) + (r+ct)G(r+ct)\right) + \frac{1}{2rc}\int_{r-ct}^{r+ct} yH(y)\,dy,$$

as long as, the domain of dependence $[r-ct, r+ct]$ is in $(0,\infty)$. Otherwise, we extend G and H to entire $\mathbb{R}$ as

$$\tilde{G}(x;r) = \begin{cases} G(x;r) & r>0\\ g(x) & r=0\\ G(x;-r) & r<0\end{cases}$$

and

$$\tilde{H}(x;r) = \begin{cases} H(x;r) & r>0\\ h(x) & r=0\\ H(x;-r) & r<0.\end{cases}$$

Invoking Lemma 4.2.3, we get $\tilde{M}(x;r,t)$ as

$$\frac{1}{2r}\Big((r-ct)\tilde{G}(r-ct) + (r+ct)\tilde{G}(r+ct)\Big) + \frac{1}{2rc}\int_{r-ct}^{r+ct} y\tilde{H}(y)\,dy$$

Since $\tilde{G}$ and $\tilde{H}$ are even functions, we have

$$\int_{r-ct}^{-(r-ct)} yH(y)\,dy = 0$$

and, therefore $\tilde{M}(x;r,t)$

$$\frac{1}{2r}\left((r+ct)\tilde{G}(ct+r)-(ct-r)\tilde{G}(ct-r)\right)+\frac{1}{2rc}\int_{ct-r}^{r+ct} y\tilde{H}(y)\,dy.$$

Since

$$u(x,t)=\lim_{r\to 0^+}\tilde{M}(x;r,t),$$

by L'Hospital's rule, we get

$$u(x,t)=\frac{d}{dt}(tG(ct))+tH(ct)=ctG'(ct)+G(ct)+tH(ct).$$

□

Lemma 4.2.3. *$r\tilde{G}, r\tilde{H}\in C^2(\mathbb{R})$.*

Proof. It follows from the definition that

$$\lim_{r\to 0+} G(x;r)=g(x) \text{ and } \lim_{r\to 0+} H(x;r)=h(x).$$

Thus, $r\tilde{G}, r\tilde{H}\in C(\mathbb{R})$. Since $G,H\in C^1(\mathbb{R})$, $r\tilde{G}, r\tilde{H}\in C^1(\mathbb{R})$. Further, since G'' and H'' are bounded as $r\to 0+$, $r\tilde{G}, r\tilde{H}\in C^2(\mathbb{R})$. □

The domain of dependence of (x,t_0) for the three dimensional wave equation is the boundary of the three dimensional sphere with radius ct.

The Hadamard's *method of descent* is the technique of finding a solution of the two dimensional wave equation using the three dimensional wave equation.

Theorem 4.2.4 (Method of Descent)**.** *Given $g\in C^3(\mathbb{R}^2)$ and $h\in C^2(\mathbb{R}^2)$ there exists a unique solution $u\in C^2(\mathbb{R}^2\times[0,\infty)$ of (4.2.5) given by the Poisson's formula*

$$\frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\left(\int_{B_{ct}(x,y)}\frac{g(\xi,\eta)}{\sqrt{c^2t^2-\rho^2}}\,d\xi\,d\eta\right)+\int_{B_{ct}(x,y)}\frac{h(\xi,\eta)}{c^2t^2-\rho^2}\,d\xi\,d\eta\right]$$

where

$$\rho=\sqrt{(\xi-x)^2+(\eta-y)^2}.$$

Proof. Let v be a solution of (4.2.5) in two dimensions with $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$. Then

$$u(x, y, z, t) := v(x, y, t)$$

is solution to (4.2.5) in three dimensions where g and h are given independent of z. Since $u(x, y, z, t) = u(x, y, 0, t) + zu_z(x, y, \varepsilon z, t)$ for $0 < \varepsilon < 1$ and $u_z = 0$, we have $v(x, y, t) = u(x, y, 0, t)$. Therefore, using the poisson formula in three dimensions, we get

$$v(x, y, t) = \frac{1}{4\pi c^2}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S_{ct}(x,y,0)} g(\xi, \eta, \zeta)\, d\sigma\right) + \frac{1}{t}\int_{S_{ct}(x,y,0)} h(\xi, \eta, \zeta)\, d\sigma\right].$$

Recall that g and h are independent of ζ, therefore $g(\xi, \eta, \zeta) = g(\xi, \eta)$ and $h(\xi, \eta, \zeta) = h(\xi, \eta)$. The sphere has the equation $(\xi - x)^2 + (\eta - y)^2 + \zeta^2 = c^2t^2$. The exterior normal is given as

$$\nu = \frac{\nabla S}{|\nabla S|} = \left(\frac{\xi - x}{ct}, \frac{\eta - y}{ct}, \frac{\zeta}{ct}\right)$$

and the surface element $d\sigma$ is given as $d\sigma = d\xi d\eta \frac{1}{|\gamma_3|}$ where

$$\gamma_3 = \pm \frac{\sqrt{c^2t^2 - (\xi - x)^2 - (\eta - y)^2}}{ct}$$

with the positive sign applying when $\zeta > 0$ and negative sign applying when $\zeta < 0$. Set

$$\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2}.$$

□

In the two dimensions, the domain of dependence is the entire disk $B_{ct_0}(x_0, y_0)$ in contrast to three dimensions which had only the boundary of the sphere as domain of dependence.

4.2.2 Odd Dimension

One can copy the idea of three dimension to any odd dimension, if we rewrite the Euler-Poisson-Darboux equation in approriate form.

Exercise 19. If n is odd, show that the correct form of M that satisfies the one dimensional wave equation is

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}(r^{n-2}M(r,t)).$$

For instance, when $n = 5$, $r^2M_r + 3rM$ satisfies the one dimensional wave equation.

We have already noted that the solution at a given point is determined by the value of intial data in a subset of the initial hypersurface. Consequently, changing initial value outside the domain of dependence does not change values of solutions.

Also, it takes time for the initial data to make influence. Suppose g and h have their support in $B_r(x_0)$. Then the support of $u(\cdot,t)$ is contained in $\cup_{y\in B_r(x_0)}B_t(y) = B_{r+ct}(x_0)$. The support of u spreads at a finite speed and is called the *finite speed propagation*.

4.2.3 Inhomogeneous Wave equation

We have already derived in Theorem 4.2.1 the formula for inhomogeneous equation (4.2.4).

Theorem 4.2.5. *For any $f \in C^1$, the solution $u(x,t)$ of (4.2.4) is given as*

$$u(x,t) = \begin{cases} \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\right) ds & n=1 \\ \frac{1}{4\pi c}\int_0^t \left(\int_{B_{c(t-s)}(x)} \frac{f(y,s)}{\sqrt{c^2(t-s)^2-r^2}}dy\right) ds & n=2 \\ \frac{1}{4\pi c^2}\int_{B_{ct}(x)} \frac{f(y,t-r/c)}{r}dy & n=3. \end{cases}$$

Proof. The Poisson's formula corresponding to the three dimension case gives the formula for

$$w^{f_s}(x,t-s) = \frac{1}{4\pi c^2(t-s)}\int_{S_{c(t-s)}(x)} f(y,s)\,d\sigma_y.$$

and

$$\begin{aligned} u(x,t) &= \int_0^t w^{f_s}(x,t-s)\,ds = \frac{1}{4\pi c^2}\int_0^t \int_{S_{c(t-s)}(x)} \frac{f(y,s)}{t-s}\,d\sigma_y\,ds \\ &= \frac{1}{4\pi c^2}\int_0^{ct}\int_{S_\tau(x)} \frac{f(y,t-\tau/c)}{\tau}\,d\sigma_y\,d\tau \quad [\text{setting } \tau = c(t-s)] \\ &= \frac{1}{4\pi c^2}\int_{B_{ct}(x)} \frac{f(y,t-r/c)}{r}\,dy \end{aligned}$$

where $r = |x-y|$. Similarly, one can derive the formulae for one and two dimensions. □

Note that in the three dimensional case the integrand is not taken at time t, but at an earlier time $t - r/c$. Thus, the integrand in this case is called *retarded potential*.

Example 4.1. Consider the wave equation

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= \sin 3x \quad \text{in } (0,\pi)\times(0,\infty) \\ u(0,t) = u(\pi,t) &= 0 \quad \text{in } (0,\infty) \\ u(x,0) = u_t(x,0) &= 0 \quad \text{in } (0,\pi). \end{cases}$$

We look for the solution of the homogeneous wave equation

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) &= 0 \quad \text{in } (0,\pi)\times(0,\infty) \\ w(0,t) = w(\pi,t) &= 0 \quad \text{in } (0,\infty) \\ w(x,0) &= 0 \quad \text{in } (0,\pi) \\ w_t(x,0) &= \sin 3x \quad \text{in } (0,\pi). \end{cases}$$

By separation of variable technique, we know that the general solution of w is

$$w(x,t) = \sum_{k=1}^{\infty} [a_k \cos(kct) + b_k \sin(kct)] \sin(kx)$$

and

$$w(x,0) = \sum_{k=1}^{\infty} a_k \sin(kx) = 0.$$

Thus, $a_k = 0$, for all $k \in \mathbb{N}$. Also,

$$w_t(x,0) = \sum_{k=1}^{\infty} b_k ck \sin(kx) = \sin 3x.$$

Hence, b_k's are all zeroes, except for $k = 3$ and $b_3 = 1/3c$. Thus,

$$w(x,t) = \frac{1}{3c}\sin(3ct)\sin(3x)$$

and

$$\begin{aligned} u(x,t) &= \int_0^t w(x,t-s)\,ds = \frac{1}{3c}\int_0^t \sin(3c(t-s))\sin 3x\,ds \\ &= \frac{\sin 3x}{3c}\int_0^t \sin(3c(t-s))\,ds = \frac{\sin 3x}{3c}\frac{\cos(3c(t-s))}{3c}\Bigg|_0^t \\ &= \frac{\sin 3x}{9c^2}\left(1-\cos 3ct\right). \end{aligned}$$

4.3 Eigenvalue Problem of Laplacian

The separation of variable technique can be used for studying wave equation on 2D Rectangle and 2D Disk etc. This leads to studying the eigen value problem of the Laplacian. For a given open bounded subset $\Omega \subset \mathbb{R}^2$, the Dirichlet eigenvalue problem,

$$\begin{cases} -\Delta u(x,y) &= \lambda u(x,y) \quad (x,y) \in \Omega \\ \quad u(x,y) &= 0 \quad\quad\quad\;\; (x,y) \in \partial\Omega. \end{cases}$$

Note that, for all $\lambda \in \mathbb{R}$, zero is a trivial solution of the Laplacian. Thus, we are interested in non-zero λ's for which the Laplacian has non-trivial solutions. Such an λ is called the *eigenvalue* and corresponding solution u_λ is called the *eigen function*.

Note that if u_λ is an eigen function corresponding to λ, then αu_λ, for all $\alpha \in \mathbb{R}$, is also an eigen function corresponding to λ. Let W be the real vector space of all $u : \Omega \to \mathbb{R}$ continuous (smooth, as required) functions such that $u(x,y) = 0$ on $\partial\Omega$. For each eigenvalue λ of the Laplacian, we define the subspace of W as

$$W_\lambda = \{u \in W \mid u \text{ solves Dirichlet EVP for given } \lambda\}.$$

Theorem 4.3.1. *There exists an increasing sequence of positive numbers $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$ with $\lambda_n \to \infty$ which are eigenvalues of the Laplacian and $W_n = W_{\lambda_n}$ is finite dimensional. Conversely, any solution u of the Laplacian is in W_n, for some n.*

Though the above theorem assures the existence of eigenvalues for Laplacian, it is usually difficult to compute them for a given Ω. In this course, we shall compute the eigenvalues when Ω is a 2D-rectangle and a 2D-disk.

4.3.1 In Rectangle

Let the rectangle be $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < a, 0 < y < b\}$. We wish to solve the Dirichlet EVP in the rectangle Ω

$$\begin{cases} -\Delta u(x,y) &= \lambda u(x,y) \quad (x,y) \in \Omega \\ u(x,y) &= 0 \qquad\qquad (x,y) \in \partial\Omega. \end{cases}$$

The boundary condition amounts to saying

$$u(x, 0) = u(a, y) = u(x, b) = u(0, y) = 0.$$

We look for solutions of the form $u(x, y) = v(x)w(y)$ (variable separated). Substituting u in separated form in the equation, we get

$$-v''(x)w(y) - v(x)w''(y) = \lambda v(x)w(y).$$

Hence

$$-\frac{v''(x)}{v(x)} = \lambda + \frac{w''(y)}{w(y)}.$$

Since LHS is function of x and RHS is function y and are equal they must be some constant, say μ. We need to solve the EVP's

$$-v''(x) = \mu v(x) \quad \text{and} \quad -w''(y) = (\lambda - \mu)w(y)$$

under the boundary conditions $v(0) = v(a) = 0$ and $w(0) = w(b) = 0$.

As seen before, while solving for v, we have trivial solutions for $\mu \leq 0$. If $\mu > 0$, then $v(x) = c_1 \cos(\sqrt{\mu}x) + c_2 \sin(\sqrt{\mu}x)$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$. Now using $v(a) = 0$, we have $c_2 \sin\sqrt{\mu}a = 0$. Thus, either $c_2 = 0$ or $\sin\sqrt{\mu}a = 0$. We have non-trivial solution, if $c_2 \neq 0$, then $\sqrt{\mu}a = k\pi$ or $\sqrt{\mu} = k\pi/a$, for $k \in \mathbb{Z}$. For each $k \in \mathbb{N}$, we have $v_k(x) = \sin(k\pi x/a)$ and $\mu_k = (k\pi/a)^2$. We solve for w for each μ_k. For each $k, l \in \mathbb{N}$, we have $w_{kl}(y) = \sin(l\pi y/b)$ and $\lambda_{kl} = (k\pi/a)^2 + (l\pi/b)^2$. For each $k, l \in \mathbb{N}$, we have

$$u_{kl}(x, y) = \sin(k\pi x/a) \sin(l\pi y/b)$$

and $\lambda_{kl} = (k\pi/a)^2 + (l\pi/b)^2$.

4.3.2 In Disk

Let the disk of radius a be $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < a^2\}$. We wish to solve the Dirichlet EVP in the disk Ω

$$\begin{cases} \frac{-1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} & = \lambda u(r, \theta) & (r, \theta) \in \Omega \\ u(\theta) & = u(\theta + 2\pi) & \theta \in \mathbb{R} \\ u(a, \theta) & = 0 & \theta \in \mathbb{R}. \end{cases}$$

We look for solutions of the form $u(r, \theta) = v(r)w(\theta)$ (variable separated). Substituting u in separated form in the equation, we get

$$-\frac{w}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \frac{v}{r^2}w''(\theta) = \lambda v(r)w(\theta).$$

Hence dividing by vw and multiplying by r^2, we get

$$-\frac{r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \frac{1}{w}w''(\theta) = \lambda r^2.$$

$$\frac{r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \lambda r^2 = \frac{-1}{w}w''(\theta) = \mu.$$

Solving for non-trivial w, using the periodicity of w, we get for $\mu_0 = 0$, $w_0(\theta) = \frac{a_0}{2}$ and for each $k \in \mathbb{N}$, $\mu_k = k^2$ and

$$w_k(\theta) = a_k \cos k\theta + b_k \sin k\theta.$$

For each $k \in \mathbb{N} \cup \{0\}$, we have the equation,

$$r\frac{d}{dr}\left(r\frac{dv}{dr}\right) + (\lambda r^2 - k^2)v = 0.$$

Introduce change of variable $x = \sqrt{\lambda}r$ and $x^2 = \lambda r^2$. Then

$$r\frac{d}{dr} = x\frac{d}{dx}.$$

rewriting the equation in new variable $y(x)) = v(r)$

$$x\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right) + (x^2 - k^2)y(x) = 0.$$

Note that this none other than the *Bessel's equation*. We already know that for each $k \in \mathbb{N} \cup \{0\}$, we have the Bessel's function J_k as a solution to the Bessel's equation. Recall the boundary condition on v, $v(a) = 0$. Thus, $y(\sqrt{\lambda}a) = 0$. Hence $\sqrt{\lambda}a$ should be a zero of the Bessel's function.

For each $k \in \mathbb{N} \cup \{0\}$, let z_{kl} be the l-th zero of J_k, $l \in \mathbb{N}$. Hence $\sqrt{\lambda}a = z_{kl}$ and so $\lambda_{kl} = z_{kl}^2/a^2$ and $y(x) = J_k(x)$. Therefore, $v(r) = J_k(z_{kl}r/a)$. For each $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$, we have

$$u_{kl}(r, \theta) = J_k(z_{kl}r/a)\sin(k\theta) \text{ or } J_k(z_{kl}r/a)\cos(k\theta)$$

and $\lambda_{kl} = z_{kl}^2/a^2$.

Chapter 5

Heat Equation

5.1 Derivation

Let a homogeneous material occupy a region $\Omega \subset \mathbb{R}^n$ with C^1 boundary. Let k denote the thermal conductivity (dimensionless quantity) and c be the heat capacity of the material. Let $u(x,t)$ be a function plotting the temperature of the material at $x \in \Omega$ in time t. The thermal energy stored at $x \in \Omega$, at time t, is $cu(x,t)$. If $\mathbf{v}(x,t)$ denotes the velocity of (x,t), by Fourier law, the thermal energy changes following the gradients of temperature, i.e.,

$$cu(x,t)\mathbf{v}(x,t) = -k\nabla u.$$

The thermal energy is the quantity that is conserved (conservation law) and satisfies the continuity equation (1.4.1). Thus, we have

$$u_t - \frac{k}{c}\Delta u = 0.$$

If the material occupying the region Ω is non-homogeneous, anisotropic, the temperature gradient may generate heat in preferred directions, which themselves may depend on $x \in \Omega$. Thus, the conductivity of such a material at $x \in \Omega$, at time t, is given by a $n \times n$ matrix $K(x,t) = (k_{ij}(x,t))$. Thus, in this case, the heat equation becomes,

$$u_t - \operatorname{div}\left(\frac{1}{c}K\nabla u\right) = 0.$$

The heat equation is an example of a second order equation in divergence form. The heat equation gives the temperature distribution $u(x,t)$ of the material with conductivity k and capacity c. In general, we may choose $k/c = 1$, since, for any k and c, we may rescale our time scale $t \mapsto (k/c)t$. The Cauchy initial value problem (IVP) of the heat equation,

$$\begin{cases} u_t(x,t) - \Delta_x u(x,t) &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= g(x) \quad \text{in } \mathbb{R}^n, \end{cases} \tag{5.1.1}$$

where $g \in C(\Omega)$. We end this section with a remark that under the transformation $t \mapsto -t$, in contrast to the wave equation, the heat equation changes to a background equation. This signifies that the heat equation describes irreversible process, i.e., it is not possible to find the distribution of temperature at an earlier time $t < t_0$, if the distribution is given at t_0.

5.2 Boundary Conditions

To make the heat equation $u_t - \Delta u = f$ in $\Omega \times (0,T)$ well-posed, where Ω is a bounded open subset of $\mathbb{R}^n$, we choose to specify the boundary condition $u(x,0) = g(x)$ on $\Omega \times \{t = 0\}$ and one of the following conditions on $\partial\Omega \times (0,T)$:

(i) (Dirichlet condition) $u(x,t) = h(x,t)$;

(ii) (Neumann condition) $\nabla_x u(x,t) \cdot \nu(x) = h(x,t)$, where $nu(x)$ is the unit outward normal of $(x,t) \in \partial\Omega \times (0,T)$;

(iii) (Robin condition) $\nabla_x u(x,t) \cdot \nu + cu(x,t) = h(x,t)$ for any $c > 0$.

(iv) (Mixed condition) $u(x,t) = h(x,t)$ on Γ_1 and $\nabla_x u(x,t) \cdot \nu = h(x,t)$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega \times (0,T)$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

5.3 Heat Flow on a Bar

The equation governing the heat propagation in a bar of length L is

$$\frac{\partial u}{\partial t} = \frac{1}{\rho(x)\sigma(x)} \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$$

where $\sigma(x)$ is the *specific heat* at x, $\rho(x)$ is *density* of bar at x and $\kappa(x)$ is the *thermal conductivity* of the bar at x. If the bar is *homogeneous*, i.e, its properties are same at every point, then

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho\sigma}\frac{\partial^2 u}{\partial x^2}$$

with ρ, σ, κ being constants.

