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Integral-norm estimates for the polar derivative of a polynomial

Abdullah Mir[1] · **Mahmood Bidkham**[2]

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Abstract
In this paper, we establish some integral-norm estimates for lacunary-type polynomials in the complex plane that are inspired by some classical Bernstein-type inequalities that relate the sup-norm of a polynomial to that of its polar derivative on the unit circle. The obtained results generalize some already known estimates that relate the L^{γ}-norm of the polar derivative and the polynomial.

Keywords Polar derivative of a polynomial · Bernstein's inequality · L^{γ}-norm · Minkwoski's inequality

Mathematics Subject Classification 30A10 · 30C10 · 30D15

1 Introduction

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n in the complex plane and $P'(z)$ its derivative. The study of inequalities for different norms of derivatives of a univariate complex polynomial in terms of the polynomial norm is a classical topic in analysis. A classical inequality that provides an estimate to the size of the derivative of a given polynomial on the unit disk, relative to size of the polynomial itself on the same disk is the famous Bernstein inequality [4]. It states that: if $P(z)$ is a polynomial of degree n, then on $|z| = 1$,

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✉ Abdullah Mir
drabmir@yahoo.com

Mahmood Bidkham
mbidkham@gmail.com

1 Department of Mathematics, University of Kashmir, Srinagar 190006, India

2 Department of Mathematics, University of Semnan, Semnan, Iran

$$|P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Over the years, this Bernstein inequality has been generalized and extended in several directions. In 1930, (see [5]) Bernstein himself revisited (1.1) and proved that, for two polynomials $P(z)$ and $Q(z)$ with degree of $P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leq |Q(z)|$ on the unit disk $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leq |Q'(z)|$ on $|z| = 1$. In fact, this inequality gives (1.1) in particular by taking $Q(z) = z^n \max_{|z|=1} |P(z)|$. It is worth mentioning that equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin, so it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials $P(z)$ having no zero in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later proved by Lax [13].

As a refinement of (1.2), Aziz and Dawood [2] established that if $P(z)$ is a polynomial of degree n not vanishing in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}. \tag{1.3}$$

In 1969 (see [14]), Malik extended (1.2) and proved that if $P(z) \neq 0$ in $|z| < k, k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Chan and Malik [7] generalized (1.4) and proved that if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \mu \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^\mu} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Further, as a generalization and refinement of (1.5), Kumar and Lal [12] considered the class of polynomials $P(z) = z^s \left(\sum_{j=\mu}^{n-s} a_j z^j \right), 1 \leq \mu \leq n-s, 0 \leq s \leq n-1,$ of degree n having a zero of order s at the origin and the remaining $n-s$ zeros in $|z| \geq k, k \geq 1$ and established that

$$\max_{|z|=1} |P'(z)| \leq \frac{n + sk^\mu}{1+k^\mu} \max_{|z|=1} |P(z)| - \frac{(n-s)}{k^s(1+k^\mu)} \min_{|z|=k} |P(z)|. \tag{1.6}$$

The above inequalities have been extended and generalized in different domains, different norms and for different classes of functions. Zygmund [22] extended the Bernstein-inequality (1.1) to L^γ-norms of $P(z)$ as

$$\left\{ \int_0^{2\pi} \left|P'(e^{i\theta})\right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}} \leq n \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}, \gamma \geq 1.$$

As an extension of (1.2) to L^{γ}-norms, de-Bruijn [6] proved an analogue of Zygmund's result for the class of polynomials not vanishing in $|z| < 1$. Govil and Rahman [10] generalized and sharpened the inequality due to de-Bruijn for polynomials of degree n not vanishing in $|z| < k, \;\; k \geq 1$ and for any $\gamma \geq 1$. Gardner and Weems [9] not only generalized the above result of Govil and Rahman to lacunary-tpye of polynomials but also validated it for $0 < \gamma < 1$ as well. As mentioned earlier, different versions of Bernstein-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomials are replaced by more general classes of functions. The one such generalization is moving from the ordinary derivative to their polar derivative. Before mentioning few such generalizations of the said inequalities, let us first introduce the concept of the polar derivative involved. For a polynomial $P(z)$ of degree n, we define

$$D_{\alpha}P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point α. The polynomial $D_{\alpha}P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left\{ \frac{D_{\alpha}P(z)}{\alpha} \right\} = P'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Aziz [1] was among the first to extend some Bernstein-type inequalities by replacing the ordinary derivative with the polar derivative of polynomial. The latest research and development on this topic can be found in the papers [11, 15–17, 19, 20]. In fact, in 1988, Aziz [1] proved that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < 1$, then for any complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_{\alpha}P(z)| \leq n\left(\frac{|\alpha| + 1}{2}\right) \max_{|z|=1} |P(z)|. \tag{1.7}$$

