Mathematical Modeling

1.1 Introduction

Real life problems arise from different disciplines including life science, social science, Health, Management and Information technology etc. Mathematical modeling is the use of mathematics to:

- describe our beliefs about how the world functions.
- investigate important questions about the observed world
- explain real world phenomena
- test ideas
- make predictions about the real world.

It is possible that we might have solved some of the problems with the help of mathematics and mathematical modeling without knowing what actually mathematical modeling is. The choice of approach to a real world problem depends on how the results are to be used. If the aim is to get knowledge for knowledge sake, then practical application is of no importance. A present day engineer/Industrialist will not undertake any strenuous task without a well-defined purpose. Anyone who likes to invest on the industrial production of the product would like to make calculations either to avoid unrealistically high cost of real scale experiments or to estimate some future situation. It is in this context a mathematical model of a real world problem gains enormous significance.

In mathematical modeling, we translate those beliefs into the language of mathematics with many advantages as

a) Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions.
b) Mathematics is a concise language, with well-defined rules for manipulations.
c) All the results that mathematicians have proved over hundreds of years are at our disposal.
d) Computers can be used to perform numerical calculations.

There is a large element of compromise in mathematical modeling. The majority of interacting systems in the real world are far too complicated to model in their entirety. Hence the first level of compromise is to identify the most important parts of the system. These will be included in the model; the rest will be excluded. The second level of compromise concerns the amount of mathematical manipulation, which is worthwhile. Although mathematics has the potential to prove general results, these results depend critically on the form of equations used. Small changes in the structure of equations may require enormous changes in the mathematical methods. Using computers to handle the model equations may never lead to elegant results, but it is much more robust against alterations.

The concept of mathematical modeling is not a new one. The Chinese, the ancient Egyptians, Indians, Babylonians and Greeks indulge in understanding and predicting the natural phenomena through their knowledge of mathematics. The architects, artisans and craftsmen based many of their works of art on geometric principles.

Mathematical modeling consists of simplifying real world problems and representing them as mathematical problems (mathematical model), solving the model and interpreting these solutions in the language of real world. In other words, we can divide the modeling process into three main steps, formulation, finding solution and interpretation and evaluation.

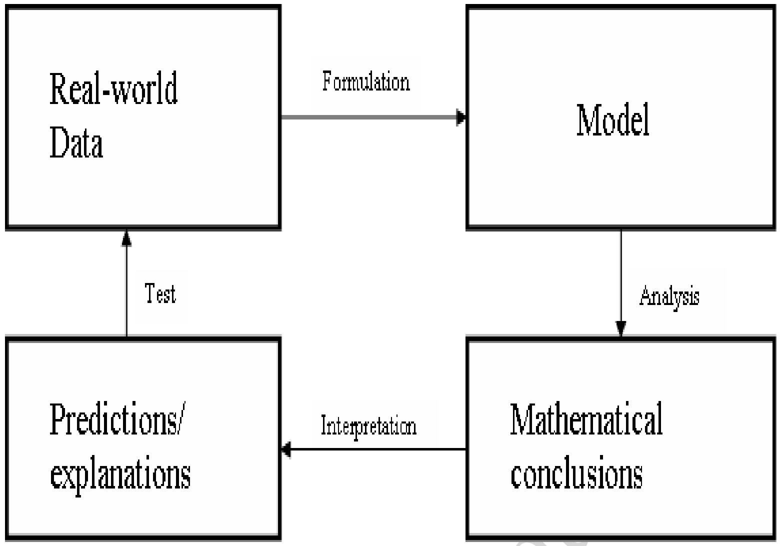

Figure-1.1: Process of mathematical modeling

1.2 Formulation of the Model

Formulation can be divided into three steps:

(i) Stating the Question: Understanding natural phenomena involves describing them. An accurate description answers such questions as; how long? How fast? How loud? etc. But the questions we start with should not be vague or too complicated. In problems drawn from the real world this should be done by describing the context of the problem and then stating the problem within this context.

(ii) Identifying relevant factors: Decided which quantities and relationships are important for the question and those that are unimportant can be neglected. The unimportant quantities are those that have very little or no effect on the process, e.g. in studying the motion of a falling body, its color is usually of little interest.

(iii) Mathematical description: Each important quantity should be represented by a suitable mathematical entity, e.g. a variable, a function, a geometric figure etc. Each relationship should be represented by an equation, inequality or other suitable mathematical assumptions.

1.3 Finding the solution

The mathematical formulation rarely gives us answer directly. We usually have to do some operations; this may involve calculations, solving an equation, providing a theorem etc.