Let L be the length of a homogeneous rod insulated along sides insulated along sides and its ends are kept at zero temperature. Then the temperature $u(x,t)$ at every point of the rod, $0 \leq x \leq L$ and time $t \geq 0$ is given by the equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

where c is a constant.

The temperature zero at the end points is given by the Dirichlet boundary condition

$$u(0,t) = u(L,t) = 0.$$

Also, given is the initial temperature of the rod at time $t = 0$, $u(x,0) = g(x)$, where g is given (or known) such that $g(0) = g(L) = 0$. Given $g : [0,L] \to \mathbb{R}$ such that $g(0) = g(L) = 0$, we look for all the solutions of the Dirichlet problem

$$\begin{cases} u_t(x,t) - c^2 u_{xx}(x,t) & = 0 \quad \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) & = 0 \quad \text{in } (0,\infty) \\ u(x,0) & = g(x) \quad \text{on } [0,L]. \end{cases}$$

We look for $u(x,t) = v(x)w(t)$ (variable separated). Substituting u in separated form in the equation, we get

$$v(x)w'(t) = c^2 v''(x)w(t)$$

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)}.$$

Since LHS is function of t and RHS is function x and are equal they must be some constant, say λ. Thus,

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)} = \lambda.$$

Thus we need to solve two ODE to get v and w,

$$w'(t) = \lambda c^2 w(t)$$

and

$$v''(x) = \lambda v(x).$$

But we already know how to solve the eigenvalue problem involving v. For each $k \in \mathbb{N}$, we have the pair (λ_k, v_k) as solutions to the EVP involving v, where $\lambda_k = -(k\pi)^2/L^2$ and $v_k(x) = \sin\left(\frac{k\pi x}{L}\right)$ some constants b_k. For each $k \in \mathbb{N}$, we solve for w_k to get

$$\ln w_k(t) = \lambda_k c^2 t + \ln \alpha$$

where α is integration constant. Thus, $w_k(t) = \alpha e^{-(kc\pi/L)^2 t}$. Hence,

$$u_k(x,t) = v_k(x) w_k(t) = \beta_k \sin\left(\frac{k\pi x}{L}\right) e^{-(kc\pi/L)^2 t},$$

for some constants β_k, is a solution to the heat equation. By superposition principle, the general solution is

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right) e^{-(kc\pi/L)^2 t}.$$

We now use the initial temperature of the rod, given as $g : [0, L] \to \mathbb{R}$ to find the particular solution of the heat equation. We are given $u(x, 0) = g(x)$. Thus,

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right)$$

Since $g(0) = g(L) = 0$, we know that g admits a Fourier Sine expansion and hence its coefficients β_k are given as

$$\beta_k = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right).$$

5.4 On a Circular Wire

We intend solve the heat equation in a circle (circular wire) of radius one which is insulated along its sides. Then the temperature $u(\theta, t)$ at every point of the circle, $\theta \in \mathbb{R}$ and time $t \geq 0$ is given by the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}$$

where c is a constant. We note that now $u(\theta,t)$ is 2π-periodic in the variable θ. Thus,

$$u(\theta+2\pi,t)=u(\theta,t)\quad \forall\theta\in\mathbb{R}, t\geq 0.$$

Let the initial temperature of the wire at time $t=0$, be $u(\theta,0)=g(\theta)$, where g is a given 2π-periodic function. Given a 2π-periodic function $g:\mathbb{R}\to\mathbb{R}$, we look for all solutions of

$$\begin{cases} u_t(\theta,t)-c^2u_{\theta\theta}(\theta,t) &= 0 & \text{in } \mathbb{R}\times(0,\infty)\\ u(\theta+2\pi,t) &= u(\theta,t) & \text{in } \mathbb{R}\times(0,\infty)\\ u(\theta,0) &= g(\theta) & \text{on } \mathbb{R}\times\{t=0\}.\end{cases}$$

We look for $u(\theta,t)=v(\theta)w(t)$ with variables separated. Substituting for u in the equation, we get

$$\frac{w'(t)}{c^2w(t)}=\frac{v''(\theta)}{v(\theta)}=\lambda.$$

For each $k\in\mathbb{N}\cup\{0\}$, the pair (λ_k,v_k) is a solution to the EVP where $\lambda_k=-k^2$ and

$$v_k(\theta)=a_k\cos(k\theta)+b_k\sin(k\theta).$$

For each $k\in\mathbb{N}\cup\{0\}$, we get $w_k(t)=\alpha e^{-(kc)^2t}$. For $k=0$

$$u_0(\theta,t)=a_0/2\quad \text{(To maintain consistency with Fourier series)}$$

and for each $k\in\mathbb{N}$, we have

$$u_k(\theta,t)=\left[a_k\cos(k\theta)+b_k\sin(k\theta)\right]e^{-k^2c^2t}$$

Therefore, the general solution is

$$u(\theta,t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos(k\theta)+b_k\sin(k\theta)\right]e^{-k^2c^2t}.$$

We now use the initial temperature on the circle to find the particular solution. We are given $u(\theta,0)=g(\theta)$. Thus,

$$g(\theta)=u(\theta,0)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos(k\theta)+b_k\sin(k\theta)\right]$$

Since g is 2π-periodic it admits a Fourier series expansion and hence

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}g(\theta)\cos(k\theta)\,d\theta,$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(k\theta)\, d\theta.$$

Note that as $t \to \infty$ the temperature of the wire approaches a constant $a_0/2$.

Exercise 20. Solve the heat equation for 2D Rectangle and 2D Disk

5.5 Inhomogeneous Equation

In this section we solve the inhomogeneous heat equation, using Duhamel's principle. The *Duhamel's principle* states that one can obtain a solution of the inhomogeneous IVP for heat from its homogeneous IVP.

For a given f, let $u(x,t)$ be the solution of the *inhomogeneous* heat equation,

$$\begin{cases} u_t(x,t) - c^2 \Delta u(x,t) &= f(x,t) \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) &= 0 \quad \text{in } \partial\Omega \times (0,\infty) \\ u(x,0) &= 0 \quad \text{in } \Omega. \end{cases}$$

As a first step, for each $s \in (0,\infty)$, consider $w(x,t;s)$ as the solution of the *homogeneous* problem (auxiliary)

$$\begin{cases} w^s_t(x,t) - c^2 \Delta w^s(x,t) &= 0 \quad \text{in } \Omega \times (s,\infty) \\ w^s(x,t) &= 0 \quad \text{in } \partial\Omega \times (s,\infty) \\ w^s(x,s) &= f(x,s) \quad \text{on } \Omega \times \{s\}. \end{cases}$$

Since $t \in (s,\infty)$, introducing a change of variable $r = t-s$, we have $w^s(x,t) = w(x,t-s)$ which solves

$$\begin{cases} w_t(x,r) - c^2 \Delta w(x,r) &= 0 \quad \text{in } \Omega \times (0,\infty) \\ w(x,r) &= 0 \quad \text{in } \partial\Omega \times (0,\infty) \\ w(x,0) &= f(x,s) \quad \text{on } \Omega. \end{cases}$$

Duhamel's principle states that

$$u(x,t) = \int_0^t w^s(x,t)\, ds = \int_0^t w(x,t-s)\, ds$$

and u solves the inhomogenous heat equation. Suppose w is C^2, we get

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\int_0^t w(x,t-s)\,ds \\ &= \int_0^t w_t(x,t-s)\,ds + w(x,t-t)\frac{d(t)}{dt} \\ &\quad - w(x,t-0)\frac{d(0)}{dt} \\ &= \int_0^t w_t(x,t-s)\,ds + w(x,0) \\ &= \int_0^t w_t(x,t-s)\,ds + f(x,t). \end{aligned}$$

Similarly,

$$\Delta u(x,t) = \int_0^t \Delta w(x,t-s)\,ds.$$

Thus,

$$\begin{aligned} u_t - c^2\Delta u &= f(x,t) + \int_0^t \big(w_t(x,t-s) - c^2\Delta w(x,t-s)\big)\,ds \\ &= f(x,t). \end{aligned}$$

5.6 Steady State Equation

Consider the Cauchy problem with inhomogeneous Dirichlet boundary conditions

$$\begin{cases} u_t(x,t) - c^2\Delta u(x,t) &= 0 \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) = &= \phi(x,t) \quad \text{in } \partial\Omega \times [0,\infty) \\ u(x,0) &= g(x) \quad \text{on } \overline{\Omega} \end{cases}$$

such that, for all $x \in \partial\Omega$,

$$g(x) = \phi(x,0), g''(x) = \phi''(x,0).$$

The *steady-state* solution of the heat equation is defined as

$$v(x) = \lim t \to \infty u(x,t).$$

Note that v satisfies the equation $\Delta v = 0$, since $v_t = 0$. Further v satisfies the boundary condition on $\partial\Omega$ as

$$v(x) = \lim t \to \infty \phi(x,t).$$

5.7 Fundamental Solution of Heat Equation

We shall now derive the fundamental solution of the heat equation (5.1.1). Taking Fourier transform both sides of the equation, we get

$$\begin{aligned}
\widehat{u_t - \Delta u} &= \hat{f} \\
\widehat{u_t}(\xi,t) - \widehat{\Delta u}(\xi,t) &= \hat{f}(\xi,t) \\
\hat{u}_t(\xi,t) - \sum_{j=1}^{n} i^2\xi_j^2\hat{u}(\xi,t) &= \hat{f}(\xi,t) \\
\hat{u}_t(\xi,t) + |\xi|^2\hat{u}(\xi,t) &= \hat{f}(\xi,t).
\end{aligned}$$

The solution of the above ODE is given by (cf. Appendix G)

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-|\xi|^2 t} + \int_0^t e^{-|\xi|^2(t-s)}\hat{f}(\xi,s)\,ds$$

where we have used the initial condition $\hat{u}(\xi,0) = \hat{g}(\xi)$ of the ODE. Therefore, by inverse Fourier formula,

$$\begin{aligned}
u(x,t) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\hat{g}(\xi)e^{-|\xi|^2 t}e^{i\xi\cdot x} + \int_0^t e^{i\xi\cdot x-|\xi|^2(t-s)}\hat{f}(\xi,s)\,ds\right)d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(y)e^{-i\xi\cdot y}\,dy\right)e^{-|\xi|^2 t}e^{i\xi\cdot x}\,d\xi \\
&\quad +(2\pi)^{-n}\int_{\mathbb{R}^n}\int_0^t\left(\int_{\mathbb{R}^n} f(y,s)e^{-i\xi\cdot y}\,dy\right)e^{i\xi\cdot x-|\xi|^2(t-s)}\,ds\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} g(y)\left(\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)-|\xi|^2 t}\,d\xi\right)dy \\
&\quad +(2\pi)^{-n}\int_{\mathbb{R}^n}\int_0^t f(y,s)\left(\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)-|\xi|^2(t-s)}\,d\xi\right)ds\,dy \\
&= \int_{\mathbb{R}^n} g(y)K(x,y,t)\,dy + \int_{\mathbb{R}^n}\int_0^t f(y,s)K(x,y,t-s)\,ds\,dy
\end{aligned}$$

where

$$K(x,y,t) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)-|\xi|^2 t}\,d\xi.$$

Note that

$$i\xi\cdot(x-y) - |\xi|^2 t = -\left(\xi\sqrt{t} - i\frac{(x-y)}{2\sqrt{t}}\right)\cdot\left(\xi\sqrt{t} - i\frac{(x-y)}{2\sqrt{t}}\right) - \frac{|x-y|^2}{4t}$$

and, set $\eta = \left(\xi\sqrt{t} - i\frac{(x-y)}{2\sqrt{t}}\right)$. Therefore, $d\eta = \sqrt{t}d\xi$. Using this substituion in K and simplifying, we get

$$K(x,y,t) = (4\pi t)^{-n/2}e^{-|x-y|^2/4t}$$

called the *heat kernel* or the fundamental solution of heat equation. The function K can be motivated in another way. Note that if $u(x,t)$ is a solution of the heat equation, then $(u \circ T_\lambda)(x,t)$ is also a solution of the heat equation, where $T_\lambda(x,t) = (\lambda x, \lambda^2 t)$ is a linear transformation for any $\lambda \neq 0$. This scaling or dilation is called the parabolic scaling. Thus, we look for a solution $u(x,t) = v(t)w(r^2/t)$, where $r = |x|$. Substituting this separation of variable in the heat equation, we derive $v(t) = t^{-n/2}$ and $w(t) = e^{-r^2/4t}$. This motivates us to define the fundamental solution as

$$K(x,t) := \begin{cases} -\frac{1}{4\pi t}^{n/2}e^{-r^2/4t} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0. \end{cases}$$

Chapter 6

The Laplacian

A general second order linear elliptic equation is of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u(x) = f(x)$$

where $A(x) = a_{ij}(x)$ is real, symmetric and positive definite. If A is a constant matrix then with a suitable transformation one can rewrite

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} = \Delta v$$

where $v(x) := u(Tx)$. We introduced (cf. Chapter 1) Laplacian to be the trace of the Hessain matrix, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. The Laplace operator usually appears in physical models associated with dissipative effects (except wave equation). The importance of Laplace operator can be realised by its appearance in various physical models. For instance, the heat equation

$$\frac{\partial}{\partial t} - \Delta,$$

the wave equation

$$\frac{\partial^2}{\partial t^2} - \Delta,$$

or the Schrödinger's equation

$$i\frac{\partial}{\partial t} + \Delta.$$

The Laplacian is a linear operator, i.e., $\Delta(u+v) = \Delta u + \Delta v$ and $\Delta(\lambda u) = \lambda \Delta u$ for any constant $\lambda \in \mathbb{R}$.

6.1 Properties of Laplacian

The Laplace operator commutes with the translation and rotation operator. For any $a \in \mathbb{R}^n$, we define the translation map $T_a : C(\Omega) \to C(\Omega)$ as $(T_a u)(x) = u(x+a)$. The invariance of Laplacian under translation means that $\Delta \circ T_a = T_a \circ \Delta$. For any $u \in C^2(\Omega)$, $(T_a u)_{x_i}(x) = u_{x_i}(x+a)$ and $(T_a u)_{x_i x_i}(x) = u_{x_i x_i}(x+a)$. Thus, $\Delta(T_a u)(x) = \Delta u(x+a)$.

For any orthogonal $n \times n$ matrix O ($O^{-1} = O^t$), we define $R : C(\Omega) \to C(\Omega)$ as $Ru(x) = u(Ox)$. The invariance of Laplacian under rotation means that $\Delta \circ R = R \circ \Delta$. Let $y = Ox$. Then, $y_j = \sum_{i=1}^n O_{ji} x_i$ and, by chain rule,

$$(Ru)_{x_i} = \sum_{j=1}^n u_{y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^n u_{y_j} O_{ji}.$$

Therefore, $\nabla_x Ru = O^t \nabla_y u$ and

$$(\Delta \circ R)u(x) = \nabla_x Ru \cdot \nabla_x (Ru) = O^t \nabla_y u \cdot O^t \nabla_y u = O O^t \nabla_y u \cdot \nabla_y u = \Delta_y u.$$

But $\Delta_y u = (\Delta u)(Ox) = (R \circ \Delta)u(x)$.

A radial function is constant on every sphere about the origin. Since Laplacian commutes with rotations, it should map the class of all radial functions to itself.

In cartesian coordiantes, the n-dimensional Laplacian is given as

$$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

In polar coordinates (2 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

where r is the magnitude component ($0 \leq r < \infty$) and θ is the direction component ($0 \leq \theta < 2\pi$). The direction component is also called the azimuth angle or polar angle.

Exercise 21. Show that the two dimensional Laplacian has the representation

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates.

Proof. Using the fact that $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial y}{\partial r} = \sin\theta \text{ and } \frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}.$$

Also,

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\theta\frac{\partial^2 u}{\partial x^2} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}.$$

Similarly,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta \text{ and } \frac{\partial u}{\partial \theta} = r\cos\theta\frac{\partial u}{\partial y} - r\sin\theta\frac{\partial u}{\partial x}.$$

Also,

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \sin^2\theta\frac{\partial^2 u}{\partial x^2} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{r}\frac{\partial u}{\partial r}.$$

Therefore,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}.$$

and hence

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

□

In cylindrical coordinates (3 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

where $r \in [0,\infty)$, $\theta \in [0, 2\pi)$ and $z \in \mathbb{R}$. In spherical coordinates (3 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$

where $r \in [0,\infty)$, $\phi \in [0,\pi]$ (zenith angle or inclination) and $\theta \in [0, 2\pi)$ (azimuth angle).

Note that in one dimension, $n = 1$, $\Delta = \frac{d^2}{dx^2}$.

Proposition 6.1.1. *Let $n \geq 2$ and u be a radial function, i.e., $u(x) = v(r)$ where $x \in \mathbb{R}^n$ and $r = |x|$, then*

$$\Delta u(x) = \frac{d^2 v(r)}{dr^2} + \frac{(n-1)}{r}\frac{dv(r)}{dr}.$$

Proof. Note that

$$\begin{aligned}
\frac{\partial r}{\partial x_i} = \frac{\partial |x|}{\partial x_i} &= \frac{\partial(\sqrt{x_1^2 + \ldots + x_n^2})}{\partial x_i} \\
&= \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{-1/2}(2x_i) \\
&= \frac{x_i}{r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta u(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u(x)}{\partial x_i}\right) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{dv(r)}{dr}\frac{x_i}{r}\right) \\
&= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\frac{dv(r)}{dr}\right) + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \sum_{i=1}^{n} \frac{x_i^2}{r}\frac{d}{dr}\left(\frac{dv(r)}{dr}\frac{1}{r}\right) + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \sum_{i=1}^{n} \frac{x_i^2}{r}\left\{\frac{1}{r}\frac{d^2 v(r)}{dr^2} - \frac{1}{r^2}\frac{dv(r)}{dr}\right\} + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \frac{r^2}{r}\left\{\frac{1}{r}\frac{d^2 v(r)}{dr^2} - \frac{1}{r^2}\frac{dv(r)}{dr}\right\} + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \frac{d^2 v(r)}{dr^2} - \frac{1}{r}\frac{dv(r)}{dr} + \frac{n}{r}\frac{dv(r)}{dr} \\
&= \frac{d^2 v(r)}{dr^2} + \frac{(n-1)}{r}\frac{dv(r)}{dr}.
\end{aligned}$$

Hence the result proved. □

More generally, the Laplacian in $\mathbb{R}^n$ may be written in polar coordinates as

$$\Delta := \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is a second order differential operator in angular variables only. The above forms of Laplacian gives an insight into how the Laplacian treats radial function and angular functions. The angular part of Laplacian is called the *Laplace-Beltrami operator* acting on $\mathbb{S}^{n-1}$ (unit sphere of $\mathbb{R}^n$) with Riemannian metric induced by the standard Euclidean metric in $\mathbb{R}^n$.

6.2 Ill-Posedness of Cauchy Problem

Recall that for a second order Cauchy problem we need to know both u and its normal derivative on a data curve Γ contained in Ω. However, the Cauchy problem for Laplacian (more generally for elliptic equations) is not well-posed. In fact, the Cauchy problem for Laplace equation on a bounded domain Ω is over-determined.

Example 6.1 (Hadamard). Consider the Cauchy problem for Laplace equation

$$\begin{cases} u_{xx} + u_{yy} &= 0 \\ u(0, y) &= \frac{\cos ky}{k^2} \\ u_x(0, y) &= 0, \end{cases}$$

where $k > 0$ is an integer. It is easy to verify that there is a unique solution

$$u_k(x, y) = \frac{\cosh(kx)\cos(ky)}{k^2}$$

of the Cauchy problem. Note that for any $x_0 > 0$,

$$|u_k(x_0, n\pi/k)| = \frac{\cosh(kx_0)}{k^2}.$$

Since, as $k \to \infty$, $n\pi/k \to 0$ and $|u_k(x_0, n\pi/k)| \to \infty$ the Cauchy problem is not stable, and hence not well-posed.