Very recently, Mir and Wani [20] extended (1.6) to the polar derivative of a polynomial and proved that if $P(z) = z^s\left(\sum_{j=\mu}^{n-s} a_j z^j\right), 1 \leq \mu \leq n - s, 0 \leq s \leq n - 1$, is a polynomial of degree n having a zero of order s at the origin and the remaining $n - s$ zeros in $|z| \geq k, k \geq 1$ then for any complex number α with $|\alpha| \geq 1$,

$$\begin{aligned} \max_{|z|=1} |D_{\alpha}P(z)| \leq &\frac{n(|\alpha| + k^{\mu}) + s(|\alpha| - 1)k^{\mu}}{1 + k^{\mu}} \max_{|z|=1} |P(z)| \\ &- \frac{(n - s)(|\alpha| - 1)}{k^s(1 + k^{\mu})} \min_{|z|=k} |P(z)|. \end{aligned} \tag{1.8}$$

Mir and Baba [18] proved the following L^{γ}-integral inequality which not only provides L^γ-analogue of (1.3) and (1.7) but also provides a refinement of (1.7) as well.

Theorem A *If $P(z)$ is a polynomial of degree n not vanishing in $|z|<1$, then for any complex numbers α, δ with $|\alpha| \geq 1$, $|\delta| \leq 1$ and $\gamma > 0$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + mn\delta\left(\frac{|\alpha|-1}{2} \right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n(|\alpha|+1) B_\gamma \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{1.9}$$

where

$$B_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{it} \right|^\gamma dt \right\}^{-\frac{1}{\gamma}} \text{ and} m = \min_{|z|=1} |P(z)|.$$

Further, Mir and Wani [17] *proved a similar type of inequality as in* (1.9) *by using a parameter β and established the following generalization of* (1.7).

Theorem B *If $P(z)$ is a polynomial of degree n not vanishing in $|z|<1$, then for any complex numbers α, β with $|\alpha| \geq 1, |\beta| \leq 1$ and $\gamma > 0$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P\left(e^{i\theta}\right) + n\beta\left(\frac{|\alpha|-1}{2} \right) P\left(e^{i\theta}\right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n\left\{ (|\alpha|+1) + |\beta|(|\alpha|-1) \right\} B_\gamma \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{1.10}$$

where B_γ is as defined in Theorem A.

Note: Taking $\delta = 0$ in Theorem A or $\beta = 0$ in Theorem B, we get the L^γ-analogue of (1.7). Dividing both sides of (1.9) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the corresponding L^γ-analogue of (1.3).

The authors are curious to know how the above mentioned inequalities in Theorems A and B, as well as some other inequalities, can be obtained from a general integral inequality. Indeed, this paper is mainly motivated by the desire to establish some more general Zygmund-type inequalities that extend (1.6) and (1.8)–(1.10).

2 Main results

Here, we prove the following generalization of Theorems A and B by considering the lacunary-type of polynomials not vanishing in a disk. As special cases, some known inequalities that relate the sup-norm of the derivative of a polynomial on the unit circle to that of the polynomial itself will be the consequences from the more fundamental inequality presented by the following theorem.

Theorem 1 *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n$, *is a polynomial of degree n having no zeros in* $|z| < k, k \ge 1$, *then for any complex numbers* α, β, δ *with* $|\alpha| \ge 1, |\delta| \le 1$ *and* $\gamma \ge 1$,

$$\left\{\int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + mn\delta\left(\frac{|\alpha| - 1}{1 + k^\mu}\right) + n\beta\left(\frac{|\alpha| - 1}{1 + k^\mu}\right) P\left(e^{i\theta}\right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\le n\{(|\alpha| + k^\mu) + |\beta|(|\alpha| - 1)\} C_\gamma(k, \mu) \left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \qquad (2.1)$$

where

$$C_\gamma(k, \mu) = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|k^\mu + e^{i\theta}\right|^\gamma d\theta \right\}^{\frac{-1}{\gamma}},$$

and $m = \min_{|z|=k} |P(z)|$.