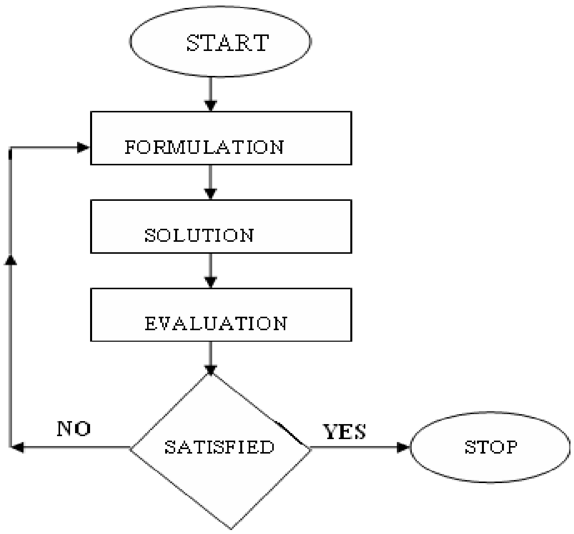

Figure-1.2: Flow chart of the mathematical model

1.4 Evaluation

Since a model is a simplified representation of real problem, but its very nature has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is good one or not, i.e., when the obtained results are interpreted physically, whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new evaluation. Thus, mathematical modeling can be a cycle of three steps shown in the flowchart of Figure- 1.2.

Example-1 Modeling speed and velocity.

By the definition, speed/ velocity is the rate of change of distance traveled. Since speed is a scalar, we model it as L/T, where L is a distance traveled and T is the time required to travel. While modeling velocity, the direction too should be specified and hence the model for velocity is $V = \vec{L}/T$, where the vector notion is used additionally. Using calculus, the model can be further improved by writing the elementary distance as $ds = (dx, dy, dz)$, so that $V = ds/dt$.

Exercise 1: Explain modeling acceleration of a particle? (Try yourself)

As we know that every branch of knowledge has two aspects, one of which is theoretical involving mathematical, statistical and computer based methods and the factors of which is empirical based on experiments and observations.

Likewise, mathematical models are basically of two kinds:

(i) Empirical Models
(ii) Theoretical Models

Empirical Models are based on experimentally founded hypothesis. They lead to the construction of an underlying theoretical framework. In other words, they more often lead to “Laws of nature” which represent a fundamental characteristic of nature. Such models are formulated by great mathematicians- Newton, Einstein etc. Typical examples are; the theory of gravitation by Sir Isaac Newton, Electromagnetic waves by Maxell, Theory of relativity by Einstein, Planetary motion by Kapler, Wave equation by Schrödinger etc. Only those hypotheses that have withstood large amounts of critical scrutiny can be elevated to the status of laws. In other words, the mere fact that the proposed model agrees well with a small of data does not spice the agreement could be justified coincidental. It should be test against a large amount of data before accepting as a law. This aspect should be clear from the fact that nearly half a century elapsed between the works of Galileo and Newton.

Theoretical Models are inspired by the formulations or guidelines provided by the modeling schemes. The objective is to apply the basic laws or ideas in small way and to particular cases.

1.5 Classifications of models

When studying models, it is helpful to identify broad categories of models. Classification of individual models into these categories tells us immediately some of the essentials of their structure. One division between models is based on the type of outcome they predict. According to the nature of the models, we can classify mathematical models into the following four types.

(i) **Linear or Non-linear Models:**

According as the resulting equations which may be algebraic, differential or difference being linear or non-linear, models are classified as linear or non-linear. For instance, consider the equation

$$\frac{dN}{dt} = \pm\lambda N \qquad \ldots(1.1)$$

For the negative sign on the right hand side of equation (1.1), i.e.,

$$\frac{dN}{dt} = -\lambda N$$

Then equation models the radioactive decay. When we assume that the rate of a decay of a radioactive atom is proportional to the number N of radioactive atoms present and $\lambda > 0$ is decay constant. For a positive sign on the right hand of equation (1.1) gives a model for the population growth. In both the cases equation (1.1) represent linear models being linear differential equations. The solution can be written as

$$N = N_0 e^{\pm\lambda t} \qquad \ldots (1.2)$$

where N_0 in the case of decay denote the original number of radioactive atoms at t = 0. This model though very simple agrees excellently with experimental results. In the case of population growth N_0 would be the initial population.

Remark: Most of the real life problems are not amenable to such simple mathematical treatment. Many a time, the resulting equation is non-linear or highly non-linear but still we are able to solve it. The example of the population growth model as:

$$\frac{dN}{dt} = \lambda N(B - N), \quad \lambda > 0,\ B > 0 \qquad \ldots (1.3)$$

where N is the size of the population and λ and B are the constants of proportionality. This is a non-linear model, but it is easy to find the solution of the model as

$$N = \frac{B}{1 + Ke^{-\lambda Bt}} \qquad \ldots (1.4)$$

where $K > 0$ is an arbitrary constant. There are numerous experimental growth data, say, that of the bacteria with which the model agrees extremely well.