Exercise 22. Show that the Cauchy problem for Laplace equation

$$\begin{cases} u_{xx} + u_{yy} &= 0 \\ u(x, 0) &= 0 \\ u_y(x, 0) &= k^{-1}\sin kx, \end{cases}$$

where $k > 0$, is not well-posed. (Hint: Compute explicit solution using separation of variable. Note that, as $k \to \infty$, the Cauchy data tends to zero uniformly, but the solution does not converge to zero for any $y \neq 0$. Therefore, a small change from zero Cauchy data (with corresponding solution being zero) may induce bigger change in the solution.)

This issue of ill-posedness of the Cauchy problem is very special to second order elliptic equations. In general, any hyperbolic equation Cauchy problem is well-posed, as long as the hyperbolicity is valid in the full neighbourhood of the data curve.

Example 6.2. Consider the Cauchy problem for the second order hyperbolic equation

$$\begin{cases} y^2 u_{xx} - y u_{yy} + \frac{1}{2} u_y & = 0 \quad y > 0 \\ u(x,0) & = f(x) \\ u_y(x,0) & = g(x). \end{cases}$$

The general solution to this problem can be computed as

$$u(x,y) = F\left(x + \frac{2}{3} y^{3/2}\right) + G\left(x - \frac{2}{3} y^{3/2}\right).$$

On $y = 0$ $u(x,0) = F(x) + G(x) = f(x)$. Further,

$$u_y(x,y) = y^{1/2} F'\left(x + \frac{2}{3} y^{3/2}\right) - y^{1/2} G'\left(x - \frac{2}{3} y^{3/2}\right)$$

and $u_y(x,0) = 0$. Thus, the Cauchy problem has no solution unless $g(x) = 0$. If $g \equiv 0$ then the solution is

$$u(x,y) = F\left(x + \frac{2}{3} y^{3/2}\right) - F\left(x - \frac{2}{3} y^{3/2}\right) + f\left(x - \frac{2}{3} y^{3/2}\right)$$

for arbitrary $F \in C^2$. Therefore, when $g \equiv 0$ the solution is not unique. The Cauchy problem is not well-posed because the equation is hyperbolic ($B^2 - AC = y^3$) not in the full neighbourhood of the data curve $\{y = 0\}$.

6.3 Boundary Conditions

To make the Poisson equation $-\Delta u = f$ in Ω well-posed, where Ω is a bounded open subset of $\mathbb{R}^n$, we choose to specify one of the following conditions on the boundary, $\partial\Omega$, of Ω:

(i) (Dirichlet condition) $u = g$;

(ii) (Neumann condition) $\nabla u \cdot \nu = g$, where $\nu(x)$ is the unit outward normal of $x \in \partial\Omega$;

(iii) (Robin condition) $\nabla u \cdot \nu + cu = g$ for any $c > 0$.

(iv) (Mixed condition) $u = g$ on Γ_1 and $\nabla u \cdot \nu = h$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

The Poisson equation with Neumann boundary condition comes with a compatibility condition. Note that, by Guass divergence theorem (cf. Corollary D.0.7), if u is a solution of the Neumann problem then u satisfies, for every connected component ω of Ω,

$$\begin{aligned} \int_\omega \Delta u &= \int_{\partial\omega} \nabla u \cdot \nu \quad \text{(Using GDT)} \\ -\int_\omega f &= \int_{\partial\omega} g. \end{aligned}$$

The second equality is called the *compatibility* condition. Thus, for a Neumann problem the given data f, g must necessarily satisfy the compatibility condition. Otherwise, the Neumann problem does not make any sense.

The aim of this chapter is solve

$$\begin{cases} -\Delta u(x) & = f(x) & \text{in } \Omega \\ \text{one of} & \text{the above} & \text{inhomogeneous boudary condition on } \partial\Omega, \end{cases}$$

for any open subset $\Omega \subset \mathbb{R}^n$. Note that, by linearity of Laplacian, $u = v + w$ where v is a harmonic function[1] that solves

$$\begin{cases} \Delta v(x) & = 0 & \text{in } \Omega \\ \text{one of} & \text{the above} & \text{inhomogeneous boudary condition on } \partial\Omega, \end{cases}$$

and w solves the Poisson equation[2]

$$\begin{cases} -\Delta w(x) & = f(x) & \text{in } \Omega \\ \text{one of} & \text{the above} & \text{homogeneous boudary condition on } \partial\Omega. \end{cases}$$

Therefore, we shall restrict our attention to solving only for v and w. We begin our analysis with v called *harmonic functions*.

[1]defined later
[2]defined later

6.4 Harmonic Functions

The *one dimensional* Laplace equation is a ODE and is solvable with solutions $u(x) = ax + b$ for some constants a and b. But in higher dimensions solving Laplace equation is not so simple. For instance, a *two dimensional* Laplace equation

$$u_{xx} + u_{yy} = 0$$

has the trivial solution, $u(x, y) = ax + by + c$, all one degree polynomials of two variables. In addition, xy, $x^2 - y^2$, $x^3 - 3xy^2$, $3x^2y - y^3$, $e^x \sin y$ and $e^x \cos y$ are all solutions to the two variable Laplace equation. In $\mathbb{R}^n$, it is trivial to check that all polynomials up to degree one, i.e.

$$\sum_{|\alpha| \leq 1} a_\alpha x^\alpha$$

is a solution to $\Delta u = 0$ in $\mathbb{R}^n$. But we also have functions of higher degree and functions not expressible in terms of elementary functions as solutions to Laplace equation. For instance, note that $u(x) = \prod_{i=1}^n x_i$ is a solution to $\Delta u = 0$ in $\mathbb{R}^n$.

Definition 6.4.1. *Let Ω be an open subset of $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is said to be* harmonic *on Ω if $\Delta u(x) = 0$ in Ω.*

We already remarked that every scalar potential is a harmonic function. Gauss was the first to deduce some important properties of harmonic functions and thus laid the foundation for Potential theory and Harmonic Analysis.

Due to the linearity of Δ, sum of any finite number of harmonic functions is harmonic and a scalar multiple of a harmonic function is harmonic. Moreover, harmonic functions can be viewed as the null-space of the Laplace operator, say from $C^2(\Omega)$ to $C(\Omega)$, the space of continuous functions.

In two dimension, one can associate with a harmonic function $u(x, y)$, a *conjugate* harmonic function, $v(x, y)$ which satisfy the first order system of PDE called the *Cauchy-Riemann* equations,

$$u_x = v_y \quad \text{and } u_y = -v_x.$$

Harmonic functions and holomorphic functions (differentiable complex functions) are related in the sense that, for any pair (u, v), harmonic and its conjugate, gives a holomorphic function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$.

Conversely, for any holomorphic function f, its real part and imaginary part are conjugate harmonic functions. This observation gives us more examples of harmonic functions, for instance, since all complex polynomials $f(z) = z^m$ are holomorphic we have (using the polar coordinates) $u(r,\theta) = r^m \cos m\theta$ and $v(r,\theta) = r^m \sin m\theta$ are harmonic functions in $\mathbb{R}^2$ for all $m \in \mathbb{N}$. Similarly, since $f(z) = \log z = \ln r + i\theta$ is holomorphic in certain region, we have $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ are harmonic in $\mathbb{R}^2 \setminus (0,0)$ and $\mathbb{R}^2 \setminus \{\theta = 0\}$, respectively.

Exercise 23. Show that there are infinitely many linearly independent harmonic functions in the vector space $C^2(\mathbb{R}^2)$.

6.4.1 Spherical Harmonics

A polynomial of degree k in n-variables is of the form

$$P_k(x) := \sum_{|\alpha| \leq k} a_\alpha x^\alpha.$$

A polynomial P is said to be homogeneous of degree k if $P(\lambda x) = \lambda^k P(x)$ for any real $\lambda \neq 0$. Note that a homogeneous polynomial of degree k is of the form

$$\sum_{|\alpha| = k} a_\alpha x^\alpha.$$

The number of possible n-tuples α such that $|\alpha| = k$ is given by $\binom{n+k-1}{k}$. Let $\mathcal{H}_k(\mathbb{R}^n)$ denote the set of all homogeneous harmonic polynomial of degree k in n variables. Note that $\mathcal{H}_k(\mathbb{R}^n)$ forms a vector space. Recall that the class of harmonic functions, call it $N(\Delta)$, is a (vector) subspace of $C^2(\mathbb{R}^n)$ and, hence, $\mathcal{H}_k(\mathbb{R}^n) \subset N(\Delta) \subset C^2(\mathbb{R}^n)$.

Two Dimensions

Consider a general homogeneous polynomial

$$P_k(x,y) := \sum_{i=0}^{k} a_i x^i y^{k-i}$$

of degree k in $\mathbb{R}^2$ (two variables). Note that P_k contains $k+1$ coefficients[3]. Then

$$\Delta P_k(x,y) = \sum_{i=2}^{k} a_i i(i-1)x^{i-2}y^{k-i} + \sum_{i=0}^{k-2} a_i(k-i)(k-i-1)x^i y^{k-i-2}$$

is a homogeneous polynomial of degree $k-2$ and, hence, contains $k-1$ coefficients. If $P_k \in \mathcal{H}_k(\mathbb{R}^2)$, i.e. $\Delta P_k(x,y) = 0$, then all the $k-1$ coefficients should vanish. Thus, we have $k-1$ equations relating the $k+1$ coefficients of P_k and, hence, $\mathcal{H}_k(\mathbb{R}^2)$ is of dimension two (since $k+1-(k-1)=2$). Let us now find the basis of the two dimensional space $\mathcal{H}_k(\mathbb{R}^2)$. In polar coordinates, $P_k(r,\theta) = r^k Q_k(\theta)$ where

$$Q_k(\theta) = \sum_{i=0}^{k} a_i (\cos\theta)^i (\sin\theta)^{k-i}.$$

Note that Q_k is the restriction of P_k to $\mathbb{S}^1$ and are called *spherical harmonics.* If $P_k \in \mathcal{H}_k(\mathbb{R}^2)$ then, using the polar form of Laplacian, we get

$$r^{k-2}\left[Q_k''(\theta) + k^2 Q_k(\theta)\right] = 0.$$

Therefore, for all $r > 0$, $Q_k(\theta)$ is a solution to the ODE

$$Q_k''(\theta) + k^2 Q_k(\theta) = 0.$$

Therefore, $Q_k(\theta) = \alpha\cos k\theta + \beta\sin k\theta$ and $P_k(r,\theta) = r^k(\alpha\cos k\theta + \beta\sin k\theta)$. Thus, P_k is a linear combination $r^k\cos k\theta$ and $r^k\sin k\theta$. In fact, if we identify each vector $(x,y) \in \mathbb{R}^2$ with the complex number $z = x+iy$, then we have shown that $\mathrm{Re}(z^k), \mathrm{Im}(z^k)$ are the basis of $\mathcal{H}_k(\mathbb{R}^2)$. If we choose α_1 and β_1 such that $\beta = -\alpha_1\sin\beta_1$ and $\alpha = \alpha_1\cos\beta_1$, then we can rewrite the polynomial as

$$P_k(r,\theta) = \alpha_1 r^k \cos(k\theta + \beta_1).$$

Thus, we immediately see that the zero set of $P_k(r,\theta)$ in $\mathbb{R}^2$ will be a family of k straight lines passing through origin such that between any two consecutive lines the angle is same.

[3] $\binom{k+1}{k} = k+1$

Three Dimensions

Consider a general homogeneous polynomial

$$P_k(x) := \sum_{|\alpha|=k} a_\alpha x^\alpha$$

of degree k in $\mathbb{R}^3$ (three variables). Note that P_k contains $\binom{k+2}{k} = \frac{(k+2)(k+1)}{2}$ coefficients. Then $\Delta P_k(x)$ is a homogeneous polynomial of degree $k-2$ and, hence, contains $\frac{k(k-1)}{2}$ coefficients. If $P_k \in \mathcal{H}_k(\mathbb{R}^3)$, i.e. $\Delta P_k(x) = 0$, then all the $\frac{k(k-1)}{2}$ coefficients should vanish. Thus, we have $\frac{k(k-1)}{2}$ equations relating the $\frac{(k+2)(k+1)}{2}$ coefficients of P_k and, hence, $\mathcal{H}_k(\mathbb{R}^3)$ is of dimension

$$\frac{(k+2)(k+1) - k(k-1)}{2} = 2k+1.$$

The basis of the $2k+1$ dimensional space $\mathcal{H}_k(\mathbb{R}^3)$ is given in terms of the Legendre functions which we shall describe now. In spherical coordinates, $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$ and $z = r\cos\phi$. Thus, $P_k(r,\phi,\theta) = r^k R(\phi)Q(\theta)$ where

$$R(\phi)Q(\theta) = \sum_{|\alpha|=k} a_\alpha (\sin\phi)^{\alpha_1+\alpha_2}(\cos\phi)^{\alpha_3}(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}.$$

The separated variable assumption above is not a issue because differential operator is linear. Note that RQ is the restriction of P_k to $\mathbb{S}^2$ and are called *spherical harmonics*. If $P_k \in \mathcal{H}_k(\mathbb{R}^2)$ then, using the spherical form of Laplacian, we get

$$r^{k-2}\left[k(k+1)\sin^2\phi + \sin^2\phi\frac{R''(\phi)}{R(\phi)} + \sin\phi\cos\phi\frac{R'(\phi)}{R(\phi)} + \frac{Q''(\theta)}{Q(\theta)}\right] = 0.$$

Therefore, for all $r > 0$, we have equality

$$k(k+1)\sin^2\phi + \sin^2\phi\frac{R''(\phi)}{R(\phi)} + \sin\phi\cos\phi\frac{R'(\phi)}{R(\phi)} = -\frac{Q''(\theta)}{Q(\theta)}.$$

Since LHS is a function of ϕ and RHS is a function of θ they must be equal to some constant λ. Then, we have to solve for the eigenvalue problem

$$-Q''(\theta) = \lambda Q(\theta)$$

where Q is 2π-periodic. This has the solution, for all $m \in \mathbb{N} \cup \{0\}$, $\lambda = m^2$ and $Q_m(\theta) = \alpha_m \cos m\theta + \beta_m \sin m\theta$. For $\lambda = m^2$ we solve for $R(\phi)$ in

$$R''(\phi) + \frac{\cos\phi}{\sin\phi} R'(\phi) = R(\phi) \left(\frac{m^2}{\sin^2\phi} - k(k+1) \right) \quad \phi \in (0, \phi).$$

Set $w = \cos\phi$. Then $\frac{dw}{d\phi} = -\sin\phi$.

$$R'(\phi) = -\sin\phi \frac{dR}{dw} \text{ and } R''(\phi) = \sin^2\phi \frac{d^2R}{dw^2} - \cos\phi \frac{dR}{dw}$$

In the new variable w, we get the Legendre equation

$$(1 - w^2)R''(w) - 2wR'(w) = \left(\frac{m^2}{1 - w^2} - k(k+1) \right) R(w) \quad w \in [-1, 1].$$

For each $k \in \mathbb{N} \cup \{0\}$, this has the Legendre polynomials, $R_{k,m}(\cos\phi)$, as its solutions. Therefore, in general,

$$P_k(r, \phi, \theta) = r^k(\alpha \cos m\theta + \beta \sin m\theta) R_{k,m}(\cos\phi).$$

However, we are interested only those $R_{k,m}$ which gives a polynomial of degree k in $\mathbb{R}^3$. Thus, for $m = 0, 1, \ldots, k$,

$$R_{k,m}(w) = (1 - w^2)^{m/2} \frac{d^{k+m}}{dw^{k+m}} (1 - w^2)^k.$$

Note that, for each fixed k and all $1 \leq m \leq k$, the collection

$$\{R_{k,0}(\cos\phi), \cos m\theta R_{k,m}(\cos\phi), \sin m\theta R_{k,m}(\cos\phi)\} \subset \mathcal{H}_k(\mathbb{R}^3)$$

is $2k + 1$ linearly independent homogeneous harmonic polynomials of degree k and forms a basis. Thus, each P_k is a linear combination of these basis elements.

The zero sets of P_k exhibit properties depending on m. For $m = 0$ the harmonic polynomial P_k is a constant multiple of $R_{k,0}(\cos\phi)$. Since $R_{k,0}(w)$ has k distinct zeros in $[-1, 1]$ arranged symmetrically about $w = 0$, there are k distince zeros of $R_{k,0}(\cos\phi)$ in $(0, \pi)$ arranged symmetrically about $\pi/2$. Thus on $\mathbb{S}^2$, the unit sphere, the function $R_{k,0}(\cos\phi)$ vanishes on k circles circumscribed in the latitudinal direction. For k odd the circle along equator is also a zero set. The function $R_{k,0}(\cos\phi)$ and its constant multiples are called *zonal harmonics*.

If $0 < m < k$, then the spherical harmonics is of the form

$$(\alpha\cos m\theta + \beta\sin m\theta)\sin^m\phi\frac{d^{k+m}}{dw^{k+m}}(1-w^2)^k.$$

If the first term is zero then $\tan m\theta = -\alpha/\beta$. This corresponds to great circle through the north pole and south pole of $\mathbb{S}^2$ and the angle between the planes containing two consecutive great circle is π/m. The second term vanishes on $\phi = 0$ and $\phi = \pi$ corresponding to the north and south pole, respectively. The third term vanishes on $k-m$ latitude circle. Thus, we have orthogonally intersecting family of circles which form the zero set which are called *tesseral harmonics*.

If $m = k$ then the spherical harmonics is of the form

$$(\alpha\cos k\theta + \beta\sin k\theta)\sin^k\phi$$

and it vanishes for $\phi = 0$, $\phi = \pi$ or $\tan k\theta = -\alpha/\beta$. The first two cases corresponds to the north and south pole, respectively, and the last case corresponds to great circles through the north pole and south pole of $\mathbb{S}^2$ and the angle between the planes containing two consecutive great circle is π/k. Thus, the great circles divide the $\mathbb{S}^2$ in to $2k$ sectors and are called *sectorial harmoics*.

Higher Dimensions

Consider a general homogeneous polynomial

$$P_k(x) := \sum_{|\alpha|=k} a_\alpha x^\alpha$$

of degree k in $\mathbb{R}^n$ (n variables). Note that P_k contains $\binom{n+k-1}{k}$ coefficients. Then $\Delta P_k(x)$ is a homogeneous polynomial of degree $k-2$ and, hence, contains $\binom{n+k-3}{k-2}$ coefficients. If $P_k \in \mathcal{H}_k(\mathbb{R}^n)$, i.e. $\Delta P_k(x) = 0$, then all the $\binom{n+k-3}{k-2}$ coefficients should vanish. Thus, we have $\binom{n+k-3}{k-2}$ equations relating $\binom{n+k-1}{k}$ coefficients of P_k and, hence, $\mathcal{H}_k(\mathbb{R}^n)$ is of dimension

$$\ell := \binom{n+k-1}{k} - \binom{n+k-3}{k-2}.$$

In polar form, $P_k(r,\theta) = r^k Q(\theta)$ where $\theta \in S^{n-1}$ and if $P_k(r,\theta) \in \mathcal{H}_k(\mathbb{R}^n)$ then

$$\Delta P_k = \frac{\partial^2 P_k}{\partial r^2} + \frac{n-1}{r}\frac{\partial P_k}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} P_k = 0$$

where $\Delta_{\mathbb{S}^{n-1}}$ is a second order differential operator in angular variables only called the Laplace-Beltrami operator. Therefore, we have

$$r^{k-2}\left[\Delta_{\mathbb{S}^{n-1}} Q(\theta) + k(n+k-2)Q(\theta)\right] = 0$$

and for $r > 0$,

$$\Delta_{\mathbb{S}^{n-1}} Q(\theta) + k(n+k-2)Q(\theta) = 0.$$

6.4.2 Properties of Harmonic Functions

In this section we shall study properties of harmonic functions. We shall assume the divergence theorems from multivariable calculus (cf. Appendix D). Also, note that if u is a harmonic function on Ω then, by Gauss divergence theorem (cf. Theorem D.0.6),

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = 0.$$

Definition 6.4.2. *Let Ω be an open subset of $\mathbb{R}^n$ and $w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ (cf. Appendix E) be the surface area of the unit sphere $S_1(0)$ of $\mathbb{R}^n$.*

(a) A function $u \in C(\Omega)$ is said to satisfy the first mean value property *(I-MVP) in Ω if*

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\, d\sigma_y \quad \text{for any } B_r(x) \subset \Omega.$$

(b) A function $u \in C(\Omega)$ is said to satisfy the second mean value property *(II-MVP) in Ω if*

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \quad \text{for any } B_r(x) \subset \Omega.$$

Exercise 24. Show that u satisfies the I-MVP iff

$$u(x) = \frac{1}{\omega_n} \int_{S_1(0)} u(x + rz)\, d\sigma_z.$$

Similarly, u satisfies II-MVP iff

$$u(x) = \frac{n}{\omega_n} \int_{B_1(0)} u(x + rz)\, dz.$$

Exercise 25. Show that the first MVP and second MVP are equivalent. That is show that u satisfies (a) iff u satisfies (b).