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality holds in (2.1) for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ and $\alpha \ge 1$ with $\beta = 0$.

Taking $\beta = 0$ in Theorem 1, we get the following generalization of Theorem A for $\gamma \ge 1$.

Corollary 1 *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n$, *is a polynomial of degree n having no zeros in* $|z| < k, k \ge 1$, *then for any complex numbers* α, δ *with* $|\alpha| \ge 1, |\delta| \le 1$ *and* $\gamma \ge 1$,

$$\left\{\int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + mn\delta\left(\frac{|\alpha| - 1}{1 + k^\mu}\right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\le n(|\alpha| + k^\mu) C_\gamma(k, \mu) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \qquad (2.2)$$

where m and $C_\gamma(k, \mu)$ *are as defined in Theorem 1.*

In the limiting case, when $\gamma \to \infty$, the result is best possible and equality holds in (2.2) for $P(z) = (z^{\mu} + k^{\mu})^{\frac{n}{\mu}}$, where n is a multiple of μ and $\alpha \geq 1$.

Remark 1 If we divide both sides of (2.2) by $|\alpha|$ and let $|\alpha| \to \infty$, we get a result of Aziz and Shah [3]. If we let $\gamma \to \infty$ in (2.2), noting that $C_{\gamma} \to \frac{1}{1+k^{\mu}}$ and choose the argument of δ with $|\delta| = 1$ suitably, we get a result of Dewan et al. [8, Corollary 1]. For $k = \mu = 1$, (2.2) reduces to (1.9).

If we take $\delta = 0$ in Theorem 1, we get the following generalization of Theorem B.

Corollary 2 *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$, is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k, k \geq 1$, then for any complex number α, β with $|\alpha| \geq 1$ and $\gamma \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_{\alpha} P\left(e^{i\theta}\right) + n\beta \left(\frac{|\alpha| - 1}{1 + k^{\mu}} \right) P\left(e^{i\theta}\right) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n\{(|\alpha| + k^{\mu}) + |\beta|(|\alpha| - 1)\} C_{\gamma}(k, \mu) \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.3}$$

where $C_{\gamma}(k, \mu)$ is as defined in Theorem 1.

Remark 2 For $k = \mu = 1$, the above corollary reduces to Theorem B when $|\beta| \leq 1$. Dividing both sides of (2.3) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 3 *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^J, 1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for any complex numbers β with and $\gamma \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} P'\left(e^{i\theta}\right) + \frac{n\beta}{1 + k^{\mu}} P\left(e^{i\theta}\right) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\beta| + k^{\mu}) C_{\gamma}(k, \mu) \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.4}$$

where $C_{\gamma}(k, \mu)$ is as defined in Theorem 1.

Remark 3 The above inequality (2.4) generalizes a result of Mir and Wani [17] and some inequalities obtained by de-Bruijn [6] as well as Rahman and Schmeisser [21].

Remark 4 If we take $\beta = \delta = 0$ in Theorem 1, we get the polar derivative analogue of an L^{γ}-inequality due to Gardner and Weems [9].

Finally, we shall prove the following L^{γ}-analogue of (1.8). As a special case our result extends (1.6) as well.

Theorem 2 *If* $P(z)=z^s\left(a_0+\sum_{j=\mu}^{n-s}a_jz^j\right), 1\leq\mu\leq n-s, 0\leq s\leq n-1$, *is a polynomial of degree n having s-fold zeros at the origin and the remaining* $n-s$ *zeros in* $|z|\geq k, k\geq 1$, *then for any complex numbers* α,δ *with* $|\alpha|\geq 1, |\delta|\leq 1$ *and* $\gamma\geq 1$,

$$\left\{\int_0^{2\pi}\left|e^{i\theta}D_\alpha P(e^{i\theta})+\frac{m(n-s)(|\alpha|-1)\delta}{k^s(1+k^\mu)}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$
$$\leq\{(n-s)(|\alpha|+k^\mu)C_\gamma(k,\mu)+s|\alpha|\}\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{2.5}$$

where m and $C_\gamma(k,\mu)$ *are as defined in Theorem 1.*

Remark 5 For $s=0$, Theorem 2 reduces to Corollary 1. If we let $\gamma\to\infty$ in (2.5), noting that $C_\gamma(k,\mu)\to\frac{1}{1+k^\mu}$ and choose the argument of δ with $|\delta|=1$ suitably, we get (1.8). If we divide both sides of (2.5) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the $L^\gamma-$analogue of (1.6).