(ii) **Static or Dynamic:**

In static systems, time does not play any part and hence the variable and relationships describing the system are time independent. In contrast, in dynamic systems, time plays a very important role with the variables and/or relationships describing the system changing with time. Consider for instance a fluid flowing through a rigid diverging tube see Figure-1.3.

Let the velocity of the fluid be V_1 at the point P_1 at which the area of the cross section of the tube is A_1. Let V_2 be the velocity at the point P_2 at which the area of cross section of the tube is A_2. The principal of conservation of mass states that the rate of flow in at P_1 is equal to the rate of flow out at P_2. Since the tube is rigid and no extra fluid is produced inside or nothing is taken out. In other words, there are no sources or sinks inside or surrounding the tube.

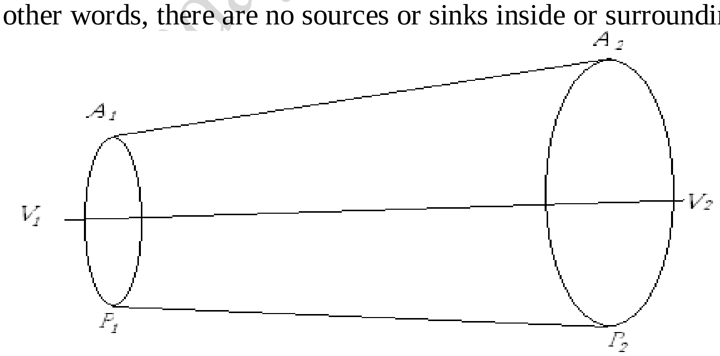

Figure-1.3: Determination of static system

Now the rate of mass entering the tube at P_1 = area x velocity = $A_1 V_1$.
Rate of mass leaving the tube at P_2 = $A_2 V_2$.

Conservation law can be written therefore, in the form of an equation

$$A_1V_1 = A_2 V_2 \qquad \ldots(1.5)$$

Rate of mass entering the tube at P_1 is equal to the rate of mass leaving the tube at P_2.
Equation (1.5) is the conservation equation corresponding to the steady state i.e., all variables are independent of time. Such a system is static system.
In the dynamic formulations, the equations describing the model involve derivatives of the dependent variables with respect to time.

Most of the real life problems e.g., the population growth (equation (1.3)), the bacterial growth, simple harmonic oscillator, rocket launch are time dependent and come under the category of dynamic systems.

(iii) **Discrete or Continuous**:

Mathematical model may be discrete or continuous according as the variables involved are discrete or continuous. In a discrete model, the dependent variable assumes a range of values and is characterized for discrete values of the independent variable. E.g., suppose a population cells divided synchronously, with each member producing a daughter cell. Let us define the number of cells in each generation with a subscript i.e., M_1, M_2, . . . , M_n are respectively the number of cells in the first, second, . . . , n^{th} generations. The number of generations, the independent variable, is the discrete here. A simple equation relating successive generations in the difference equation

$$M_{n+1} = a\, M_n \;, \quad a > 0 \qquad \text{...(1.6)}$$

If initially, there are M_0 cells after n generations, the population will be

$M_{n+1} = a\, M_n = a\,(a\, M_{n-1}) = \ldots = a^{n+1} M_0$
If $|a| > 1$, M_n increases over successive generations.
If $|a| <1$, M_n decreases over successive generations.
and if $a = 1$, M_n is constant.

Most of the discrete models result in difference equations similar to equation (1.6). Models based on continuous variable are continuous models. The problem of radioactive decay is best described by treating the time element as being continuous with a variable of the system description, i.e., number N of radioactive atoms produced (equation (1.1)). Most of the continuous models result in differential equations, ordinary or partial, the derivatives being instantaneous rates of change. Continuous models appear to be easier to handle than the discrete models due to the development of calculus and differential equations. However, continuous models are simpler only when analytical solutions are available. Otherwise we have to approximate a continuous model also by a discrete model so that these can be handled numerically.

(iv) **Deterministic or Stochastic**:

A system is said to be deterministic if values assumed by the variables (for a static system) or the changes to the variable (for a dynamic system) are predictable with certainty. Consider for example, the well-known example of the simple pendulum; the variables of the system are the position and the velocity of the bob of the pendulum. Since the laws of classical dynamics describe the motion fairly, accurately, the changes in position and velocity can be predicted with a high degree of certainty. Hence, in this case we can view the system as being deterministic.