Owing to the above exercise we shall, henceforth, refer to the I-MVP and II-MVP as just mean value property (MVP).

We shall now prove a result on the smoothness of a function satisfying MVP.

Theorem 6.4.3. *If $u \in C(\Omega)$ satisfies the MVP in Ω, then $u \in C^\infty(\Omega)$.*

Proof. We first consider $u_\varepsilon := \rho_\varepsilon * u$, the convolution of u with mollifiers, as introduced in Theorem F.0.10. where

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

We shall now show that $u = u_\varepsilon$ for all $\varepsilon > 0$, due to the MVP of u and the radial nature of ρ. Let $x \in \Omega_\varepsilon$. Consider

$$\begin{aligned}
u_\varepsilon(x) &= \int_\Omega \rho_\varepsilon(x - y) u(y)\, dy \\
&= \int_{B_\varepsilon(x)} \rho_\varepsilon(x - y) u(y)\, dy \quad \text{(Since supp}(\rho_\varepsilon)\text{ is in } B_\varepsilon(x)) \\
&= \int_0^\varepsilon \rho_\varepsilon(r) \left(\int_{S_r(x)} u(y)\, d\sigma_y \right) dr \quad \text{(cf. Theorem E.0.8)} \\
&= u(x)\omega_n \int_0^\varepsilon \rho_\varepsilon(r) r^{n-1}\, dr \quad \text{(Using MVP of } u) \\
&= u(x) \int_0^\varepsilon \rho_\varepsilon(r) \left(\int_{S_r(0)} d\sigma_y \right) dr \\
&= u(x) \int_{B_\varepsilon(0)} \rho_\varepsilon(y)\, dy = u(x).
\end{aligned}$$

Thus $u_\varepsilon(x) = u(x)$ for all $x \in \Omega_\varepsilon$ and for all $\varepsilon > 0$. Since $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$ (cf. Theorem F.0.10), we have $u \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$. □

Theorem 6.4.4. *Let u be a harmonic function on Ω. Then u satisfies the MVP in Ω.*

Proof. Let $B_r(x) \subset \Omega$ be any ball with centre at $x \in \Omega$ and for some $r > 0$. For the given harmonic function u, we set

$$v(r) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\, d\sigma_y.$$

Note that v is not defined at 0, since $r > 0$. We have from Exercise 24 that

$$v(r) = \frac{1}{\omega_n} \int_{S_1(0)} u(x + rz)\, d\sigma_z.$$

Now, differentiating both sides w.r.t r, we get

$$\begin{aligned} \frac{dv(r)}{dr} &= \frac{1}{\omega_n} \int_{S_1(0)} \nabla u(x + rz) \cdot z\, d\sigma_z \\ &= \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} \nabla u(y) \cdot \frac{(y - x)}{r}\, d\sigma_y \end{aligned}$$

Since $|x - y| = r$, by setting $\nu := (y - x)/r$ as the unit vector, and applying the Gauss divergence theorem along with the fact that u is harmonic, we get

$$\frac{dv(r)}{dr} = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} \nabla u(y) \cdot \nu\, d\sigma_y = \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y)\, dy = 0.$$

Thus, v is a constant function of $r > 0$ and hence

$$v(r) = v(\varepsilon) \quad \forall \varepsilon > 0.$$

Moreover, since v is continuous (constant function), we have

$$\begin{aligned} v(r) &= \lim_{\varepsilon \to 0} v(\varepsilon) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\omega_n} \int_{S_1(0)} u(x + \varepsilon z)\, d\sigma_z \\ &= \frac{1}{\omega_n} \int_{S_1(0)} \lim_{\varepsilon \to 0} u(x + \varepsilon z)\, d\sigma_z \quad (u \text{ is continuous on } S_1(0)) \\ &= \frac{1}{\omega_n} \int_{S_1(0)} u(x)\, d\sigma_z \\ &= u(x) \quad (\text{Since } \omega_n \text{ is the surface area of } S_1(0)). \end{aligned}$$

Thus, u satisfies I-MVP and hence the II-MVP. □

Corollary 6.4.5. *If u is harmonic on Ω, then $u \in C^\infty(\Omega)$.*

The above corollary is a easy consequence of Theorem 6.4.4 and Theorem 6.4.3. We shall now prove that any function satisfying MVP is harmonic.

Theorem 6.4.6. *If $u \in C(\Omega)$ satisfies the MVP in Ω, then u is harmonic in Ω.*

Proof. Since u satisfies MVP, by Theorem 6.4.3, $u \in C^\infty(\Omega)$. Thus, Δu makes sense. Now, suppose u is not harmonic in Ω, then there is a $x \in \Omega$ such that $\Delta u(x) \neq 0$. Without loss of generality, let's say $\Delta u(x) > 0$. Moreover, since Δu is continuous there is a $s > 0$ such that, for all $y \in B_s(x)$, $\Delta u(y) > 0$. As done previously, we set for $r > 0$,

$$v(r) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\, d\sigma_y.$$

Thus, $v(r) = u(x)$ for all $r > 0$ and hence v is a constant function of r and $v'(s) = 0$. But

$$0 = \frac{dv(s)}{dr} = \frac{1}{\omega_n r^{n-1}} \int_{B_s(x)} \Delta u(y)\, dy > 0$$

is a contradiction. Therefore, u is harmonic in Ω. □

Above results leads us to conclude that a function is harmonic iff it satisfies the MVP.

Exercise 26. If u_m is a sequence of harmonic functions in Ω converging to u uniformly on compact subsets of Ω, then show that u is harmonic in Ω.

Theorem 6.4.7 (Strong Maximum Principle)**.** *Let Ω be an open, connected (domain) subset of $\mathbb{R}^n$. Let u be harmonic in Ω and $M := \max_{y \in \overline{\Omega}} u(y)$. Then*

$$u(x) < M \quad \forall x \in \Omega$$

or $u \equiv M$ is constant in Ω.

Proof. We define a subset S of Ω as follows,

$$S := \{x \in \Omega \mid u(x) = M\}.$$

If $S = \emptyset$, we have $u(x) < M$ for all $x \in \Omega$. Suppose $S \neq \emptyset$. Then S is closed subset of Ω, since u is continuous. Now, for any $x \in S$, by MVP

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \quad \text{for every } r \text{ such that } B_r(x) \subset \Omega.$$

Thus, we have

$$M = u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \leq M$$

Hence equality will hold above only when $u(y) = M$ for all $y \in B_r(x)$. Thus, we have shown that for any $x \in S$, we have $B_r(x) \subset S$. Therefore, S is open. Since Ω is connected, the only open and closed subsets are $\emptyset$ or Ω. Since S was assumed to be non-empty, we should have $S = \Omega$. Thus, $u \equiv M$ is constant in Ω. □

Corollary 6.4.8 (Weak maximum Principle)**.** *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be harmonic in Ω. Then*

$$\max_{y \in \overline{\Omega}} u(y) = \max_{y \in \partial\Omega} u(y).$$

Proof. Let $M := \max_{y \in \overline{\Omega}} u(y)$. If there is a $x \in \Omega$ such that $u(x) = M$, then $u \equiv M$ is constant on the connected component of Ω containing x. Thus, $u = M$ on the boundary of the connected component which is a part of $\partial\Omega$. □

Aliter. Since $\partial\Omega \subset \overline{\Omega}$, we have $\max_{\partial\Omega} u \leq \max_{\overline{\Omega}} u$. It only remains to prove the other equality. For the given harmonic function u and for a fixed $\varepsilon > 0$, we set $v_\varepsilon(x) = u(x) + \varepsilon|x|^2$, for each $x \in \overline{\Omega}$. For each $x \in \Omega$, $\Delta v_\varepsilon = \Delta u + 2n\varepsilon > 0$. Recall that[4] if a function v attains local maximum at a point $x \in \Omega$, then in each direction its second order partial derivative $v_{x_i x_i}(x) \leq 0$, for all $i = 1, 2, \ldots, n$. Therefore $\Delta v(x) \leq 0$. Thus, we argue that v_ε does not attain (even a local) maximum in Ω. But v_ε has to have a maximum in $\overline{\Omega}$, hence it should be attained at some point $x^\star \in \partial\Omega$, on the boundary. For all $x \in \overline{\Omega}$,

$$u(x) \leq v_\varepsilon(x) \leq v_\varepsilon(x^\star) = u(x^\star) + \varepsilon|x^\star|^2 \leq \max_{x \in \partial\Omega} u(x) + \varepsilon \max_{x \in \partial\Omega} |x|^2.$$

The above inequality is true for all $\varepsilon > 0$. Thus, $u(x) \leq \max_{x \in \partial\Omega} u(x)$, for all $x \in \overline{\Omega}$. Therefore, $\max_{\overline{\Omega}} u \leq \max_{x \in \partial\Omega} u(x)$. and hence we have equality. □

[4] $v \in C^2(a, b)$ has a local maximum at $x \in (a, b)$ then $v'(x) = 0$ and $v''(x) \leq 0$

Theorem 6.4.9 (Estimates on derivatives). *If u is harmonic in Ω, then*

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \|u\|_{1,B_r(x)} \quad \forall B_r(x) \subset \Omega \text{ and each } |\alpha| = k$$

where the constants $C_0 = \frac{n}{\omega_n}$ and $C_k = C_0(2^{n+1}nk)^k$ for $k = 1, 2, \ldots$.

Proof. We prove the result by induction on k. Let $k = 0$. Since u is harmonic, by II-MVP we have, for any $B_r(x) \subset \Omega$,

$$\begin{aligned}
|u(x)| &= \frac{n}{\omega_n r^n} \left| \int_{B_r(x)} u(y)\, dy \right| \\
&\leq \frac{n}{\omega_n r^n} \int_{B_r(x)} |u(y)|\, dy \\
&= \frac{n}{\omega_n r^n} \|u\|_{1,B_r(x)} = \frac{C_0}{r^n} \|u\|_{1,B_r(x)}.
\end{aligned}$$

Now, let $k = 1$. Observe that if u is harmonic then by differentiating the Laplace equation and using the equality of mixed derivatives, we have $u_{x_i} := \frac{\partial u}{\partial x_i}$ is harmoic, for all $i = 1, 2, \ldots, n$. Now, by the II-MVP of u_{x_i}, we have

$$\begin{aligned}
|u_{x_i}(x)| &= \frac{n2^n}{\omega_n r^n} \left| \int_{B_{r/2}(x)} u_{x_i}(y)\, dy \right| \\
&= \frac{n2^n}{\omega_n r^n} \left| \int_{S_{r/2}(x)} u\nu_i\, d\sigma_y \right| \quad \text{(by Gauss-Green theorem)} \\
&\leq \frac{2n}{r} \|u\|_{\infty,S_{r/2}(x)}.
\end{aligned}$$

Thus, it now remains to estimate $\|u\|_{\infty,S_{r/2}(x)}$. Let $z \in S_{r/2}(x)$, then

$$B_{r/2}(z) \subset B_r(x) \subset \Omega.$$

But, using $k = 0$ result, we have

$$|u(z)| \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_{r/2}(z)} \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_r(x)}.$$

Therefore, $\|u\|_{\infty,S_{r/2}(x)} \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_r(x)}$ and using this in the estimate of u_{x_i}, we get

$$|u_{x_i}(x)| \leq \frac{C_0 n 2^{n+1}}{r^{n+1}} \|u\|_{1,B_r(x)}.$$

Hence

$$|D^\alpha u(x)| \leq \frac{C_1}{r^{n+1}} \|u\|_{1,B_r(x)} \quad \text{for } |\alpha| = 1.$$

Let now $k \geq 2$ and α be a multi-index such that $|\alpha| = k$. We assume the induction hypothesis that the estimate to be proved is true for $k-1$. Note that $D^\alpha u = \frac{\partial D^\beta u}{\partial x_i}$ for some $i \in \{1, 2, \dots, n\}$ and $|\beta| = k-1$. Moreover, if u is harmonic then by differentiating the Laplace equation and using the equality of mixed derivatives, we have $\frac{\partial D^\beta u}{\partial x_i}$ is harmoic for $i = 1, 2, \dots, n$. Thus, following an earlier argument, we have

$$\begin{aligned} |D^\alpha u(x)| = \left| \frac{\partial D^\beta u(x)}{\partial x_i} \right| &= \frac{nk^n}{\omega_n r^n} \left| \int_{B_{r/k}(x)} \frac{\partial D^\beta u(y)}{\partial x_i} \, dy \right| \\ &= \frac{nk^n}{\omega_n r^n} \left| \int_{S_{r/k}(x)} D^\beta u \nu_i \, d\sigma_y \right| \\ &\leq \frac{nk}{r} \|D^\beta u\|_{\infty, S_{r/k}(x)}. \end{aligned}$$

It now only remains to estimate $\|D^\beta u\|_{\infty, S_{r/k}(x)}$. Let $z \in S_{r/k}(x)$, then $B_{(k-1)r/k}(z) \subset B_r(x) \subset \Omega$. But, using induction hypothesis for $k-1$, we have

$$|D^\beta u(z)| \leq \frac{C_{k-1} k^{n+k-1}}{((k-1)r)^{n+k-1}} \|u\|_{1, B_{(k-1)r/k}(z)} \leq \frac{C_{k-1} k^{n+k-1}}{((k-1)r)^{n+k-1}} \|u\|_{1, B_r(x)}.$$

Therefore, using the above estimate for $D^\alpha u$, we get

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{C_{k-1} n k^{n+k}}{(k-1)^{n+k-1} r^{n+k}} \|u\|_{1, B_r(x)} \\ &= \frac{C_0 2^{(n+1)(k-1)} n^k (k-1)^{k-1} k^{n+k}}{(k-1)^{n+k-1} r^{n+k}} \|u\|_{1, B_r(x)} \\ &= \frac{C_0 (2^{n+1} nk)^k}{r^{n+k}} \left(\frac{k}{k-1} \right)^n \left(\frac{1}{2^{n+1}} \right) \|u\|_{1, B_r(x)} \\ &= \frac{C_0 (2^{n+1} nk)^k}{r^{n+k}} \left(\frac{k}{2(k-1)} \right)^n \left(\frac{1}{2} \right) \|u\|_{1, B_r(x)} \\ &\leq \frac{C_k}{r^{n+k}} \|u\|_{1, B_r(x)} \quad \text{since } \left(\frac{k}{2(k-1)} \right)^n \left(\frac{1}{2} \right) \leq 1. \end{aligned}$$

Hence

$$|D^\alpha u(x)| \le \frac{C_k}{r^{n+k}}\|u\|_{1,B_r(x)} \quad \text{for } |\alpha| = k, \forall k \ge 2.$$

□

Theorem 6.4.10 (Liouville's Theorem)**.** *If u is bounded and harmonic on $\mathbb{R}^n$, then u is constant.*

Proof. For any $x \in \mathbb{R}^n$ and $r > 0$, we have the estimate on the first derivative as,

$$\begin{aligned}
|\nabla u(x)| &\le \frac{C_1}{r^{n+1}}\|u\|_{1,B_r(x)} \\
&= \frac{2^{n+1}n}{\omega_n r^{n+1}}\|u\|_{1,B_r(x)} \\
&\le \frac{2^{n+1}n}{\omega_n r^{n+1}}\|u\|_{\infty,\mathbb{R}^n}\omega_n r^n \\
&= \frac{2^{n+1}}{r}\|u\|_{\infty,\mathbb{R}^n} \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Thus, $\nabla u \equiv 0$ in $\mathbb{R}^n$ and hence u is constant. □

Exercise 27. Show that if u is harmonic in Ω, then u is analytic in Ω. (Hint: Use the estimates on derivatives with Stirling's formula and Taylor expansion).

We end our discussion on the properties of harmonic function with Harnack inequality. The Harnack inequality states that non-negative harmonic functions cannot be very large or very small at any point without being so everywhere in a compact set containing that point.

Theorem 6.4.11 (Harnack's Inequality)**.** *Let u be harmonic in Ω and $u \ge 0$ in Ω, then for each connected open subset $\omega \subset\subset \Omega$ there is a constant $C > 0$ (depending only on ω) such that*

$$\sup_{x\in\omega} u(x) \le C \inf_{x\in\omega} u(x).$$

In particular,

$$\frac{1}{C}u(y) \le u(x) \le Cu(y) \quad \forall x, y \in \omega.$$

Proof. Set $r := \frac{1}{4}\text{dist}(\omega, \partial\Omega)$. Let $x, y \in \omega$ such that $|x-y| < r$. By II-MVP,

$$\begin{aligned} u(x) &= \frac{n}{\omega_n 2^n r^n}\int_{B_{2r}(x)} u(z)\,dz \\ &\geq \frac{n}{\omega_n 2^n r^n}\int_{B_r(y)} u(z)\,dz = \frac{1}{2^n}u(y). \end{aligned}$$

Thus, $1/2^n u(y) \leq u(x)$. Interchanging the role of x and y, we get $1/2^n u(x) \leq u(y)$. Thus, $1/2^n u(y) \leq u(x) \leq 2^n u(y)$ for all $x, y \in \omega$ such that $|x-y| \leq r$.

Now, let $x, y \in \omega$. Since $\overline{\omega}$ is compact and connected in Ω, we can pick points $x = x_1, x_2, \ldots, x_m = y$ such that $\cup_{i=1}^m B_i \supset \overline{\omega}$, where $B_i := B_{r/2}(x_i)$ and are sorted such that $B_i \cap B_{i+1} \neq \emptyset$, for $i = 1, 2, \ldots, m-1$. Hence, note that $|x_{i+1} - x_i| \leq r$. Therefore,

$$u(x) = u(x_1) \geq \frac{1}{2^n}u(x_2) \geq \frac{1}{2^{2n}}u(x_3) \geq \ldots \geq \frac{1}{2^{(m-1)n}}u(x_m) = \frac{1}{2^{(m-1)n}}u(y).$$

Thus, C can be chosen to be $\frac{1}{2^{(m-1)n}}$. □

6.5 Existence and Uniqueness

A consequence of the maximum principle is the uniqueness of the harmonic functions.

Theorem 6.5.1 (Uniqueness of Harmonic Functions)**.** *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω such that $u_1 = u_2$ on $\partial\Omega$, then $u_1 = u_2$ in Ω.*

Proof. Note that $u_1 - u_2$ is a harmonic function and hence, by weak maximum principle, should attain its maximum on $\partial\Omega$. But $u_1 - u_2 = 0$ on $\partial\Omega$. Thus $u_1 - u_2 \leq 0$ in Ω. Now, repeat the argument for $u_2 - u_1$, we get $u_2 - u_1 \leq 0$ in Ω. Thus, we get $u_1 - u_2 = 0$ in Ω. □

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Dirichlet problem

$$\begin{cases} \Delta u(x) &= 0 \quad x \in \Omega \\ u(x) &= g(x) \quad x \in \partial\Omega. \end{cases} \tag{6.5.1}$$

By the strong maximum principle (cf. Theorem 6.4.7), if Ω is connected and $g \geq 0$ and $g(x) > 0$ for some $x \in \partial\Omega$ then $u(x) > 0$ for all $x \in \Omega$.

Theorem 6.5.2. *Let Ω be an open bounded connected subset of $\mathbb{R}^n$ and $g \in C(\partial\Omega)$. Then the Dirichlet problem (6.5.1) has atmost one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, if u_1 and u_2 are solution to the Dirichlet problem corresponding to g_1 and g_2 in $C(\partial\Omega)$, respectively, then*

(a) (Comparison) $g_1 \geq g_2$ on $\partial\Omega$ and $g_1(x_0) > g_2(x_0)$ for some $x \in \partial\Omega$ implies that $u_1 > u_2$ in Ω.

(b) (Stability) $|u_1(x) - u_2(x)| \leq \max_{y \in \partial\Omega} |g_1(y) - g_2(y)|$ for all $x \in \Omega$.