Remark 6 The inequality (2.2) was also recently proved by Mir [16] and for $\delta=0$, the inequality (2.5) was established by Mir [15].

3 Auxiliary results

We need the following lemmas to prove our theorems. The first lemma is due to Aziz and Shah [3].

Lemma 1 *If* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j, 1\leq\mu\leq n$, *is a polynomial of degree n having no zeros in* $|z|<k, k\geq 1$, *then on* $|z|=1$

$$k^\mu|P'(z)|\leq|Q'(z)|-n\min_{|z|=k}|P(z)|,$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

Lemma 2 *Let p, q be any two positive real numbers such that* $(q-h)\geq(p+h)x$, *where* $h\geq 0$ *and* $x\geq 1$. *If* β *is any real number such that* $0\leq\beta<2\pi$, *then*

$$[(q-h)+(p+h)y]\left|x+e^{i\beta}\right|\leq(x+y)\left|q+pe^{i\beta}\right|, \tag{3.1}$$

for any $y\geq 1$.

The above lemma was recently proved by the first author [16].

Lemma 3 *If* $P(z)$ *is a polynomial of degree n and* $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, *then for every* $\gamma>0$ *and* β *real,*

$$\int_0^{2\pi}\int_0^{2\pi}\left|Q'\left(e^{i\theta}\right)+e^{i\beta}P'\left(e^{i\theta}\right)\right|^{\gamma}d\theta d\beta\leq 2\pi n^{\gamma}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^{\gamma}d\theta.$$

The above lemma is again due to Aziz and Shah [3].

4 Proofs of Theorems

Proof of Theorem 1 Recall that $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j\neq 0$ in $|z|<k, k\geq 1$. If $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$, then $P(z)=z^n\overline{Q(\frac{1}{\bar{z}})}$ and it can be easily verified that for $0\leq\theta<2\pi$,

$$nP\left(e^{i\theta}\right)-e^{i\theta}P'\left(e^{i\theta}\right)=e^{i(n-1)\theta}\overline{Q'\left(e^{i\theta}\right)}. \tag{4.1}$$

For any complex number α and $0\leq\theta<2\pi$, we have

$$D_{\alpha}P\left(e^{i\theta}\right)=nP\left(e^{i\theta}\right)+\left(\alpha-e^{i\theta}\right)P'\left(e^{i\theta}\right),$$

which on using (4.1) gives,

$$\begin{aligned}\left|D_{\alpha}P\left(e^{i\theta}\right)\right|&\leq\left|nP\left(e^{i\theta}\right)-e^{i\theta}P'\left(e^{i\theta}\right)\right|+|\alpha|\left|P'\left(e^{i\theta}\right)\right|\\&=\left|Q'\left(e^{i\theta}\right)\right|+|\alpha|\left|P'\left(e^{i\theta}\right)\right|.\end{aligned} \tag{4.2}$$

Now for $\gamma\geq 1$ and $\beta,\delta\in\mathbb{C}$ with $|\delta|\leq 1$ and real t, we have by Minkwoski's inequality,

$$\left\{\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma dt\int_0^{2\pi}\left|e^{i\theta}D_\alpha P\left(e^{i\theta}\right)+\delta mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right.\right.$$
$$\left.\left.+n\beta\left(\frac{|\alpha|-1}{1+k^\mu}\right)P\left(e^{i\theta}\right)\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$
$$=\left\{\int_0^{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma\left|e^{i\theta}D_\alpha P\left(e^{i\theta}\right)+\delta mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right.\right.$$
$$\left.\left.+n\beta\left(\frac{|\alpha|-1}{1+k^\mu}\right)P\left(e^{i\theta}\right)\right|^\gamma d\theta dt\right\}^{\frac{1}{\gamma}}$$
$$\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma\left|e^{i\theta}D_\alpha P\left(e^{i\theta}\right)+\delta mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^\gamma d\theta dt\right\}^{\frac{1}{\gamma}}$$
$$+n|\beta|(|\alpha|-1)\left\{\int_0^{2\pi}\int_0^{2\pi}\left|\frac{k^\mu+e^{it}}{1+k^\mu}\right|^\gamma\left|P(e^{i\theta})\right|^\gamma d\theta dt\right\}^{\frac{1}{\gamma}}$$
$$\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma\left|e^{i\theta}D_\alpha P(e^{i\theta})+\delta mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|^\gamma d\theta dt\right\}^{\frac{1}{\gamma}}$$
$$+n|\beta|(|\alpha|-1)(2\pi)^{\frac{1}{\gamma}}\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \tag{4.3}$$