If the values assumed by the variables or the changes to the variables are not predictable with certainty, then uncertainty is a significant feature of the system. Such systems are called Probabilistic or Stochastic systems. For example, if one drops a rubber ball from a given height and measures the height of a bounce with sufficient accuracy, it will be found that if the same process is repeated many times, the height of bounces are not same every time, even if all the conditions associated with laboratory experiments are carefully maintained, the results show lot of variability. In such cases the system must be viewed as a stochastic system.

Remark: Every real system must be considered to be subject to randomness of one type or another, all of which are ignored in the formulation of a deterministic model. Hence deterministic models generally present few mathematical difficulties but can be only considered to describe system behavior in same average sense. Stochastic models are required whenever it is necessary to explicitly account for the randomness of underlying events.

Most of the discrete and stochastic models lead to difference/algebraic equations whereas linear, static/dynamic and continuous models require the knowledge of algebraic/differential equations. With the advent of fast computers, it should be possible (whenever analytic solutions are not available) to solve these equations numerically. Apart from these, the success of mathematical modeling will also depend on the skills you have in algebra, calculus, geometry, trigonometry, transcendental equations, integral equations, integro-differential equations etc.

Exercise: Which type of modeling will be used for the launching of a rocket/ satellite for meteorological purposes?

Modeling used for the said purpose is dynamic, continuous and deterministic. It is dynamic and continuous because the flight velocity will continuously depend on time. It is deterministic because equations describing the flight can be set up based on established laws and the path of the satellite/rocket can be predicted with certainty.

1.6 Objectives of the modeling

Mathematical modeling can be used for a number of different reasons. How well any particular objective is achieved depends on both the state of knowledge about a system and how well the modeling is done. Examples of the range of objectives are:

- Developing scientific understanding - through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also show up what we do not know)
- test the effect of changes in a system
- aid decision making, including
    - tactical decisions by managers
    - Strategic decisions by planners.

1.7 Limitations of a Mathematical Model

Mathematical modeling is a multi-stage activity requiring a variety of concepts and techniques. Utmost caution is required in framing proper models, otherwise an absurd model lead to a strange solution. If the basic formulation is wrong, no amount of sophistication in the treatment of resulting equations can lead to a right answer. It is important to remember that the model is only a simplification of the real world problem and that the two are not the same. In fact lack of distinction between models and the reality has often slowed down the progress in modeling. It is paradoxical that some models, which were very successful initially in understanding, the problems have become stumbling blocks to progress. The reason is we get used to a model and continue to use it even after it is discredited. For instance consider the solar system, till 16th centaury it was believed that earth was the centre of the universe and all the other planets and sun move around the earth. Because of this theory the model used to study the solar system were circular paths with earth as the centre. It was called the **Geocentric** model. This model was successful in explaining night, day, seasons etc. But, there were many observations, the model could not explain. Later in 16th centuryCopernicus proposed another theory called **Heliocentric** theory which describes that the sun is the centre of the universe, and that all planets moved around the sun in elliptical paths. So in this case model used is an elliptical path with sun as the centre. This model successfully explained most of the problems connected with solar system but people simply refused to accept the model, initially. One of the reasons for this is that the geocentric model put the earth as the centre of the universe and people were unwilling to discard such a favorite notion.

1.8 Formulation of the problem

In this step of modeling- given a real world problem, we proceed, how do we convert it to model abstraction leading to a mathematical equation? We have to also take into account, how to:

(i) Identify the problem with all its complexities
(ii) Identify the essential characteristics of the problem, which have to be incorporated into the model.
(iii) Simplify the model by neglecting features, which are of secondary or lesser importance.
(iv) Write the basic equations based on the basic laws of nature or intuitive logic, which retain the essential characteristics of the model.

Primarily, mathematical modeling utilizes analogy to help you understand the behavior of complex system. e.g., the phrase "cool as cucumber" introduces a conceptual model of "cool" into our minds. Modeling is an activity, which is fundamental to the scientific methods. Models rarely replicate a system. Also, they are not unique in representation and so can mean different things for different people. Consider how businessman and a biologist view a mango tree:

A Businessman's view: Wealth, Orchard, Timber!!!
A Biologist's view: A living thing, A large plant, Nutritious food!!!

Their conceptual views of the same object are rather different since they are heavily influenced by their own environment, background and objectives. The same is true when we come to the mathematical modeling of any system or process.

Thus there is no hard and fast approach to develop a model. But, we need to broadly follow the following steps in the beginning:

(i) **Establish a main purpose for the model:**

Real situations are quite complex. If one wishes to develop a model, which will explain and account for all aspects of a phenomenon, such a model will most likely be difficult to develop, very complex and unmanageable. On the other hand, a model with limited purpose will be easy to handle and still many important conclusions related to the main purpose can be drawn. Thus, before developing a model we must be clear about the purpose of doing it. For example, in the case of a problem concerned with simple pendulum, what is our main purpose? It is to find the period of the oscillation of the pendulum.