Proof. The fact that there is atmost one solution to the Dirichlet problem follows from the Theorem 6.5.1. Let $w = u_1 - u_2$. Then w is harmonic.

(a) Note that $w = g_1 - g_2 \geq 0$ on $\partial\Omega$. Since $g_1(x_0) > g_2(x_0)$ for some $x_0 \in \partial\Omega$, then $w(x) > 0$ for all $x \in \partial\Omega$. This proves the comparison result.

(b) Again, by maximum principle, we have

$$\pm w(x) \leq \max_{y \in \partial\Omega} |g_1(y) - g_2(y)| \forall x \in \Omega.$$

This proves the stability result.

□

We remark that the uniqueness result is not true for unbounded domains.

Example 6.3. Consider the problem (6.5.1) with $g \equiv 0$ in the domain $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$. Obviously, $u = 0$ is a solution. But we also have a non-trivial solution

$$u(x) = \begin{cases} \ln |x| & n = 2 \\ |x|^{2-n} - 1 & n \geq 3. \end{cases}$$

Example 6.4. Consider the problem (6.5.1) with $g \equiv 0$ in the domain $\Omega = \{x \in \mathbb{R}^n \mid x_n > 0\}$. Obviously, $u = 0$ is a solution. But we also have a non-trivial solution $u(x) = x_n$.

We have shown above that if a solution exists for (6.5.1) then it is unique (cf. Theorem 6.5.1). So the question that remains to be answered is on the existence of solution of (6.5.1), for any given domain Ω. In the modern theory, there are three different methods to address this question of existence, viz., Perron's Method, Layer Potential (Integral Equations) and L^2 methods.

6.6 Perron's Method

Definition 6.6.1. *We say a function $w \in C(\overline{\Omega})$ is a* barrier *at $x_0 \in \partial\Omega$ if there is a neighbourhood U of x_0 such that*

1. *w is superharmonic in $\Omega \cap U$*
2. *$w > 0$ in $(\overline{\Omega} \cap U) \setminus \{x_0\}$ and $w(x_0) = 0$.*

Definition 6.6.2. *Any point on $\partial\Omega$ is said to be* regular *(w.r.t Laplacian) if there exists a barrier at that point.*

A necessary and sufficient condition for the existence of solution to (6.5.1) is given by the following result:

Theorem 6.6.3. *The Dirichlet problem (6.5.1) is solvable for any arbitrary bounded domain Ω and for any arbitrary g on $\partial\Omega$ iff all the points in $\partial\Omega$ are regular.*

Proof. One way is obvious. If (6.5.1) is solvable and $x_0 \in \partial\Omega$. Then, the solution to

$$\begin{cases} \Delta w &= 0 \quad \text{in } \Omega \\ w &= g \quad \text{on } \partial\Omega, \end{cases}$$

where $g(x) = |x - x_0|$, is a barrier function at x_0. Thus, any $x_0 \in \partial\Omega$ is a regular point. The converse is proved using the Perron's method for subharmonic functions. □

Definition 6.6.4. *A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the* exterior sphere condition *if for every point $x_0 \in \partial\Omega$ there is a ball $B := B_R(y)$ such that $\overline{B} \cap \overline{\Omega} = x_0$.*

Lemma 6.6.5. *If Ω satisfies the exterior sphere condition then all boundary points of Ω are regular.*

Proof. For any $x_0 \in \partial\Omega$, we define the barrier function at $x_0 \in \partial\Omega$ as

$$w(x) = \begin{cases} R^{2-n} - |x - y|^{2-n} & \text{for } n \geq 3 \\ \ln\left(\frac{|x-y|}{R}\right) & \text{for } n = 2. \end{cases}$$

□

Theorem 6.6.6. *Any bounded domain with C^2 boundary satisfies the exterior sphere condition.*

Definition 6.6.7. *A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the* exterior cone condition *if for every point $x_0 \in \partial\Omega$ there is a finite right circular cone K with vertex at x_0 such that $\overline{K} \cap \overline{\Omega} = x_0$.*

Exercise 28. Any domain satisfying the exterior sphere condition also satisfies the exterior cone condition.

Exercise 29. Every bounded Lipschitz domain satisfies the exterior cone condition.

Lemma 6.6.8. *If Ω satisfies the exterior cone condition then all boundary points of Ω are regular.*

6.6.1 Non-existence of Solutions

In 1912, Lebesgue gave an example of a domain on which the classical Dirichlet problem is not solvable. Let

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid r^2 + z^2 < 1; r > e^{-1/2z} \text{ for } z > 0\}.$$

Note that Ω is the unit ball in $\mathbb{R}^3$ with a sharp inward cusp, called *Lebesgue spine*, at $(0, 0, 0)$. The origin is a not regular point of Ω.

However, there do exist domains with inward cusps for which the classical problem is solvable, for instance, consider

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid r^2 + z^2 < 1; r > z^{2k} \text{ for } z > 0\},$$

for any positive integer k. The proof of this fact involves the theory of capacities.

6.6.2 Characterizing regular points

The *Wiener's criterion* gives the necessary and sufficient condition for the regularity of the boundary points. For $n \geq 3$ and a fixed $\lambda \in (0, 1)$, the *Wiener's criterion* states that a point $x_0 \in \partial\Omega$ is regular iff the series

$$\sum_{i=0}^{\infty} \frac{C_i}{\lambda^{i(n-2)}}$$

diverges, where $C_i := \text{cap}_2\{x \notin \Omega \mid |x - x_0| \leq \lambda^i\}$.

6.7 Ω with Simple Geometry

The method of separation of variables was introduced by d'Alembert (1747) and Euler (1748) for the wave equation. This technique was also employed by Laplace (1782) and Legendre (1782) while studying the Laplace equation and also by Fourier while studying the heat equation. The motivation behind the "separation of variable" technique will be highlighted while studying wave equation.

Theorem 6.7.1 (2D Rectangle)**.** *Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < a$ and $0 < y < b\}$ be a rectangle in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ which vanishes on three sides of the rectangle, i.e., $g(0, y) = g(x, 0) = g(a, y) = 0$ and $g(x, b) = h(x)$ where h is a continuous function $h(0) = h(a) = 0$. Then there is a unique solution to (6.5.1) on this rectangle with given boundary value g.*

Proof. We begin by looking for solution $u(x, y)$ whose variables are separated, i.e., $u(x, y) = v(x)w(y)$. Substituting this form of u in the Laplace equation, we get

$$v''(x)w(y) + v(x)w''(y) = 0.$$

Hence

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}.$$

Since LHS is function of x and RHS is function y, they must equal a constant, say λ. Thus,

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda.$$

Using the boundary condition on u, $u(0, y) = g(0, y) = g(a, y) = u(a, y) = 0$, we get $v(0)w(y) = v(a)w(y) = 0$. If $w \equiv 0$, then $u \equiv 0$ which is not a solution to (6.5.1). Hence, $w \not\equiv 0$ and $v(0) = v(a) = 0$. Thus, we need to solve,

$$\begin{cases} v''(x) & = \lambda v(x), \quad x \in (0, a) \\ v(0) & = v(a) \quad = 0, \end{cases}$$

the *eigen value problem* for the second order differential operator. Note that the λ can be either zero, positive or negative.

If $\lambda = 0$, then $v'' = 0$ and the general solution is $v(x) = \alpha x + \beta$, for some constants α and β. Since $v(0) = 0$, we get $\beta = 0$, and $v(a) = 0$ and $a \neq 0$ implies that $\alpha = 0$. Thus, $v \equiv 0$ and hence $u \equiv 0$. But, this can not be a solution to (6.5.1).

If $\lambda > 0$, then $v(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$. Equivalently,

$$v(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

such that $\alpha = (c_1 + c_2)/2$ and $\beta = (c_1 - c_2)/2$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$ and hence

$$v(x) = c_2 \sinh(\sqrt{\lambda}x).$$

Now using $v(a) = 0$, we have $c_2 \sinh \sqrt{\lambda}a = 0$. Thus, $c_2 = 0$ and $v(x) = 0$. We have seen this cannot be a solution.

If $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$. We need to solve

$$\begin{cases} v''(x) + \omega^2 v(x) &= 0 \quad x \in (0, a) \\ v(0) &= v(a) \quad = 0. \end{cases} \tag{6.7.1}$$

The general solution is

$$v(x) = \alpha \cos(\omega x) + \beta \sin(\omega x).$$

Using the boundary condition $v(0) = 0$, we get $\alpha = 0$ and hence $v(x) = \beta \sin(\omega x)$. Now using $v(a) = 0$, we have $\beta \sin \omega a = 0$. Thus, either $\beta = 0$ or $\sin \omega a = 0$. But $\beta = 0$ does not yield a solution. Hence $\omega a = k\pi$ or $\omega = k\pi/a$, for all non-zero $k \in \mathbb{Z}$. Hence, for each $k \in \mathbb{N}$, there is a solution (v_k, λ_k) for (6.7.1), with

$$v_k(x) = \beta_k \sin\left(\frac{k\pi x}{a}\right),$$

for some constant β_k and $\lambda_k = -(k\pi/a)^2$. We now solve w corresponding to each λ_k. For each $k \in \mathbb{N}$, we solve for w_k in the ODE

$$\begin{cases} w_k''(y) &= \left(\frac{k\pi}{a}\right)^2 w_k(y), \quad y \in (0, b) \\ w(0) &= 0. \end{cases}$$

Thus, $w_k(y) = c_k \sinh(k\pi y/a)$. Therefore, for each $k \in \mathbb{N}$,

$$u_k = \delta_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right)$$

is a solution to (6.5.1). The general solution is of the form (principle of superposition) (convergence?)

$$u(x,y) = \sum_{k=1}^{\infty} \delta_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right).$$

The constant δ_k are obtained by using the boundary condition $u(x,b) = h(x)$ which yields

$$h(x) = u(x,b) = \sum_{k=1}^{\infty} \delta_k \sinh\left(\frac{k\pi b}{a}\right) \sin\left(\frac{k\pi x}{a}\right).$$

Since $h(0) = h(a) = 0$, we know that h admits a Fourier Sine series. Thus $\delta_k \sinh\left(\frac{k\pi b}{a}\right)$ is the k-th Fourier sine coefficient of h, i.e.,

$$\delta_k = \left(\sinh\left(\frac{k\pi b}{a}\right)\right)^{-1} \frac{2}{a} \int_0^a h(x) \sin\left(\frac{k\pi x}{a}\right).$$

□

Theorem 6.7.2 (2D Disk)**.** *Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}$ be the disk of radius R in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ is a continuous function. Then there is a unique solution to* (6.5.1) *on the unit disk with given boundary value g.*

Proof. Given the nature of the domain, we shall use the Laplace operator in polar coordinates,

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

where r is the magnitude component and θ is the direction component. Then $\partial\Omega$ is the circle of radius one. Then, solving for $u(x,y)$ in the Dirichlet problem is to equivalent to finding $U(r,\theta) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} & = 0 & \text{in } \Omega \\ U(r, \theta + 2\pi) & = U(r,\theta) & \text{in } \Omega \\ U(R,\theta) & = G(\theta) & \text{on } \partial\Omega \end{cases} \tag{6.7.2}$$

where $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, $G : [0, 2\pi) \to \mathbb{R}$ is $G(\theta) = g(\cos\theta, \sin\theta)$. Note that both U and G are 2π periodic w.r.t θ. We will look for solution

$U(r,\theta)$ whose variables can be separated, i.e., $U(r,\theta) = v(r)w(\theta)$ with both v and w non-zero. Substituting it in the polar form of Laplacian, we get

$$\frac{w}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \frac{v}{r^2}\frac{d^2w}{d\theta^2} = 0$$

and hence

$$\frac{-r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{1}{w}\left(\frac{d^2w}{d\theta^2}\right).$$

Since LHS is a function of r and RHS is a function of θ, they must equal a constant, say λ. We need to solve the eigen value problem,

$$\begin{cases} w''(\theta) - \lambda w(\theta) & = 0 \quad \theta \in \mathbb{R} \\ w(\theta + 2\pi) & = w(\theta) \quad \forall\theta. \end{cases}$$

Note that the λ can be either zero, positive or negative. If $\lambda = 0$, then $w'' = 0$ and the general solution is $w(\theta) = \alpha\theta + \beta$, for some constants α and β. Using the periodicity of w,

$$\alpha\theta + \beta = w(\theta) = w(\theta + 2\pi) = \alpha\theta + 2\alpha\pi + \beta$$

implies that $\alpha = 0$. Thus, the pair $\lambda = 0$ and $w(\theta) = \beta$ is a solution. If $\lambda > 0$, then

$$w(\theta) = \alpha e^{\sqrt{\lambda}\theta} + \beta e^{-\sqrt{\lambda}\theta}.$$

If either of α and β is non-zero, then $w(\theta) \to \pm\infty$ as $\theta \to \infty$, which contradicts the periodicity of w. Thus, $\alpha = \beta = 0$ and $w \equiv 0$, which cannot be a solution. If $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$ and the equation becomes

$$\begin{cases} w''(\theta) + \omega^2 w(\theta) & = 0 \quad \theta \in \mathbb{R} \\ w(\theta + 2\pi) & = w(\theta) \quad \forall\theta \end{cases}$$

Its general solution is

$$w(\theta) = \alpha\cos(\omega\theta) + \beta\sin(\omega\theta).$$

Using the periodicity of w, we get $\omega = k$ where k is an integer. For each $k \in \mathbb{N}$, we have the solution (w_k, λ_k) where

$$\lambda_k = -k^2 \quad \text{and } w_k(\theta) = \alpha_k\cos(k\theta) + \beta_k\sin(k\theta).$$

For the λ_k's, we solve for v_k, for each $k = 0, 1, 2, \ldots$,

$$r\frac{d}{dr}\left(r\frac{dv_k}{dr}\right) = k^2 v_k.$$

For $k = 0$, we get $v_0(r) = \alpha \ln r + \beta$. But $\ln r$ blows up as $r \to 0$, but any solution U and, hence v, on the closed unit disk (compact subset) has to be bounded. Thus, we must have the $\alpha = 0$. Hence $v_0 \equiv \beta$. For $k \in \mathbb{N}$, we need to solve for v_k in

$$r\frac{d}{dr}\left(r\frac{dv_k}{dr}\right) = k^2 v_k.$$

Use the change of variable $r = e^s$. Then $e^s\frac{ds}{dr} = 1$ and $\frac{d}{dr} = \frac{d}{ds}\frac{ds}{dr} = \frac{1}{e^s}\frac{d}{ds}$. Hence $r\frac{d}{dr} = \frac{d}{ds}$. $v_k(e^s) = \alpha e^{ks} + \beta e^{-ks}$. $v_k(r) = \alpha r^k + \beta r^{-k}$. Since r^{-k} blows up as $r \to 0$, we must have $\beta = 0$. Thus, $v_k = \alpha r^k$. Therefore, for each $k = 0, 1, 2, \ldots$,

$$U_k(r, \theta) = a_k r^k \cos(k\theta) + b_k r^k \sin(k\theta).$$

The general solution is

$$U(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k r^k \cos(k\theta) + b_k r^k \sin(k\theta)\right).$$

To find the constants, we must use $U(R, \theta) = G(\theta)$. If $G \in C^1[0, 2\pi]$, then G admits Fourier series expansion. Therefore,

$$G(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[R^k a_k \cos(k\theta) + R^k b_k \sin(k\theta)\right]$$

where

$$a_k = \frac{1}{R^k\pi}\int_{-\pi}^{\pi} G(\theta)\cos(k\theta)\,d\theta,$$
$$b_k = \frac{1}{R^k\pi}\int_{-\pi}^{\pi} G(\theta)\sin(k\theta)\,d\theta.$$

Using this in the formula for U and the uniform convergence of Fourier series, we get

$$\begin{aligned} U(r, \theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi} G(\eta)\left[\frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k (\cos k\eta \cos k\theta + \sin k\eta \sin k\theta)\right] d\eta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} G(\eta)\left[\frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \cos k(\eta - \theta)\right] d\eta. \end{aligned}$$

Using the relation

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \cos k(\eta-\theta) &= \operatorname{Re}\left[\sum_{k=1}^{\infty}\left(\frac{r}{R}e^{i(\eta-\theta)}\right)^k\right] = \operatorname{Re}\left[\frac{\frac{r}{R}e^{i(\eta-\theta)}}{1-\frac{r}{R}e^{i(\eta-\theta)}}\right]\\ &= \frac{R^2 - rR\cos(\eta-\theta)}{R^2+r^2-2rR\cos(\eta-\theta)} - 1\\ &= \frac{rR\cos(\eta-\theta)-r^2}{R^2+r^2-2rR\cos(\eta-\theta)}\end{aligned}$$

in $U(r,\theta)$ we get

$$U(r,\theta) = \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{G(\eta)}{R^2+r^2-2rR\cos(\eta-\theta)}d\eta.$$

□

Note that the formula derived above for $U(r,\theta)$ can be rewritten in Cartesian coordinates and will have the form

$$u(x) = \frac{R^2-|x|^2}{2\pi R}\int_{S_R(0)}\frac{g(y)}{|x-y|^2}dy.$$

This can be easily seen, by setting $y = R(x_0^1\cos\eta + x_0^2\sin\eta)$, we get $dy = Rd\eta$ and $|x-y|^2 = R^2+r^2-2rR\cos(\eta-\theta)$. This is called the *Poisson* formula. More generally, the unique solution to the Dirichlet problem on a ball of radius R centred at x_0 in $\mathbb{R}^n$ is given by *Poisson formula*

$$u(x) = \frac{R^2-|x-x_0|^2}{\omega_n R}\int_{S_R(x_0)}\frac{g(y)}{|x-y|^n}dy.$$

We will derive this general form later (cf. (6.8.7)).

Theorem 6.7.3 (3D Sphere)**.** *Let $\Omega = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2<1\}$ be the unit sphere in $\mathbb{R}^3$. Let $g:\partial\Omega\to\mathbb{R}$ is a continuous function. Then there is a unique solution to* (6.5.1) *on the unit sphere with given boundary value g.*

Proof. Given the nature of domain, the Laplace operator in spherical coordinates,

$$\Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}.$$

where r is the magnitude component, ϕ is the inclination (zenith or elevation) in the vertical plane and θ is the azimuth angle (in the direction in horizontal plane). Solving for u in (6.5.1) is equivalent to finding $U(r,\phi,\theta):\Omega\to\mathbb{R}$ such that

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial U}{\partial\phi}\right) & \\ \qquad\qquad +\frac{1}{r^2\sin^2\phi}\frac{\partial^2 U}{\partial\theta^2} = 0 & \text{in } \Omega \\ \qquad\qquad U(1,\phi,\theta) = G(\phi,\theta) & \text{on } \partial\Omega \end{cases} \tag{6.7.3}$$

where $U(r,\phi,\theta)$ and $G(\phi,\theta)$ are appropriate spherical coordinate function corresponding to u and g. We will look for solution $U(r,\phi,\theta)$ whose variables can be separated, i.e., $U(r,\phi,\theta)=v(r)w(\phi)z(\theta)$ with v,w and z non-zero. Substituting it in the spherical form of Laplacian, we get

$$\frac{wz}{r^2}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right)+\frac{vz}{r^2\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dw}{d\phi}\right)+\frac{vw}{r^2\sin^2\phi}\frac{d^2z}{d\theta^2}=0$$

and hence

$$\frac{1}{v}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right)=\frac{-1}{w\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dw}{d\phi}\right)-\frac{1}{z\sin^2\phi}\frac{d^2z}{d\theta^2}.$$

Since LHS is a function of r and RHS is a function of (ϕ,θ), they must equal a constant, say λ. If Azimuthal symmetry is present then $z(\theta)$ is constant and hence $\frac{dz}{d\theta}=0$. We need to solve for w,

$$\sin\phi w''(\phi)+\cos\phi w'(\phi)+\lambda\sin\phi w(\phi)=0,\quad \phi\in(0,\pi)$$

Set $x=\cos\phi$. Then $\frac{dx}{d\phi}=-\sin\phi$.

$$w'(\phi)=-\sin\phi\frac{dw}{dx}\text{ and }w''(\phi)=\sin^2\phi\frac{d^2w}{dx^2}-\cos\phi\frac{dw}{dx}$$

In the new variable x, we get the Legendre equation

$$(1-x^2)w''(x)-2xw'(x)+\lambda w(x)=0\quad x\in[-1,1].$$

We have already seen that this is a singular problem (while studying S-L problems). For each $k\in\mathbb{N}\cup\{0\}$, we have the solution (w_k,λ_k) where

$$\lambda_k=k(k+1)\quad\text{and } w_k(\phi)=P_k(\cos\phi).$$

For the λ_k's, we solve for v_k, for each $k = 0, 1, 2, \ldots$,

$$\frac{d}{dr}\left(r^2 \frac{dv_k}{dr}\right) = k(k+1)v_k.$$

For $k = 0$, we get $v_0(r) = -\alpha/r + \beta$. But $1/r$ blows up as $r \to 0$ and U must be bounded in the closed sphere. Thus, we must have the $\alpha = 0$. Hence $v_0 \equiv \beta$. For $k \in \mathbb{N}$, we need to solve for v_k in

$$\frac{d}{dr}\left(r^2 \frac{dv_k}{dr}\right) = k(k+1)v_k.$$

Use the change of variable $r = e^s$. Then $e^s \frac{ds}{dr} = 1$ and $\frac{d}{dr} = \frac{d}{ds}\frac{ds}{dr} = \frac{1}{e^s}\frac{d}{ds}$. Hence $r\frac{d}{dr} = \frac{d}{ds}$. Solving for m in the quadratic equation $m^2 + m = k(k+1)$. $m_1 = k$ and $m_2 = -k-1$. $v_k(e^s) = \alpha e^{ks} + \beta e^{(-k-1)s}$. $v_k(r) = \alpha r^k + \beta r^{-k-1}$. Since r^{-k-1} blows up as $r \to 0$, we must have $\beta = 0$. Thus, $v_k = \alpha r^k$. Therefore, for each $k = 0, 1, 2, \ldots$,

$$U_k(r, \phi, \theta) = a_k r^k P_k(\cos\phi).$$

The general solution is

$$U(r, \phi, \theta) = \sum_{k=0}^{\infty} a_k r^k P_k(\cos\phi).$$

Since we have azimuthal symmetry, $G(\phi, \theta) = G(\phi)$. To find the constants, we use $U(1, \phi, \theta) = G(\phi)$, hence

$$G(\phi) = \sum_{k=0}^{\infty} a_k P_k(\cos\phi).$$

Using the orthogonality of P_k, we have

$$a_k = \frac{2k+1}{2}\int_0^{\pi} G(\phi)P_k(\cos\phi)\sin\phi\, d\phi.$$

□

Now that we have sufficient understanding of harmonic functions, solution of homogeneous Laplace equation, with Dirichlet boundary conditions we next attempt to solve the inhomogeneous Laplace equation (called Poisson equation) with homogeneous boundary conditions.