Again, since $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j\neq 0$ in $|z|<k, k\geq 1$, by Lemma 1, we have for $0\leq\theta<2\pi$,

$$k^\mu\left|P'\left(e^{i\theta}\right)\right|\leq\left|Q'\left(e^{i\theta}\right)\right|-mn,$$

which is equivalent to

$$k^\mu\left\{\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k^\mu}\right\}\leq\left|Q'\left(e^{i\theta}\right)\right|-\frac{mn}{1+k^\mu}.$$

This gives by taking $q=|Q'(e^{i\theta})|, p=|P'(e^{i\theta})|, h=\frac{mn}{1+k^\mu}, x=k^\mu\geq 1$ and $y=|\alpha|\geq 1$ in Lemma 2, we get for t real that

$$\left\{\left(\left|Q'\left(e^{i\theta}\right)\right|-\frac{mn}{1+k^\mu}\right)+|\alpha|\left(\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k^\mu}\right)\right\}\left|k^\mu+e^{it}\right|$$
$$\leq(k^\mu+|\alpha|)\left|\left|Q'\left(e^{i\theta}\right)\right|+e^{it}\left|P'\left(e^{i\theta}\right)\right|\right|. \tag{4.4}$$

On applying (4.2) and (4.4), we get for each $\gamma\geq 1, \delta\in\mathbb{C}$ with $|\delta|\leq 1$ and t real,

$$\begin{aligned}
&\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma dt\int_0^{2\pi}\left[\left|D_\alpha P\left(e^{i\theta}\right)\right|+mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right]^\gamma d\theta\\
&\quad=\int_0^{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma\left[\left|D_\alpha P\left(e^{i\theta}\right)\right|+mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right]^\gamma dtd\theta\\
&\quad\leq\int_0^{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma\left[\left|Q'\left(e^{i\theta}\right)\right|+|\alpha|\left|P'\left(e^{i\theta}\right)\right|+\ mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right]^\gamma dtd\theta\\
&\quad=\int_0^{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma\left[\left(\left|Q'\left(e^{i\theta}\right)\right|-\frac{mn}{1+k^\mu}\right)+|\alpha|\left(\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k^\mu}\right)\right]^\gamma dtd\theta\\
&\quad\leq\left(k^\mu+|\alpha|\right)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left|\left|Q'\left(e^{i\theta}\right)\right|+e^{it}\left|P'\left(e^{i\theta}\right)\right|\right|^\gamma dtd\theta.
\end{aligned}\tag{4.5}$$

Observe that for every $\gamma\geq 1$ and $a,b\in\mathbb{C}$ with t real (see [11]), we have

$$\int_0^{2\pi}\left|a+e^{it}b\right|^\gamma dt=\int_0^{2\pi}\left||a|+e^{it}|b|\right|^\gamma dt.\tag{4.6}$$

Inequality (4.5) gives with the help of (4.6) and Lemma 3 for each $\gamma\geq 1$, t real and $|\alpha|\geq 1$,

$$\begin{aligned}
&\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma dt\int_0^{2\pi}\left[\left|D_\alpha P\left(e^{i\theta}\right)\right|+mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right]^\gamma d\theta\\
&\quad\leq\left(k^\mu+|\alpha|\right)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left|Q'\left(e^{i\theta}\right)+e^{it}P'\left(e^{i\theta}\right)\right|^\gamma dtd\theta\\
&\quad\leq\left(k^\mu+|\alpha|\right)^\gamma 2\pi n^\gamma\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^\gamma d\theta.
\end{aligned}\tag{4.7}$$