(ii) **Observe the real world situation** and understand what is going on. These observations may be direct, as with using one of our senses or indirect, in which case we may use elaborate scientific equipment. This step allows us to gather data and inform well about the problem. We then analyze the observations and know facts about the system or phenomenon being modeled and identify possible elements (observations, measurements, ideas) related to the purpose. This step is crucial to the development of a realistic model since we will get an idea what to expect.

(iii) **Shift the essentials from the non-essentials** of the problem. The degree of detail needed to describe a system appropriately depends on various factors. If all the details are included in the description, it can become unmanageable and hence of limited use. On the other hand, if significant details are omitted, the description is incomplete and, once again of limited use in carrying out the study. We need to find a sensible compromise.

(iv) **The search for essentials of the problem** is related to the main purpose of the model. We may be dealing with the same system but the objective of our study related to the system may be different in each study. For example, consider modeling the blood flow in the circulatory system. The blood cells are of diameter approximately 10^{-6} cms and hence their individual motion or rotation may not contribute much to the fluid mechanics of blood flows in large arteries whose diameter range from 1mm to 1 cm. But in small capillaries of diameter 1 micro metre, the cell sizes are comparable to the area of cross section of the capillaries and in such a situation; the individual cell motion becomes very important. In other words, the mathematical model trying to depict the flow of blood in large arteries can assume blood flow to be homogeneous whereas a model of blood in capillaries has to emphasize the individual cell motion.

1.9 Mathematical formulation

The mathematical modeling is relating real world problem to a suitable abstract mathematical formulation. In order to carry out this step, we need a good understanding of the various mathematical formulations available. We also need to develop the skill to select the most appropriate formulation. This is very important for often, one can choose more than one type of formulation. What is most appropriate can be identified from how much detail we want to find out about the problem or the facilities we have to study a problem. If we have a limited purpose, say, we want to have a rough idea about the problem, and then a simple model will suffice. i.e., the limitations and approximations are acceptable for our purpose. If the problem has to be studies in depth, an appropriate model would be the one with finer details.

Remark: Consider the problem of finding the period of oscillation of a simple pendulum; we shall consider here two formulations:

Formulation 1: First we make a preliminary model based on a dimensional analysis to understand the oscillation of a simple pendulum. Let us see if we can make something of the dependence of the period on the length of the pendulum. We need to consider the variables, the period T_0, the string length ℓ, and the gravitational constant g, since it is obviously gravity that makes the pendulum swing.

Remark: The symbol, g, is in fact, the gravitational acceleration of the surface of the earth. The value of g depends upon the precise location of its measurement, but it is nearly constant. Dimension of g = $[LT^{-2}]$ and its value in SI system is 9.8 m/s^2

We start with

$$T_0 = T_0(l, g)$$

i.e., T_0 is a function of ℓ and g.

It is clear that if we leave out some important quantities, we shall be in error. Similarly, if we have included some quantities, which are in reality irrelevant to the problem, we will not only make the problem un-necessary sophisticated but we also arrive at an unreal answer. Very clear understanding of the problem can only help us in making a correct choice of these quantities.

Since T_0 has the dimension of time, the right hand side should have the same dimension. Since the length dimension appears in a linear fashion in both ℓ and g, it follows that

$$T_0 = T_0(\ell / g) \qquad \dots (1.7)$$

This is because $[T_0]$ = time and [g] = L/T^2

Mass (gms)	Length (cms)	Time (sec)
385	275 225	3.371 3.056
230	275 225	3.352 3.042

Table: Periods obtained experimentally for four different pendulums

Now, if we want that length should not appear on right hand side also, then ℓ and g should appear as the ratio ℓ / g. Also since $[T_0]$ = time and [ℓ / g] = $(\text{time})^2$, it follows that

$$T_0 (g / \ell)^{1/2} = A \qquad \dots (1.8)$$

where A is the constant to be determined.

We use experimental values to determine this constant A.

In the above table, we have given the results obtained from experiments with two different masses, 230 gms and 385 gms respectively, attached in turn to two strings of lengths equal to 275 cm and 225 cm. The results are for small oscillations of the four pendulums obtained by permuting the two strings.

For $\ell = 275$ cm, one measured value of the period is 3.371 sec with $g = 9.8 m/\sec^2$ or $980 cm/\sec^2$, we can use this data in equation (3) to find the constant A.

i.e., $$A = (3.371)\sqrt{\frac{980}{275}} = 6.35 \qquad \ldots (1.9)$$

which is approximately 2π.