6.8 Poisson Equation

We now wish to solve the Poisson equation, for any given f (under some hypothesis) find u such that

$$-\Delta u = f \text{ in } \mathbb{R}^n. \tag{6.8.1}$$

Recall that we have already introduced the notion convolution of functions (cf. Appendix F) while discussing C^∞ properties of harmonic functions. We also observed that the differential operator can be accumulated on either side of the convolution operation. Suppose there is a function K with the property that ΔK is the identity of the convolution operation, i.e., $f * \Delta K = f$, then we know that $u := f * K$ is a solution of (6.8.1).

Definition 6.8.1. *We shall say a function K to be the* fundamental solution *of the Laplacian, Δ, if ΔK is the identity with respect to the convolution operation.*

We caution that the above definition is not mathematically precise because we made no mention on what the "function" K could be and its differentiability, even its existence is under question. We shall just take it as a informal definition.

We note the necessry condition for any K to be a fundamental solution. Observe that K is such that $f * \Delta K$ for all f in the given space of functions in $\mathbb{R}^n$. In particular, one can choose $f \equiv 1$. Thus, the necessary condition for a fundamental solution is $1 * \Delta K = 1$, i.e.,

$$\int_{\mathbb{R}^n} \Delta K(x)\, dx = 1.$$

Equivalently, the necessary condition for K is

$$\lim_{r\to\infty} \int_{B_r(0)} \Delta K(x)\, dx = 1,$$

which by Gauss divergence theorem (all informally) means

$$\lim_{r\to\infty} \int_{S_r(0)} \nabla K(y)\cdot \nu(y)\, d\sigma_y = 1.$$

6.8.1 Fundamental Solution of Laplacian

The invariance of Laplacian under rotation motivates us to look for a radial fundamental solution. Recall how Laplacian treats radial functions (cf. Proposition 6.1.1) and, consequently, we have

Corollary 6.8.2. *The function $u(x) = ax + b$ solves $\Delta u = 0$ in $\mathbb{R}$. For $n \geq 2$, if u is a radial function on $\mathbb{R}^n$ then $\Delta u = 0$ on $\mathbb{R}^n \setminus \{0\}$ iff*

$$u(x) = \begin{cases} a + b \ln |x| & \text{if } n = 2, \\ a + \frac{b}{2-n}|x|^{2-n} & \text{if } n \geq 3 \end{cases}$$

where a, b are some constants.

Proof. For radial functions $u(x) = v(r)$ where $r = |x|$. Observe that $\Delta u(x) = 0$ iff $v''(r) + \frac{(n-1)}{r}v'(r) = 0$. Now, integrating both sides w.r.t r, we get

$$\begin{aligned} \frac{v''(r)}{v'(r)} &= \frac{(1-n)}{r} \\ \ln v'(r) &= (1-n)\ln r + \ln b \\ v'(r) &= br^{(1-n)} \end{aligned}$$

Integration both sides, once again, yields

$$v(r) = \begin{cases} b \ln r + a & \text{if } n = 2 \\ \frac{b}{2-n} r^{2-n} + a & \text{if } n \neq 2. \end{cases}$$

□

The reason to choose the domain of the Laplacian as $\mathbb{R}^n \setminus \{0\}$ is because the operator involves a 'r' in the denominator. However, for one dimensional case we can let zero to be on the domain of Laplacian, since for $n = 1$, the Laplace operator is unchanged. Thus, for $n = 1$, $u(x) = a + bx$ is a harmonic function in $\mathbb{R}^n$.

Note that as $r \to 0$, $v(r) \to \infty$. Thus, u has a singularity at 0. In fact, for any given vector $x_0 \in \mathbb{R}^n$, $\Delta u(x - x_0) = 0$ for all $x \in \mathbb{R}^n \setminus \{x_0\}$. We shall choose a, b such that for every sphere $S_r(0)$ about the origin, we have

$$\int_{S_r(0)} v'(r)\, d\sigma = 1.$$

Thus,

$$1 = \int_{S_r(0)} v'(r)\, d\sigma \quad = \begin{cases} \frac{b}{r}(2\pi r) & \text{for } n = 2 \\ br^{1-n}(r^{n-1}\omega_n) & \text{for } n \geq 3. \end{cases}$$

This is possible only for the choice

$$b = \begin{cases} \frac{1}{2\pi} & \text{for } n = 2 \\ \frac{1}{\omega_n} & \text{for } n \geq 3. \end{cases}$$

The constant a can be chosen arbitrarily, but to keep things simple, we choose $a \equiv 0$ for $n \geq 2$. For convention sake, we shall add minus ("$-$") sign (notice the minus sign in (6.8.1)).

Definition 6.8.3. *For any fixed $x_0 \in \mathbb{R}^n$ We say $K(x_0, x)$, defined as*

$$K(x_0, x) := \begin{cases} -\frac{1}{2\pi} \ln |x - x_0| & (n = 2) \\ \frac{|x-x_0|^{2-n}}{\omega_n(n-2)} & (n \geq 3), \end{cases}$$

is the fundamental solution *of Δ at any given $x_0 \in \mathbb{R}^n$.*

We end this section by emphasising that the notion of fundamental solution has a precise definition in terms of the Dirac measure. The Dirac measure, at a point $x \in \mathbb{R}^n$, is defined as,

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

for all measurable subsets E of the measure space $\mathbb{R}^n$. The Dirac measure has the property that

$$\int_E d\delta_x = 1$$

if $x \in E$ and zero if $x \notin E$. Also, for any integrable function f,

$$\int_{\mathbb{R}^n} f(y)\, d\delta_x = f(x).$$

In this new set-up a fundamental solution $K(x_0, \cdot)$ can be defined as the solution corresponding to δ_{x_0}, i.e.,

$$-\Delta K(x_0, x) = \delta_{x_0} \text{ in } \mathbb{R}^n.$$

Note that the above equation, as such, makes no sense because the RHS is a set-function taking subsets of $\mathbb{R}^n$ as arguments, whereas K is a function on $\mathbb{R}^n$. To give meaning to above equation, one needs to view δ_x as a distribution (introduced by L. Schwartz) and the equation should be interpreted in the distributional derivative sense. The Dirac measure is the distributional limit of the sequence of mollifiers, ρ_ε, in the space of distributions.

6.8.2 Existence and Uniqueness

In this section, we shall give a formula for the solution of the Poisson equation (6.8.1) in $\mathbb{R}^n$ in terms of the fundamental solution.

Theorem 6.8.4. *For any given $f \in C_c^2(\mathbb{R}^n)$, $u := K * f$ is a solution to the Poisson equation (6.8.1).*

Proof. By the property of convolution (cf. proof of Theorem F.0.10), we know that $D^\alpha u(x) = (K * D^\alpha f)(x)$ for all $|\alpha| \leq 2$. Since $f \in C_c^2(\mathbb{R}^n)$, we have $u \in C^2(\mathbb{R}^n)$. The difficulty arises due to the singularity of K at the origin. Consider, for any fixed $m > 0$,

$$\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^n} K(y)\Delta_x f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B_m(0)} K(y)\Delta_x f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B_m(0)} K(y)\Delta_y f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot\nu\,d\sigma_y \\
&\quad -\int_{\mathbb{R}^n\setminus B_m(0)} \nabla_y K(y)\cdot\nabla_y f(x-y)\,dy \quad \text{(cf. Corollary D.0.7)} \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot\nu\,d\sigma_y \\
&\quad +\int_{\mathbb{R}^n\setminus B_m(0)} \Delta_y K(y) f(x-y)\,dy \\
&\quad -\int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot\nu\,d\sigma_y \quad \text{(cf. Corollary D.0.7)} \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot\nu\,d\sigma_y \\
&\quad -\int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot\nu\,d\sigma_y \\
&:= I_m(x) + J_m(x) + K_m(x).
\end{aligned}$$

But, due to the compact support of f, we have

$$|I_m(x)| \le \|D^2 f\|_{\infty,\mathbb{R}^n} \int_{B_m(0)} |K(y)|\,dy.$$

Thus, for $n = 2$,

$$|I_m(x)| \le \frac{m^2}{2}\left(\frac{1}{2} + |\ln m|\right)\|D^2 f\|_{\infty,\mathbb{R}^n}$$

and for $n \ge 3$, we have

$$|I_m(x)| \le \frac{m^2}{2(n-2)}\|D^2 f\|_{\infty,\mathbb{R}^n}.$$

Hence, as $m \to 0$, $|I_m(x)| \to 0$. Similarly,

$$\begin{aligned} |J_m(x)| &\leq \int_{S_m(0)} |K(y) \nabla_y f(x-y) \cdot \nu| \, d\sigma_y \\ &\leq \|\nabla f\|_{\infty, \mathbb{R}^n} \int_{S_m(0)} |K(y)| \, d\sigma_y. \end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq m |\ln m| \|\nabla f\|_{\infty, \mathbb{R}^n}$$

and for $n \geq 3$, we have

$$|J_m(x)| \leq \frac{m}{(n-2)} \|\nabla f\|_{\infty, \mathbb{R}^n}.$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Now, to tackle the last term $K_m(x)$, we note that a simple computation yields that $\nabla_y K(y) = \frac{-1}{\omega_n |y|^n} y$. Since we are in the m radius sphere $|y| = m$. Also the unit vector ν outside of $S_m(0)$, as a boundary of $\mathbb{R}^n \setminus B_m(0)$, is given by $-y/|y| = -y/m$. Therefore,

$$\nabla_y K(y) \cdot \nu = \frac{1}{\omega_n m^{n+1}} y \cdot y = \frac{1}{\omega_n m^{n-1}}.$$

$$\begin{aligned} K_m(x) &= -\int_{S_m(0)} f(x-y) \nabla_y K(y) \cdot \nu \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}} \int_{S_m(0)} f(x-y) \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}} \int_{S_m(x)} f(y) \, d\sigma_y \end{aligned}$$

Since f is continuous, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\begin{aligned} |K_m(x) - (-f(x))| &= \left| f(x) - \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} f(y) \, d\sigma_y \right| \\ &= \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} |f(x) - f(y)| \, d\sigma_y < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to -f(x)$. Hence, u solves (6.8.1). □

Remark 6.8.5. Notice that in the proof above, we have used the Green's identity eventhough our domain is not bounded (which is a hypothesis for Green's identity). This can be justified by taking a ball bigger than $B_m(0)$ and working in the annular region, and later letting the bigger ball approach all of $\mathbb{R}^n$.

A natural question at this juncture is: Is every solution of the Poisson equation (6.8.1) of the form $K * f$. We answer this question in the following theorem.

Theorem 6.8.6. *Let $f \in C_c^2(\mathbb{R}^n)$ and $n \geq 3$. If u is a solution of (6.8.1) and u is bounded, then u has the form $u(x) = (K*f)(x)+C$, for any $x \in \mathbb{R}^n$, where C is some constant.*

Proof. We know that (cf. Theorem 6.8.4) $u'(x) := (K * f)(x)$ solves (6.8.1), the Poisson equation in $\mathbb{R}^n$. Moreover, u' is bounded for $n \geq 3$, since $K(x) \to 0$ as $|x| \to \infty$ and f has compact support in $\mathbb{R}^n$. Also, since u is given to be a bounded solution of (6.8.1), $v := u - u'$ is a bounded harmonic function. Hence, by Liouville's theorem, v is constant. Therefore $u = u' + C$, for some constant C. □

We turn our attention to studying Poisson equation in proper subsets of $\mathbb{R}^n$. Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$.

Theorem 6.8.7 (Uniqueness)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$. For the Poisson equation $\Delta u = f$ with one of Dirichlet, Robin or Mixed conditions on $\partial\Omega$, there exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. In the Neumann problem two solutions differ by a constant.*

Proof. Let u and v be solutions of the Poisson equation with same boundary conditions on $\partial\Omega$. Then $w := u - v$ is a harmonic function, $\Delta w = 0$, with homogeneous boundary condition on $\partial\Omega$. By Green's identity D.0.7, we have

$$\int_\Omega |\nabla w|^2 \, dx = \int_{\partial\Omega} w(\nabla w \cdot \nu) \, d\sigma.$$

For the Drichlet, Neumann and Mixed case, the RHS is zero. For the Robin condition the RHS is negative,

$$\int_{\partial\Omega} w(\nabla w \cdot \nu) \, d\sigma = -c \int_{\partial\Omega} w^2 \, d\sigma \leq 0.$$

Thus, in all the four boundary conditions

$$\int_\Omega |\nabla w|^2 \, dx \leq 0$$

and $\nabla w = 0$. Therefore, $w = u - v$ is constant in the connected components of Ω. In the case of Dirichlet, mixed and Robin the constant has to be zero, by Maximum principle[5]. Thus, $u = v$ in these three cases. □

To begin with we shall focus on the study of Dirichlet problem. The Dirichlet problem is stated as follows: Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{on } \partial\Omega \end{cases} \tag{6.8.2}$$

Lemma 6.8.8. *Let f be bounded and locally Hölder continuous[6] with exponent $\gamma \leq 1$ in Ω. Then $u := K * f \in C^2(\Omega)$, $-\Delta u = f$ in Ω.*

Theorem 6.8.9 (Existence)**.** *Let Ω be a bounded domain with all boundary points being regular w.r.t Laplacian. The classical Dirichlet problem (6.8.2) is solvable (hence uniquely) for any bounded, locally Hölder continuous function f in Ω and continuous function g on $\partial\Omega$.*

Proof. Recall that K is a fundamental solution of $-\Delta$. Set $w(x) := f * K$ in $\mathbb{R}^n$ then $-\Delta w = f$. Set $v = u - w$. Then (6.8.2) is solvable iff

$$\begin{cases} -\Delta v &= 0 \quad \text{in } \Omega \\ \quad v &= g - w \quad \text{on } \partial\Omega \end{cases}$$

is solvable. The equation for v is solvable by Theorem 6.6.3. □

6.8.3 Green's Function

We shall now attempt to solve the Poisson equation on a proper open subset Ω of $\mathbb{R}^n$. This is done via the Green's function. We begin by motivating the Green's function. For any $x \in \Omega$, choose $m > 0$ such that $B_m(x) \subset \Omega$. Set $\Omega_m := \Omega \setminus B_m(x)$. Now applying the second identity of Corollary D.0.7 for

[5] or, simply, from the fact that a non-zero c will contradict the continuous extension of w to boundary.

[6] Hölder continuous in each compact subset of Ω

any $u \in C^2(\overline{\Omega})$ and $v_x(y) = K(y-x)$, the fundamental solution on $\mathbb{R}^n \setminus \{x\}$, on the domain Ω_m, we get

$$\begin{aligned}
\int_{\Omega_m} u(y)\Delta_y v_x(y)\,dy & \\
-\int_{\Omega_m} v_x(y)\Delta_y u(y)\,dy &= \int_{\partial\Omega_m}\left(u(y)\frac{\partial v_x}{\partial\nu}(y) - v_x(y)\frac{\partial u(y)}{\partial\nu}\right)d\sigma_y \\
-\int_{\Omega_m} v_x(y)\Delta_y u(y)\,dy &= \int_{\partial\Omega_m}\left(u(y)\frac{\partial v_x}{\partial\nu}(y) - v_x(y)\frac{\partial u(y)}{\partial\nu}\right)d\sigma_y \\
-\int_{\Omega} + \int_{B_m(x)} &= \int_{\partial\Omega} + \int_{S_m(x)} \\
\int_{B_m(x)} v_x(y)\Delta_y u(y)\,dy & \\
-\int_{S_m(x)} u(y)\frac{\partial v_x}{\partial\nu}(y)\,d\sigma_y & \\
+\int_{S_m(x)} v_x(y)\frac{\partial u(y)}{\partial\nu}\,d\sigma_y &= \int_{\partial\Omega}\left(u(y)\frac{\partial v_x}{\partial\nu}(y) - v_x(y)\frac{\partial u(y)}{\partial\nu}\right)d\sigma_y \\
&\quad + \int_{\Omega} v_x(y)\Delta_y u(y)\,dy \\
I_m(x) + K_m(x) + J_m(x) &= \int_{\partial\Omega}\left(u(y)\frac{\partial K}{\partial\nu}(y-x) - K(y-x)\frac{\partial u(y)}{\partial\nu}\right)d\sigma_y \\
&\quad + \int_{\Omega} K(y-x)\Delta_y u(y)\,dy
\end{aligned}$$

The LHS is handled exactly as in the proof of Theorem 6.8.4, since u is a continuous function on the compact set $\overline{\Omega}$ and is bounded. We repeat the arguments below for completeness sake. Consider the term I_m.

$$|I_m(x)| \leq \|D^2u\|_{\infty,\Omega}\int_{B_m(x)} |K(y-x)|\,dy.$$

Thus,

$$|I_m(x)| \leq \begin{cases} \frac{m^2}{2}\left(\frac{1}{2} + |\ln m|\right)\|D^2u\|_{\infty,\Omega} & \text{for } n = 2 \\ \frac{m^2}{2(n-2)}\|D^2u\|_{\infty,\Omega} & \text{for } n \geq 3. \end{cases}$$

Hence, as $m \to 0$, $|I_m(x)| \to 0$. Next, consider the term $K_m(x)$. Note that $\nabla_y K(y-x) = \frac{-1}{\omega_n|y-x|^n}(y-x)$. Since we are in the m radius sphere $|y-x| = m$.