On using the obvious inequality

$$\left|e^{i\theta}D_\alpha P(e^{i\theta})+\delta mn\left(\frac{|\alpha|-1}{1+k^\mu}\right)\right|\leq|D_\alpha P(e^{i\theta})|+mn\left(\frac{|\alpha|-1}{1+k^\mu}\right),$$

for $|\delta|\leq 1$ in (4.7) and raising the power $\frac{1}{\gamma}$ on both sides and then using in (4.3) gives (2.1). This completes the proof of Theorem 1. □

Proof of Theorem 2 If $P(z)=z^s\phi(z)$, where $\phi(z)=a_0+\sum_{j=\mu}^{n-s}a_jz^j$, $1\leq\mu\leq n-s$. Applying inequality (4.7) to the polynomial $\phi(z)$, we get for $|\alpha|\geq 1$ and $\gamma\geq 1$,

$$\left\{\int_0^{2\pi}\left[\left|D_\alpha\phi(z)\right|+\frac{m'(n-s)(|\alpha|-1)}{(1+k^\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leq \frac{(n-s)(|\alpha|+k^\mu)}{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma dt\right\}^{\frac{1}{\gamma}}}\left\{\int_0^{2\pi}\left|\phi(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{4.8}$$

where $m' = \min_{|z|=k}|\phi(z)| = \frac{1}{k^s}\min_{|z|=k}|P(z)|$.

Now

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\ &= z^s D_\alpha\phi(z) + \alpha s z^{s-1}\phi(z), \end{aligned}$$

which implies

$$zD_\alpha P(z) = z^{s+1}D_\alpha\phi(z) + \alpha s P(z). \tag{4.9}$$

Hence for $0 \leq \theta < 2\pi$, we get from (4.9) that

$$\begin{aligned} \left|D_\alpha P\left(e^{i\theta}\right)\right| &= \left|e^{i(s+1)\theta}D_\alpha\phi\left(e^{i\theta}\right) + \alpha s P\left(e^{i\theta}\right)\right| \\ &\leq \left|D_\alpha\phi\left(e^{i\theta}\right)\right| + s|\alpha|\left|P\left(e^{i\theta}\right)\right|, \end{aligned}$$

which gives by using Minkowski's inequality for $\gamma \geq 1$,

$$\begin{aligned} &\left\{\int_0^{2\pi}\left[\left|D_\alpha P\left(e^{i\theta}\right)\right| + \frac{m(n-s)(|\alpha|-1)}{k^s(1+k^\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\ &\leq \left\{\int_0^{2\pi}\left[\left|D_\alpha\phi\left(e^{i\theta}\right)\right| + \frac{m(n-s)(|\alpha|-1)}{k^s(1+k^\mu)} + s|\alpha|\left|P\left(e^{i\theta}\right)\right|\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\ &\leq \left\{\int_0^{2\pi}\left[\left|D_\alpha\phi\left(e^{i\theta}\right)\right| + \frac{m(n-s)(|\alpha|-1)}{k^s(1+k^\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} + s|\alpha|\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \end{aligned} \tag{4.10}$$

Using (4.8) in (4.10) and noting that $|\phi(e^{i\theta})| = |e^{is\theta}\phi(e^{i\theta})| = |P(e^{i\theta})|$, it follows that for every $|\alpha| \geq 1$ and $\gamma \geq 1$

$$\left\{\int_0^{2\pi}\left[\left|D_\alpha P\left(e^{i\theta}\right)\right|+\frac{m(n-s)(|\alpha|-1)}{k^s(1+k^\mu)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$
$$\leq\left\{\frac{(n-s)(|\alpha|+k^\mu)}{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|k^\mu+e^{it}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}}+s|\alpha|\right\}\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \tag{4.11}$$

Now using the fact for $|\delta|\leq 1$,

$$\left|e^{i\theta}D_\alpha P(e^{i\theta})+\frac{\delta m(n-s)(|\alpha|-1)}{k^s(1+k^\mu)}\right|\leq|D_\alpha P(e^{i\theta})|+\frac{m(n-s)(|\alpha|-1)}{k^s(1+k^\mu)},$$

in (4.11), we get (2.5). This completes the proof of Theorem 2. □

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Declarations

Conflicts of interest The authors declare that they have no conflicts of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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