If we assume from this similarity that the period of the pendulum is in fact given by

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}} \qquad \ldots(1.10)$$

Then we can calculate periods for strings of lengths used in the experiment.

Formulation 2: Formulation 1 was helpful in finding the period of oscillation of a simple pendulum. But, what if we want to know more about the pendulum for instance the tension on the string? We find that formulation 1 is not enough. Hence we need to formulate a model, which will improve our understanding of the problem beyond equation (1.10).

In the present formulation, we take recourse to the Newton's laws of motion. Here since we are concentrating on the tension on the string, we shall assume that the string has no little mass of its own that it can be neglected in the model. We shall also assume that the air offers little resistance. Then the only forces acting on the mass are the tension T in the string and the gravitational force mg. The tension in the string must act along the line of the string, while the gravitational force acts vertically downward along the y-axis where we have assumed that the y-axis is roughly perpendicular to the earth's surface as shown in the Figure-1.4.

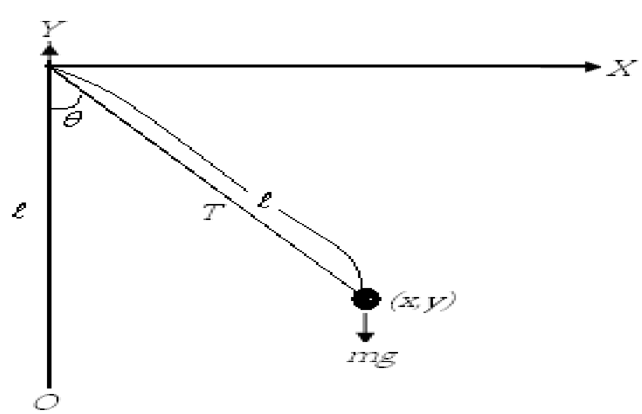

Figure-1.4: Simple pendulum perpendicular to the earth's surface

Newtons'second law tells us that the net force on a particle causes the particle to be accelerated in direct proportion to its mass. Here the forces acting on the particle are its weight mg and the tension T. If F denotes the total force acting on the system, then we would write

$$\sum F_x = m\frac{d^2x}{dt^2}, \qquad \sum F_y = m\frac{d^2y}{dt^2} \qquad \ldots(1.11)$$

where $\sum F_x$ and $\sum F_y$ are the net forces acting on the mass in directions parallel to the x and y axis and the terms d^2x/dt^2 and d^2y/dt^2 are the components of the acceleration of the mass parallel to the axes.

The component of T acting in the x-axis is $-T\sin\theta$ (Note that the negative sign is because T acts upward and the resolved components falls in the negative x-direction). Also, the components of T acting in the y-direction is $T\cos\theta - mg$, then it follows that

$$\sum F_x = -T\sin\theta \qquad \ldots(1.12)$$

$$\sum F_y = T\cos\theta - mg \qquad \ldots(1.13)$$

Also, note that

$$x = \ell\sin\theta \quad \text{and} \quad y = \ell(1-\cos\theta) \qquad \ldots(1.14)$$

Combining equation (1.11), (1.12) and (1.13), we obtain the following pair of differential equations.

$$m\frac{d^2x}{dt^2} = -T\sin\theta \qquad \ldots(1.15)$$

$$m\frac{d^2y}{dt^2} = T\cos\theta - mg \qquad \ldots(1.16)$$

Equation (1.15) and (1.16) can be solved to obtain the values of x and y by eliminating T. we shall not go into the details of solving these equations here. On solving, this formulation helps us not only to find the period of oscillation and the tension in the string, but also the position vector of the bob at different time t.

Remark: On comparing the two formulations we find that the formulation 1 based on dimensional analysis is quick and gives a first guess about the nature of the solution or main purpose of the study. But formulation 2, though more lengthy, gives a deeper insight into the problem. Thus the choice of a formulation depends on how far we can proceed, how much details we can gather about the problem in hand. There can be two factors that can be used to rank different models to indicate the best.

a) A model M_1 is preferred to a model M_2 if M_1 has fewer parameters. Thus models can be ranked in terms of the number of parameters in the model. Estimation of the parameters and design of experiments are not only costly but also very tedious and hence to be avoided.

b) If a model response is highly sensitive to the parameters of the model, then the model is of limited use for prediction purposes, as small errors in parameters will result in large errors in the model response. Thus, the models can be ranked in terms of the sensitivity of the response to changes in parameters.

1.10 Solution and Interpretation of the model

A mathematical model is complete only when we interpret the mathematical solution of the model. Now, we shall discuss this aspect of mathematical modelling, namely interpreting/ evaluating the solution. We can see that the interpretation helps us to gauge how effective the model is?