Also the unit vector ν inside of $S_m(x)$, as a boundary of $\Omega \setminus B_m(x)$, is given by $-(y-x)/|y-x| = -(y-x)/m$. Therefore,

$$\nabla_y K(y-x) \cdot \nu = \frac{1}{\omega_n m^{n+1}}(y-x)\cdot(y-x) = \frac{1}{\omega_n m^{n-1}}.$$

Thus,

$$\begin{aligned} K_m(x) &= -\int_{S_m(x)} u(y)\nabla_y K(y-x)\cdot\nu \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}}\int_{S_m(x)} u(y)\, d\sigma_y \end{aligned}$$

Since u is continuous, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|u(x)-u(y)| < \varepsilon$ whenever $|x-y| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\begin{aligned} |K_m(x) - (-u(x))| &= \left| u(x) - \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} u(y)\, d\sigma_y \right| \\ &= \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} |u(x)-u(y)|\, d\sigma_y < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to -u(x)$. Finally, we consider the term $J_m(x)$,

$$\begin{aligned} |J_m(x)| &\leq \int_{S_m(x)} |K(y-x)\nabla_y u(y)\cdot\nu|\, d\sigma_y \\ &\leq \|\nabla_y u\|_{\infty,\Omega}\int_{S_m(x)} |K(y-x)|\, d\sigma_y. \end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq \begin{cases} m|\ln m| \|\nabla_y u\|_{\infty,\Omega} & \text{for } n = 2 \\ |J_m(x)| \leq \frac{m}{(n-2)}\|\nabla_y u\|_{\infty,\Omega} & \text{for } n \geq 3. \end{cases}$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Therefore, letting $m \to 0$, we have the identity

$$u(x) = \int_{\partial\Omega}\left(K(y-x)\frac{\partial u(y)}{\partial\nu} - u(y)\frac{\partial K}{\partial\nu}(y-x)\right) d\sigma_y - \int_{\Omega} K(y-x)\Delta_y u(y)\, dy \tag{6.8.3}$$

For the Dirichlet problem, Δu is known in Ω and u is known on $\partial\Omega$. Thus, (6.8.3) gives an expression for the solution u, provided we know the normal derivative $\frac{\partial u(y)}{\partial \nu}$ along $\partial\Omega$. But this quantity is usually an unknown for Dirichlet problem. Thus, we wish to rewrite (6.8.3) such that the knowledge of the normal derivative is not necessary. To do so, we introduce a function $\psi_x(y)$, for a fixed $x \in \Omega$, as the solution of the boundary-value problem,

$$\left\{ \begin{aligned} \Delta\psi_x(y) &= 0 \text{ in } \Omega \\ \psi_x(y) &= K(y-x) \text{ on } \partial\Omega. \end{aligned} \right. \tag{6.8.4}$$

Now applying the second identity of Corollary D.0.7 for any $u \in C^2(\overline{\Omega})$ and $v(y) = \psi_x(y)$, we get

$$\int_{\partial\Omega} \left(u \frac{\partial \psi_x}{\partial \nu} - \psi_x \frac{\partial u}{\partial \nu} \right) d\sigma_y = \int_{\Omega} \left(u \Delta_y \psi_x - \psi_x \Delta_y u \right) dy.$$

Therefore, substituting the following identity

$$\int_{\partial\Omega} K(y-x) \frac{\partial u(y)}{\partial \nu} \, d\sigma_y = \int_{\Omega} \psi_x(y) \Delta_y u(y) \, dy + \int_{\partial\Omega} u(y) \frac{\partial \psi_x(y)}{\partial \nu} \, d\sigma_y$$

in (6.8.3), we get

$$u(x) = \int_{\Omega} \left(\psi_x(y) - K(y-x) \right) \Delta_y u \, dy + \int_{\partial\Omega} u \nabla \left(\psi_x(y) - K(y-x) \right) \cdot \nu \, d\sigma_y.$$

The identity above motivates the definition of what is called the *Green's function*.

Definition 6.8.10. *For any given open subset $\Omega \subset \mathbb{R}^n$ and $x, y \in \Omega$ such that $x \neq y$, we define the Green's function as*

$$G(x,y) := \psi_x(y) - K(y-x).$$

Rewriting (6.8.3) in terms of Green's function,we get

$$u(x) = \int_{\Omega} G(x,y) \Delta_y u(y) \, dy + \int_{\partial\Omega} u(y) \frac{\partial G(x,y)}{\partial \nu} \, d\sigma_y.$$

Thus, in the arguments above we have proved the following theorem.

Theorem 6.8.11. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Also, given $f \in C(\Omega)$ and $g \in C(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ solves the Dirichlet problem (6.8.2), then u has the representation*

$$u(x) = -\int_{\Omega} G(x,y) f(y)\, dy + \int_{\partial\Omega} g(y) \frac{\partial G(x,y)}{\partial \nu}\, d\sigma_y. \tag{6.8.5}$$

Observe that we have solved the Dirichlet problem (6.8.2) provided we know the Green's function. The construction of Green's function depends on the construction of ψ_x for every $x \in \Omega$. In other words, (6.8.2) is solved if we can solve (6.8.4). Ironically, computing ψ_x is usually possible when Ω has simple geometry. We shall identify two simple cases of Ω, half-space and balll, where we can explicitly compute G.

The Green's function is the analogue of the fundamental solution K for the boundary value problem. This is clear by observing that, for a fixed $x \in \Omega$, G satisfies (informally) the equation,

$$\begin{cases} -\Delta G(x,\cdot) &= \delta_x \text{ in } \Omega \\ G(x,\cdot) &= 0 \text{ on } \partial\Omega, \end{cases}$$

where δ_x is the Dirac measure at x.

Theorem 6.8.12. *For all $x, y \in \Omega$ such that $x \neq y$, we have $G(x,y) = G(y,x)$, i.e., G is symmetric in x and y.*

Proof. Let us fix $x, y \in \Omega$. For a fixed $m > 0$, set $\Omega_m = \Omega \setminus (B_m(x) \cup B_m(y))$ and applying Green's identity for $v(\cdot) := G(x,\cdot)$ and $w(\cdot) := G(y,\cdot)$, we get

$$\begin{aligned}
\int_{\partial\Omega_m} \left(v(z)\frac{\partial w(z)}{\partial\nu} - w(z)\frac{\partial v(z)}{\partial\nu} \right) d\sigma_z &= \int_{\Omega_m} v(z)\Delta_z w(z)\, dz \\
&\quad - \int_{\Omega_m} w(z)\Delta_z v(z)\, dz \\
\int_{\partial\Omega_m} \left(v(z)\frac{\partial w(z)}{\partial\nu} - w(z)\frac{\partial v(z)}{\partial\nu} \right) d\sigma_z &= 0 \\
\int_{S_m(x)} \left(v\frac{\partial w}{\partial\nu} - w\frac{\partial v}{\partial\nu} \right) d\sigma_z &= \int_{S_m(y)} \left(w\frac{\partial v}{\partial\nu} - v\frac{\partial w}{\partial\nu} \right) d\sigma_z \\
J_m(x) + K_m(x) &= J_m(y) + K_m(y).
\end{aligned}$$

$$\begin{aligned} |J_m(x)| &\leq \int_{S_m(x)} |v(z)\nabla_z w(z)\cdot\nu|\,d\sigma_z \\ &\leq \|\nabla w\|_{\infty,\Omega}\int_{S_m(x)} |v(z)|\,d\sigma_z \\ &= \|\nabla w\|_{\infty,\Omega}\int_{S_m(x)} |\psi_x(z)-K(z-x)|\,d\sigma_z. \end{aligned}$$

Thus, for $n=2$,

$$|J_m(x)| \leq (2\pi m\|\psi_x\|_{\infty,\Omega} + m|\ln m|)\,\|\nabla w\|_{\infty,\Omega}$$

and for $n\geq 3$, we have

$$|J_m(x)| \leq \left(\omega_n m^{n-1}\|\psi_x\|_{\infty,\Omega} + \frac{m}{(n-2)}\right)\|\nabla w\|_{\infty,\Omega}.$$

Hence, as $m\to 0$, $|J_m(x)|\to 0$. Now, consider the term $K_m(x)$,

$$\begin{aligned} K_m(x) &= -\int_{S_m(x)} w(z)\frac{\partial v(z)}{\partial\nu}\,d\sigma_z \\ &= \int_{S_m(x)} w(z)\frac{\partial K}{\partial\nu}(z-x)\,d\sigma_z - \int_{S_m(x)} w(z)\frac{\partial\psi_x(z)}{\partial\nu}\,d\sigma_z. \end{aligned}$$

The second term goes to zero by taking the sup-norm outside the integral. To tackle the first term, we note that $\nabla_z K(z-x) = \frac{-1}{\omega_n|z-x|^n}(z-x)$. Since we are in the m radius sphere $|z-x|=m$. Also the unit vector ν outside of $S_m(x)$, as a boundary of $\Omega\setminus B_m(x)$, is given by $-(z-x)/|z-x| = -(z-x)/m$. Therefore,

$$\nabla_z K(z-x)\cdot\nu = \frac{1}{\omega_n m^{n+1}}(z-x)\cdot(z-x) = \frac{1}{\omega_n m^{n-1}}.$$

$$\int_{S_m(x)} w(z)\nabla_z K(z-x)\cdot\nu\,d\sigma_z = \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} w(z)\,d\sigma_z$$

Since w is continuous in $\Omega\setminus\{y\}$, for every $\varepsilon>0$, there is a $\delta>0$ such that $|w(z)-w(x)|<\varepsilon$ whenever $|x-z|<\delta$. When $m\to 0$, we can choose m such that $m<\delta$ and for this m, we see that Now, consider

$$\left|\frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} w(z)\,d\sigma_z - w(x)\right| = \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} |w(z)-w(x)|\,d\sigma_z < \varepsilon.$$

Thus, as $m \to 0$, $K_m(x) \to w(x)$. Arguing similarly, for $J_m(y)$ and $K_m(y)$, we get $G(y,x) = G(x,y)$. □

6.8.4 Green's Function for half-space

In this section, we shall compute explicitly the Green's function for positive half-space. Thus, we shall have

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

and

$$\partial\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n \mid x_n = 0\}.$$

To compute the Green's function, we shall use the *method of reflection*. The reflection technique ensures that the points on the boundary (along which the reflection is done) remains unchanged to respect the imposed Dirichlet condition.

Definition 6.8.13. *For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, we define its* reflection *along its boundary $\mathbb{R}^{n-1}$ as $x^\star = (x_1, x_2, \ldots, -x_n)$.*

It is obvious from the above definition that, for any $y \in \partial\mathbb{R}^n_+$, $|y - x^\star| = |y - x|$. Given a fixed $x \in \mathbb{R}^n_+$, we need to find a harmonic function ψ_x in $\mathbb{R}^n_+$, as in (6.8.4). But $K(\cdot - x)$ is harmonic in $\mathbb{R}^n_+ \setminus \{x\}$. Thus, we use the method of reflection to shift the singularity of K from $\mathbb{R}^n_+$ to the negative half-space and define

$$\psi_x(y) = K(y - x^\star).$$

By definition, ψ_x is harmonic in $\mathbb{R}^n_+$ and on the boundary $\psi_x(y) = K(y - x)$. Therefore, we define the Green's function to be $G(x,y) = K(y - x^\star) - K(y - x)$, for all $x, y \in \mathbb{R}^n_+$ and $x \neq y$. It now only remains to compute the normal derivative of G. Recall that $\nabla_y K(y - x) = \frac{-1}{\omega_n |y-x|^n}(y - x)$. Thus,

$$\nabla_y G(x,y) = \frac{-1}{\omega_n}\left(\frac{y - x^\star}{|y - x^\star|^n} - \frac{y - x}{|y - x|^n}\right)$$

Therefore, when $y \in \partial\mathbb{R}^n_+$, we have

$$\nabla_y G(x,y) = \frac{-1}{\omega_n |y - x|^n}(x - x^\star).$$

Since the outward unit normal of $\partial\mathbb{R}^n_+$ is $\nu = (0, 0, \ldots, 0, -1)$, we get

$$\nabla_y G(x, y) \cdot \nu = \frac{2x_n}{\omega_n |y - x|^n}.$$

Definition 6.8.14. *For all $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$, the map*

$$P(x, y) := \frac{2x_n}{\omega_n |y - x|^n}$$

is called the Poisson kernel *for $\mathbb{R}^n_+$.*

Now substituing for G in (6.8.5), we get the *Poisson formula* for u,

$$u(x) = \int_{\mathbb{R}^n_+} [K(y - x) - K(y - x^\star)] f(y)\, dy + \frac{2x_n}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|y - x|^n}\, d\sigma_y. \quad (6.8.6)$$

It now remains to show that the u as defined above is, indeed, a solution of (6.8.2) for $\mathbb{R}^n_+$.

Exercise 30. Let $f \in C(\mathbb{R}^n_+)$ be given. Let $g \in C(\mathbb{R}^{n-1})$ be bounded. Then u as given in (6.8.6) is in $C^2(\mathbb{R}^n_+)$ and solves (6.8.2).

6.8.5 Green's Function for a disk

In this section, we shall compute explicitly the Green's function for a ball of radius $r > 0$ and centred at $a \in \mathbb{R}^n$, $B_r(a)$. As usual, we denote the surface of the disk as $S_r(a)$, the circle of radius r centred at a. We, once again, use the *method of reflection* but, this time reflected along the boundary of the disk.

Definition 6.8.15. *For any $x \in \mathbb{R}^n \setminus \{a\}$, we define its* reflection *along the circle $S_r(a)$ as $x^\star = \frac{r^2(x-a)}{|x-a|^2} + a$.*

The idea behind reflection is clear for the unit disk, i.e., when $a = 0$ and $r = 1$, as $x^\star = \frac{x}{|x|^2}$. The above definition is just the shift of origin to a and dilating the unit disk by r.

Now, for any $y \in S_r(a)$ and $x \neq a$, consider

$$
\begin{aligned}
|y - x^\star|^2 &= |y-a|^2 - 2(y-a)\cdot(x^\star - a) + |x^\star - a|^2 \\
&= r^2 - 2r^2(y-a)\cdot\left(\frac{x-a}{|x-a|^2}\right) + \left|\frac{r^2(x-a)}{|x-a|^2}\right|^2 \\
&= \frac{r^2}{|x-a|^2}(|x-a|^2 - 2(y-a)\cdot(x-a) + r^2) \\
&= \frac{r^2}{|x-a|^2}(|x-a|^2 - 2(y-a)\cdot(x-a) + |y-a|^2) \\
&= \frac{r^2}{|x-a|^2}|y-x|^2
\end{aligned}
$$

Therefore, $\frac{|x-a|}{r}|y-x^\star| = |y-x|$ for all $y \in S_r(a)$. For each fixed $x \in B_r(a)$, we need to find a harmonic function ψ_x in $B_r(a)$ solving (6.8.4). Since $K(\cdot - x)$ is harmonic in $B_r(a) \setminus \{x\}$, we use the method of reflection to shift the singularity of K at x to the complement of $B_r(a)$. Thus, we define

$$\psi_x(y) = K\left(\frac{|x-a|}{r}(y - x^\star)\right) \quad x \neq a.$$

For $n \geq 3$, $K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{|x-a|^{2-n}}{r^{2-n}}K(y-x^\star)$. Thus, for $n \geq 3$, ψ_x solves (6.8.4), for $x \neq a$. For $n = 2$,

$$K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{-1}{2\pi}\ln\left(\frac{|x-a|}{r}\right) + K(y-x^\star).$$

Hence ψ_x solves (6.8.4) for $n = 2$. Note that we are yet to identify a harmonic function ψ_a corresponding to $x = a$. We do this by setting ψ_a to be the constant function

$$\psi_a(y) := \begin{cases} -\frac{1}{2\pi}\ln r & (n=2) \\ \frac{r^{2-n}}{\omega_n(n-2)} & (n \geq 3). \end{cases}$$

Thus, ψ_a is harmonic and solves (6.8.4) for $x = a$. Therefore, we define the Green's function to be

$$G(x,y) := K\left(\frac{|x-a|}{r}(y-x^\star)\right) - K(y-x) \quad \forall x, y \in B_r(a), x \neq a \text{ and } x \neq y$$

and

$$G(a,y) := \begin{cases} -\frac{1}{2\pi}\ln\left(\frac{r}{|y-a|}\right) & (n=2) \\ \frac{1}{\omega_n(n-2)}\left(r^{2-n}-|y-a|^{2-n}\right) & (n\geq 3). \end{cases}$$

We shall now compute the normal derivative of G. Recall that

$$\nabla_y K(y-x) = \frac{-1}{\omega_n|y-x|^n}(y-x)$$

and one can compute $\nabla_y K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{-|x-a|^{2-n}}{r^{2-n}\omega_n|y-x^\star|^n}(y-x^\star)$. Therefore,

$$\nabla_y G(x,y) = \frac{-1}{\omega_n}\left[\frac{|x-a|^{2-n}(y-x^\star)}{r^{2-n}|y-x^\star|^n} - \frac{y-x}{|y-x|^n}\right].$$

If $y \in S_r(a)$, we have

$$\begin{aligned} \nabla_y G(x,y) &= \frac{-1}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2}(y-x^\star)-(y-x)\right] \\ &= \frac{-1}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2}-1\right](y-a) \end{aligned}$$

Since the outward unit normal at any point $y \in S_r(a)$ is $\frac{1}{r}(y-a)$, we have

$$\begin{aligned} \nabla_y G(x,y)\cdot\nu &= \frac{-1}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2}-1\right]\sum_{i=1}^n \frac{1}{r}(y_i-a_i)^2 \\ &= \frac{-r}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2}-1\right]. \end{aligned}$$

Definition 6.8.16. *For all $x \in B_r(a)$ and $y \in S_r(a)$, the map*

$$P(x,y) := \frac{r^2-|x-a|^2}{r\omega_n|y-x|^n}$$

is called the Poisson kernel *for $B_r(a)$.*

Now substituing for G in (6.8.5), we get the *Poisson formula* for u,

$$u(x) = -\int_{B_r(a)} G(x,y)f(y)\,dy + \frac{r^2-|x-a|^2}{r\omega_n}\int_{S_r(a)}\frac{g(y)}{|y-x|^n}\,d\sigma_y. \quad (6.8.7)$$

It now remains to show that the u as defined above is, indeed, a solution of (6.8.2) for $B_r(a)$.

Exercise 31. Let $f \in C(B_r(a))$ be given. Let $g \in C(S_r(a))$ be bounded. Then u as given in (6.8.7) is in $C^2(B_r(a))$ and solves (6.8.2).

6.8.6 Conformal Mapping and Green's Function

In two dimensions, the Green's function has a nice connection with conformal mapping. Let $w = f(z)$ be a conformal mapping from an open domain (connected) $\Omega \subset \mathbb{R}^2$ onto the interior of the unit circle. The Green's function of Ω is

$$G(z, z_0) = \frac{1}{2\pi} \ln \left| \frac{1 - f(z)\overline{f(z_0)}}{f(z) - f(z_0)} \right|$$

where $z = x_1 + ix_2$ and $z_0 = y_1 + iy_2$.