Solutions of formulated problem

The concept of mathematical model to be developed must depend on the purpose for which the model is required. As discussed in the formulation,we saw that if the purpose of studying the movements of a simple pendulum is to find its period of oscillation, a quick solution based on dimensional analysis will serve our purpose. But if the objective of the study is to have a deeper insight into the problem we have to use a different model. In this case a model based on Newton's law's by resolving the forces acting on the bob of the pendulum will serve the purpose.

1.11 Motion of a Simple Pendulum

We have already formulated the model on the motion of a simple pendulum and the formulation resulted in two differential equations, given below

$$m\frac{d^2x}{dt^2} = -T\sin\theta \quad \ldots(1.17)$$

$$m\frac{d^2y}{dt^2} = T\cos\theta - mg \quad \ldots(1.18)$$

We have to find the position of the pendulum and the tension in the string at any instant of time. This is possible if either we know the position (x, y) of the bob at that instant or the angle θ, the string makes with the vertical at that instant as given in the Figure-1.5.
We know that x, y and θ are connected by the relation $x = \ell\sin\theta$ and $y = \ell(1-\cos\theta)$, where ℓ being the length of the pendulum. Eliminating the terms x and y to solve θ, from (1.1) and (1.2), we have

$$\frac{dx}{dt} = \frac{dx}{d\theta}.\frac{d\theta}{dt}$$

By repeated application of chain method, we get

$$\frac{d^2x}{dt^2} = \ell\cos\theta.\frac{d^2\theta}{dt^2} - \ell\sin\theta\left(\frac{d\theta}{dt}\right)^2 \quad \ldots(1.19)$$

$$\frac{d^2y}{dt^2} = \ell \sin\theta . \frac{d^2\theta}{dt^2} + \ell\cos\theta\left(\frac{d\theta}{dt}\right)^2 \qquad \ldots(1.20)$$

So (1.17) and (1.18) becomes

$$m\ell\cos\theta . \frac{d^2\theta}{dt^2} + \left[T - m\ell\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta = 0 \qquad \ldots(1.21)$$

$$m\ell\sin\theta . \frac{d^2\theta}{dt^2} - \left[T - m\ell\left(\frac{d\theta}{dt}\right)^2\right]\cos\theta = -mg \qquad \ldots(1.22)$$

Multiplying respectively (1.21) and (1.22) by $\cos\theta$ and $\sin\theta$, then adding the resulting equation, we get

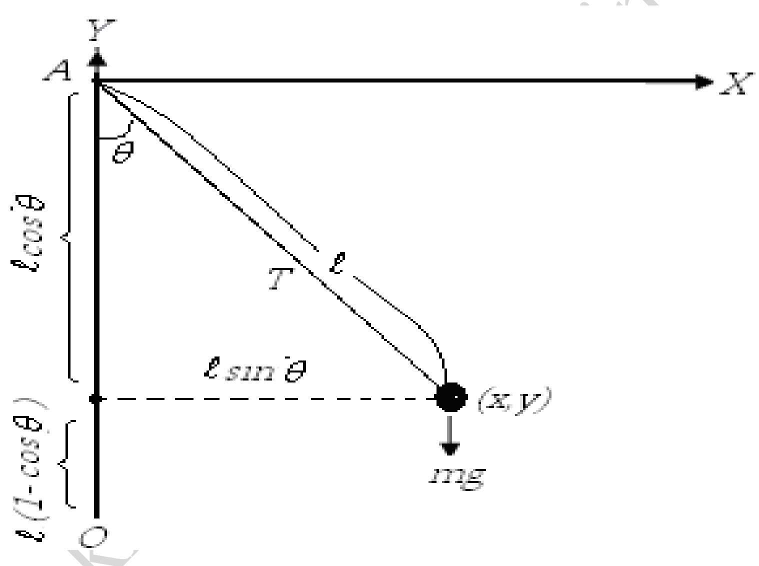

Figure-1.5: Motion of the Simple Pendulum

$$m\ell\left(\cos^2\theta + \sin^2\theta\right).\frac{d^2\theta}{dt^2} = -mg\sin\theta \qquad \ldots(1.23)$$

$$m\ell.\frac{d^2\theta}{dt^2} + mg\sin\theta = 0 \qquad \ldots(1.24)$$

Thus, we have found the equation in terms of θ alone as a function of T.