6.8.7 Dirichlet Principle

The Dirichlet principle (formulated, independently by Gauss, Lord Kelvin and Dirichlet) states that the solution of the Dirichlet problem minimizes the corresponding energy functional.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$ and let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given. For convenience, recall the Dirichlet problem ((6.8.2)),

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{on } \partial\Omega. \end{cases}$$

Any solution u of (6.8.2) is in $V = \{v \in C^2(\overline{\Omega}) \mid v = g \text{ on } \partial\Omega\}$. The *energy functional* $J : V \to \mathbb{R}$ is defined as

$$J(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega f v \, dx$$

Theorem 6.8.17 (Dirichlet's principle)**.** *A $C^2(\overline{\Omega})$ function u solves* (6.8.2) *iff u minimises the functional J on V, i.e.,*

$$J(u) \le J(v) \quad \forall v \in V.$$

Proof. Let $u \in C^2(\overline{\Omega})$ be a solution of (6.8.2). For any $v \in V$, we multiply both sides of (6.8.2) by $u - v$ and integrating we get,

$$
\begin{aligned}
\int_\Omega (-\Delta u)(u-v)\,dx &= \int_\Omega f(u-v)\,dx \\
\int_\Omega \nabla u \cdot \nabla(u-v)\,dx &= \int_\Omega f(u-v)\,dx \\
\int_\Omega \left(|\nabla u|^2 - fu\right)\,dx &= \int_\Omega (\nabla u \cdot \nabla v - fv)\,dx \\
&\leq \int_\Omega |\nabla u \cdot \nabla v| - \int_\Omega fv\,dx \\
&\leq \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\,dx - \int_\Omega fv\,dx \\
&\quad (\text{since } 2ab \leq a^2 + b^2\ ) \\
J(u) &\leq J(v).
\end{aligned}
$$

Thus, u minimises J in V. Conversely, let u minimise J in V. Thus,

$$
\begin{aligned}
J(u) &\leq J(v) \quad \forall v \in V \\
J(u) &\leq J(u + t\phi) \quad (\text{for any } \phi \in C^2(\Omega) \text{ such that } \phi = 0 \text{ on } \partial\Omega) \\
0 &\leq \frac{1}{t}\left(J(u+t\phi) - J(u)\right) \\
0 &\leq \frac{1}{t}\left(\frac{1}{2}\int_\Omega \left(t^2|\nabla\phi|^2 + 2t\nabla\phi\cdot\nabla u\right)\,dx - t\int_\Omega f\phi\,dx\right)
\end{aligned}
$$

Taking limit $t \to 0$ both sides, we get

$$
0 \leq \int_\Omega \nabla\phi \cdot \nabla u\,dx - \int_\Omega f\phi\,dx \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega.
$$

Choosing $-\phi$ in place of ϕ we get the reverse inequality, and we have equality in the above. Thus,

$$
\begin{aligned}
\int_\Omega \nabla u \cdot \nabla\phi\,dx &= \int_\Omega f\phi\,dx \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega \\
\int_\Omega (-\Delta u - f)\phi\,dx &= 0 \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega.
\end{aligned}
$$

Thus u solves (6.8.2). □

6.9 Neumann Boundary Condition

The Neumann problem is stated as follows: Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega \end{cases} \tag{6.9.1}$$

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ and $\nu = (\nu_1, \ldots, \nu_n)$ is the outward pointing unit normal vector field of $\partial\Omega$. Thus, the boundary imposed is called the Neumann boundary condition. The solution of a Neumann problem is not necessarily unique. If u is any solution of (6.9.1), then $u + c$ for any constant c is also a solution of (6.9.1). More generally, for any v such that v is constant on the connected components of Ω, $u + v$ is a solution of (6.9.1).

Appendices

Appendix A

The Gamma Function

The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad \forall x \in (0, \infty).$$

Note that the gamma function is defined as an improper integral and its existence has to be justified. Observe that for a fixed $x > 0$,

$$|e^{-t} t^{x-1}| = e^{-t} t^{x-1} \leq t^{x-1} \quad \forall t > 0,$$

since for $t > 0$, $|e^{-t}| \leq 1$. Now, since $\int_0^1 t^{x-1}\, dt$ exists, we have by comparison test the existence of the integral $\int_0^1 e^{-t} t^{x-1}\, dt$. Now, for $t \to \infty$, $e^{-t} t^{x-1} \to 0$ and hence the is a constant $C > 0$ such that

$$t^{x-1} e^{-t} \leq C/t^2 \quad \forall t \geq 1.$$

Since $\int_1^\infty 1/t^2\, dt$ exists, we again have using comparison test the existence of the integral $\int_1^\infty e^{-t} t^{x-1}\, dt$. In fact, the gamma function can be defined for any complex number $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$.

It is worth noting that the gamma function Γ generalises the notion of factorial of positive integers. This would be the first property we shall prove.

Exercise 32. Show that $\Gamma(x + 1) = x\Gamma(x)$. In particular, for any positive integer n, $\Gamma(n + 1) = n!$. Also, show that $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. Further, for any positive integer n,

$$\Gamma(n + 1/2) = (n - 1/2)(n - 3/2) \dots (1/2)\sqrt{\pi}.$$

(Hint: Use integration by parts and change of variable).

Exercise 33. Show that Γ is continuous on $(0, \infty)$.

Exercise 34. Show that the logarithm of Γ is convex on $(0, \infty)$.

We shall now show that Γ is the only possible generalisation of the notion of factorial of positive integers satisfying the above properties.

Theorem A.0.1. *Let f be positive and continuous on $(0, \infty)$ and let $\log f$ be convex on $(0, \infty)$. Also, let f satisfy the recursive equation*

$$f(x+1) = xf(x) \quad \forall x > 0$$

and $f(1) = 1$, then $f(x) = \Gamma(x)$ for all $x > 0$.

Appendix B

Normal Vector of a Surface

Let $S(x, y, z) = 0$ be the equation of a surface S in $\mathbb{R}^3$. Let us a fix a point $p_0 = (x_0, y_0, z_0) \in S$. We need to find the normal vector at p_0 for the surface S. Let us fix an arbitrary curve C lying on the surface passing through the point p_0. Let the parametrized form of the curve C be given as $r(t) = (x(t), y(t), z(t))$ such that $r(t_0) = p_0$. Since the curve $C \equiv r(t)$ lies on the surface for all t, we have $S(r(t)) = 0$. Thus, $S(x(t), y(t), z(t)) = 0$. Differentiating w.r.t t (using chain rule), we get

$$\begin{aligned} \frac{\partial S}{\partial x}\frac{dx(t)}{dt} + \frac{\partial S}{\partial y}\frac{dy(t)}{dt} + \frac{\partial S}{\partial z}\frac{dz(t)}{dt} &= 0 \\ (S_x, S_y, S_z) \cdot (x'(t), y'(t), z'(t)) &= 0 \\ \nabla S(r(t)) \cdot r'(t) &= 0. \end{aligned}$$

In particular, the above computation is true for the point p_0. Since $r'(t_0)$ is the slope of the tangent at t_0 to the curve C, we see that the vector $\nabla S(p_0)$ is perpendicular to the tangent vector at p_0. Since this argument is true for any curve that passes through p_0. We have that $\nabla S(p_0)$ is normal vector to the tangent plane at p_0. If, in particular, the equation of the surface is given as $S(x, y, z) = u(x, y) - z$, for some $u : \mathbb{R}^2 \to \mathbb{R}$, then

$$\begin{aligned} \nabla S(p_0) &= (S_x(p_0), S_y(p_0), S_z(p_0)) \\ &= (u_x(x_0, y_0), u_y(x_0, y_0), -1) = (\nabla u(x_0, y_0), -1). \end{aligned}$$

Appendix C

Implicit Function Theorem

Theorem C.0.2 (Implicit Function Theorem)**.** *Let* $\Omega \subset \mathbb{R}^m \times \mathbb{R}^n$ *be an open subset and* $f : \Omega \to \mathbb{R}^n$ *such that* f *is continuously differentiable* (C^1) *in* Ω*. Let* $(x_0, y_0) \in \Omega$ *be such that* $f(x_0, y_0) = 0$ *and the* $n \times n$ *matrix*

$$D_y f(x_0, y_0) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial f_1}{\partial y_n}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial f_n}{\partial y_n}(x_0, y_0) \end{pmatrix}$$

is non-singular, then there is a neighbourhood $U \subset \mathbb{R}^m$ *of* x_0 *and a unique function* $g : U \to \mathbb{R}^n$ *such that* $g(x_0) = y_0$ *and, for all* $x \in U$, $f(x, g(x)) = 0$*. Further* g *is continuously differentiable in* U*.*

Appendix D

Divergence Theorem

Definition D.0.3. *For an open set* $\Omega \subset \mathbb{R}^n$ *we say that its boundary* $\partial\Omega$ *is* C^k $(k \geq 1)$*, if for every point* $x \in \partial\Omega$*, there is a* $r > 0$ *and a* C^k *diffeomorphism* $\gamma : B_r(x) \to B_1(0)$ *(i.e.* γ^{-1} *exists and both* γ *and* γ^{-1} *are k-times continuously differentiable) such that*

1. $\gamma(\partial\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ *and*

2. $\gamma(\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n \mid x_n > 0\}$

We say $\partial\Omega$ *is* C^∞ *if* $\partial\Omega$ *is* C^k *for all* $k = 1, 2, \ldots$ *and* $\partial\Omega$ *is analytic if* γ *is analytic.*

Equivalently, a workable definition of C^k boundary would be the following: if for every point $x \in \partial\Omega$, there exists a neighbourhood B_x of x and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B_x = \{x \in B_x \mid x_n > \gamma(x_1, x_2, \ldots, x_{n-1})\}.$$

The divergence of a vector field is the measure of the magnitude (outgoing nature) of all source (of the vector field) and absorption in the region. The divergence theorem was discovered by C. F. Gauss in 1813[1] which relates the outward flow (flux) of a vector field through a closed surface to the behaviour of the vector field inside the surface (sum of all its "source" and "sink"). The divergence theorem is, in fact, the mathematical formulation of the conservation law.

[1] J. L. Lagrange seems to have discovered this, before Gauss, in 1762

Theorem D.0.4. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega \frac{\partial v}{\partial x_i}\,dx = \int_{\partial\Omega} v\nu_i\,d\sigma$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward pointing unit normal vector field and $d\sigma$ is the surface measure of $\partial\Omega$.

The domain Ω need not be bounded provided $|v|$ and $\left|\frac{\partial v}{\partial x_i}\right|$ decays as $|x| \to \infty$. The field of geometric measure theory attempts to identify the precise condition on $\partial\Omega$ and v for which divergence theorem or integration by parts hold.

Corollary D.0.5 (Integration by parts)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $u, v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega u\frac{\partial v}{\partial x_i}\,dx + \int_\Omega v\frac{\partial u}{\partial x_i}\,dx = \int_{\partial\Omega} uv\nu_i\,d\sigma.$$

Theorem D.0.6 (Gauss)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. Given a vector field $V = (v_1, \ldots, v_n)$ on Ω such that $v_i \in C^1(\overline{\Omega})$ for all $1 \le i \le n$, then*

$$\int_\Omega \nabla \cdot V\,dx = \int_{\partial\Omega} V \cdot \nu\,d\sigma. \tag{D.0.1}$$

Corollary D.0.7 (Green's Identities)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. Let $u, v \in C^2(\overline{\Omega})$, then*

(i)

$$\int_\Omega (v\Delta u + \nabla v \cdot \nabla u)\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\,d\sigma,$$

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ and $\Delta := \nabla \cdot \nabla$.

(ii)

$$\int_\Omega (v\Delta u - u\Delta v)\,dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right)\,d\sigma.$$

Proof. Apply divergence theorem to $V = v\nabla u$ to get the first formula. To get second formula apply divergence theorem for both $V = v\nabla u$ and $V = u\nabla v$ and subtract one from the other. □

Appendix E

Surface Area and Volume of Disk in $\mathbb{R}^n$

Theorem E.0.8 (Polar coordinates)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and integrable. Then*

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \left(\int_{S_r(a)} f(y)\,d\sigma_y \right) dr$$

for each $a \in \mathbb{R}^n$. In particular, for each $r > 0$,

$$\frac{d}{dr}\left(\int_{B_r(a)} f(x)\,dx \right) = \int_{S_r(a)} f(y)\,d\sigma_y.$$

Theorem E.0.9. *Prove that*

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx = 1.$$

Further, prove that the surface area ω_n of $S_1(0)$ in $\mathbb{R}^n$ is

$$\frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and the volume of the ball $B_1(0)$ in $\mathbb{R}^n$ is ω_n/n. Consequently, for any $x \in \mathbb{R}^n$ and the $r > 0$, the surface area of $S_r(x)$ is $r^{n-1}\omega_n$ and the volume of $B_r(x)$ is $r^n\omega_n/n$.

Proof. We first observe that

$$e^{-\pi|x|^2} = e^{-\pi\left(\sum_{i=1}^n x_i^2\right)} = \prod_{i=1}^n e^{-\pi x_i^2}.$$

Therefore,

$$\begin{aligned} I_n := \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx &= \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\pi x_i^2}\,dx \\ &= \prod_{i=1}^n \int_{\mathbb{R}} e^{-\pi t^2}\,dt \\ &= \left(\int_{\mathbb{R}} e^{-\pi t^2}\,dt\right)^n = (I_1)^n \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx &= \left(\int_{\mathbb{R}} e^{-\pi t^2}\,dt\right)^{2(n/2)} = \left((I_1)^2\right)^{n/2} = (I_2)^{n/2} \\ &= \left(\int_{\mathbb{R}^2} e^{-\pi|y|^2}\,dy\right)^{n/2} \\ &= \left(\int_0^{2\pi}\int_0^\infty e^{-\pi|y|^2}\,dy\right)^{n/2} \\ &= \left(\int_0^{2\pi}\int_0^\infty e^{-\pi r^2} r\,dr\,d\theta\right)^{n/2} \quad \text{(since jacobian is } r\text{)} \\ &= \left(2\pi\int_0^\infty e^{-\pi r^2} r\,dr\right)^{n/2} \\ &= \left(\pi\int_0^\infty e^{-\pi s}\,ds\right)^{n/2} \quad \text{(by setting } r^2 = s\text{)} \\ &= \left(\int_0^\infty e^{-q}\,dq\right)^{n/2} \quad \text{(by setting } \pi s = q\text{)} \\ &= (\Gamma(1))^{n/2} = 1. \end{aligned}$$

Let ω_n denote the surface area of the unit sphere $S_1(0)$ in $\mathbb{R}^n$, i.e.,

$$\omega_n = \int_{S_1(0)} d\sigma,$$

where $d\sigma$ is the $n-1$-dimensional surface measure. Now, consider

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx \\ &= \int_{S_1(0)}\int_0^\infty e^{-\pi r^2} r^{n-1}\,dr\,d\sigma \\ &= \omega_n \int_0^\infty e^{-\pi r^2} r^{n-1}\,dr \\ &= \frac{\omega_n}{2\pi^{n/2}}\int_0^\infty e^{-s}s^{(n/2)-1}\,ds \quad \text{(by setting } s=\pi r^2\text{)} \\ &= \frac{\omega_n\Gamma(n/2)}{2\pi^{n/2}}. \end{aligned}$$

Thus, $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. We shall now compute the volume of the disk $B_1(0)$. Consider,

$$\int_{B_1(0)} dx = \omega_n \int_0^1 r^{n-1}\,dr = \frac{\omega_n}{n}.$$

For any $x \in \mathbb{R}^n$ and $r > 0$, we observe by the shifting of origin that the surface area of $S_r(x)$ is same as the surface area of $S_r(0)$. Let $S_r(0) = \{s \in \mathbb{R}^n \mid |s| = r\}$. Now

$$\int_{S_r(0)} d\sigma_s = \int_{S_1(0)} r^{n-1}\,d\sigma_t = r^{n-1}\omega_n,$$

where $t = s/r$. Thus, the surface area of $S_r(x)$ is $r^{n-1}\omega_n$. Similarly, volume of a disk $B_r(x)$ is $r^n\omega_n/n$. □

Appendix F

Mollifiers and Convolution

Exercise 35. Show that the Cauchy's exponential function, $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \exp(-x^{-2}) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable, i.e., is in $C^\infty(\mathbb{R})$.

Using the above Cauchy's exponential function, one can construct functions in $C_c^\infty(\mathbb{R}^n)$.

Exercise 36. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, show that $\rho : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\rho(x) = \begin{cases} c\exp(\frac{-1}{1-|x|^2}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

is in $C_c^\infty(\mathbb{R}^n)$ with $\text{supp}(\rho) = B_1(0)$, ball with centre 0 and radius 1, where $c^{-1} = \int_{|x|\leq 1} \exp(\frac{-1}{1-|x|^2})\, dx$.

Thus, one can introduce a sequence of functions in $C_c^\infty(\mathbb{R}^n)$, called *mollifiers*. For $\varepsilon > 0$, we set

$$\rho_\varepsilon(x) = \begin{cases} c\varepsilon^{-n}\exp(\frac{-\varepsilon^2}{\varepsilon^2-|x|^2}) & \text{if } |x| < \varepsilon \\ 0 & \text{if } |x| \geq \varepsilon, \end{cases} \tag{F.0.1}$$

Equivalently, $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$.

Exercise 37. Show that $\rho_\varepsilon \geq 0$ and $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1$ and is in $C_c^\infty(\mathbb{R}^n)$ with support in $B_\varepsilon(0)$.

Let $f, g \in L^1(\mathbb{R}^n)$. Their convolution $f * g$ is defined as, for $x \in \mathbb{R}^n$,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

The integral on RHS is well-defined, since by Fubini's Theorem and the translation invariance of the Lebesgue measure, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)|\,dx\,dy = \int_{\mathbb{R}^n} |g(y)|\,dy \int_{\mathbb{R}^n} |f(x-y)|\,dx = \|g\|_1\|f\|_1 < \infty.$$

Thus, for a fixed x, $f(x-y)g(y) \in L^1(\mathbb{R}^n)$.

Theorem F.0.10. *Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let*

$$\Omega_\varepsilon := \{x \in \Omega \mid dist(x, \partial\Omega) > \varepsilon\}.$$

If $u : \Omega \to \mathbb{R}$ is locally integrable, i.e., for every compact subset $K \subset \Omega$, $\int_K |u| < +\infty$, then $u_\varepsilon := \rho_\varepsilon * u$ is in $C^\infty(\Omega_\varepsilon)$.

Proof. Fix $x \in \Omega_\varepsilon$. Consider

$$\begin{aligned}
\frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h} &= \frac{1}{h}\int_\Omega [\rho_\varepsilon(x+he_i-y) - \rho_\varepsilon(x-y)]\,u(y)\,dy \\
&= \int_{B_\varepsilon(x)} \frac{1}{h}[\rho_\varepsilon(x+he_i-y) - \rho_\varepsilon(x-y)]u(y)\,dy.
\end{aligned}$$

Now, taking $\lim_{h\to 0}$ both sides, we get

$$\begin{aligned}
\frac{\partial u_\varepsilon(x)}{\partial x_i} &= \lim_{h\to 0}\int_{B_\varepsilon(x)} \frac{1}{h}[\rho_\varepsilon(x+he_i-y) - \rho_\varepsilon(x-y)]u(y)\,dy \\
&= \int_{B_\varepsilon(x)} \frac{\partial\rho_\varepsilon(x-y)}{\partial x_i}u(y)\,dy \\
&\qquad \text{(interchange of limits is due to the uniform convergence)} \\
&= \int_\Omega \frac{\partial\rho_\varepsilon(x-y)}{\partial x_i}u(y)\,dy = \frac{\partial\rho_\varepsilon}{\partial x_i} * u.
\end{aligned}$$

Similarly, one can show that, for any tuple α, $D^\alpha u_\varepsilon(x) = (D^\alpha\rho_\varepsilon * u)(x)$. Thus, $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. □

Appendix G

Duhamel's Principle

Consider the first order inhomogeneous ODE

$$\begin{cases} x'(t) + ax(t) &= f(t) \quad \text{in } (0, \infty) \\ x(0) &= x_0. \end{cases} \tag{G.0.1}$$

Multiplying the integration factor e^{at} both sides, we get

$$[e^{at}x(t)]' = e^{at}f(t)$$

and

$$x(t) = e^{-at}\int_0^t e^{as}f(s)\,ds + ce^{-at}.$$

Using the initial condition $x(0) = x_0$, we get

$$x(t) = x_0 e^{-at} + \int_0^t e^{a(s-t)}f(s)\,ds.$$

Notice that $x_0 e^{-at}$ is a solution of the homogeneous ODE. Thus, the solution $x(t)$ can be given as

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,ds$$

where $S(t)$ is a solution operator of the linear equation, given as $S(t) = e^{-at}$.

Consider the second order inhomogeneous ODE

$$\begin{cases} x''(t) + a^2x(t) &= f(t) \quad \text{in } (0, \infty) \\ x(0) &= x_0 \\ x'(0) &= x_1. \end{cases} \tag{G.0.2}$$

We introduce a new function y such that

$$x'(t) = ay(t).$$

Then

$$y'(t) = \frac{f(t)}{a} - ax(t)$$

and the second order ODE can be rewritten as a system of first order ODE

$$X'(t) + AX(t) = F(t)$$

where $X = (x, y)$, $F = (0, f/a)$ and

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

with the initial condition $X_0 := X(0) = (x_0, x_1/a)$. We introduce the matrix exponential $e^{At} = \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$. Then, multiplying the integration factor e^{At} both sides, we get

$$[e^{At}X(t)]' = e^{At}F(t)$$

and

$$X(t) = X_0 e^{-At} + \int_0^t e^{A(s-t)}F(s)\, ds.$$

Notice that $X_0 e^{-At}$ is a solution of the homogeneous ODE. Thus, the solution $X(t)$ can be given as

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s)\, ds$$

where $S(t)$ is a solution operator of the linear equation, given as $S(t) = e^{-At}$.

Bibliography

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