In order to find the formula for tension T on a string, we multiply (1.21) by $\sin\theta$ and (1.22) by $\cos\theta$ and taking the difference we have

$$\left[T - m\ell\left(\frac{d\theta}{dt}\right)^2\right]\left(\cos^2\theta + \sin^2\theta\right) = mg\cos\theta \qquad \ldots(1.25)$$

i.e.,

$$T = m\ell\left(\frac{d\theta}{dt}\right)^2 + mg\cos\theta \qquad \ldots(1.26)$$

This equation of motion in the direction along the string determines the tension once $\theta(t)$ has been determined from equation (1.26).

Solution using linear model

To begin with, let us assume that the oscillations are small which means that θ is small. This will enable us to approximate $\sin\theta$ by θ since $\theta \to 0$, $\sin\theta \to \theta$. This will certainly reduce the accuracy in our calculations. But the mathematics involved gets much reduced. In fact, even for fairly large angles, i.e., angles whose magnitude may be anywhere up to 30^0, i.e., $-30^0 \le \theta \le 30^0$, we can take

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1$$

As we expect, these approximations will introduce some errors. For example, let $\theta = 15^0$, Then from the table of sine, we can find that $\sin 15^0 = 0.25881$. To compare this with the given values of θ, we have to find θ in radian measure. The radian measure of θ = 15^0 is 0.26196. The error in this approximation is 0.26196 – 0.25881 = 0.00315.
Using the approximations, we can write equation (1.26) as

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \qquad \ldots(1.27)$$

and equation (1.26), as

$$T = mg\left[1 + \frac{\ell}{g}\left(\frac{d\theta}{dt}\right)^2\right] \qquad \ldots(1.28)$$

we can further simplify equation (1.28) by using the argument that when θ is small $\left(\frac{d\theta}{dt}\right)^2 < 1$ and hence the second term in the bracket is much smaller than the first term.

Therefore, we can neglect the second term. This would imply

$$T = mg \qquad \ldots(1.29)$$

Isn't this an interesting result? Even for swings of the pendulum up to $\pm 30^0$, the tension is a constant.

Let us now go back to equation (1.27), it is nothing but the classical simple harmonic equation. Equation (1.27) is a simple second order ordinary differential equation with constant coefficients. From the knowledge of ordinary differential equations, we know that

$$\theta = A\cos\left(\sqrt{\frac{g}{\ell}}t\right) + B\sin\left(\sqrt{\frac{g}{\ell}}t\right) \quad \ldots(1.30)$$

where A and B are arbitrary constants.
These constants will depend on the initial position of the bob and the velocity with which it is started. Let us assume that

$$\theta = \theta_0 \text{ at } t = 0, \text{ where } \theta_0 \text{ is some arbitrary angle} \quad \ldots(1.30\text{-a})$$

$$d\theta/dt = 0 \text{ at } t = 0 \quad \ldots(1.30\text{-b})$$

Condition (1.30-a) would imply that at $t = 0$, the initial amplitude of motion of the pendulum is θ_0 . (30-b) implies that the initial speed of the pendulum is zero. Thus, conditions (1.30-a) and (1.30-b) correspond to initially holding the pendulum at rest at any arbitrary angle θ_0 and then letting it go.

When we put $t = 0$ and apply (30-a) in equation (30), we get $A = \theta_0$

Then, we obtain $\dfrac{d\theta}{dt}$ from equation (1.30) and apply conditions $d\theta/dt = 0$, $t = 0$ to get

$$-\theta_0 \sin\left(\sqrt{\frac{g}{\ell}}t\right)\sqrt{\frac{g}{\ell}} + B\cos\left(\sqrt{\frac{g}{\ell}}t\right)\sqrt{\frac{g}{\ell}} = 0, \text{ when } t = 0.$$

This implies that $B = 0$.
Therefore, the solution is given by

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \quad \ldots(1.31)$$

Instead of equation (1.30-a) and (1.30-b), suppose we assume that

$$\theta = 0 \text{ at } t = 0 \text{ and } \quad \frac{d\theta}{dt} = \omega, \text{ at } t = 0$$

This means that at $t = 0$, the initial amplitude of the motion is 0 i.e., the bob is at the equilibrium position and the initial speed is ω. Now, we can easily check that the solution in this case is given by

$$\theta = \omega\sqrt{\frac{g}{\ell}} \sin\left(\sqrt{\frac{g}{\ell}}t\right) \quad \ldots(1.32)$$

Thus, individually equation (1.31) and (1.32) are both solutions of equation (1.27), of course, under different conditions. The sum of equations (1.31) and (1.32) is also a solution of equation (1.27), being the solution of a linear differential equation.
Therefore,

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) + \omega\sqrt{\frac{\ell}{g}}\sin\left(\sqrt{\frac{g}{\ell}}t\right) \qquad \ldots(1.33)$$

is the solution of (11) with the conditions

$$\theta = \theta_0, \text{ at } t = 0 \text{ and } \frac{d\theta}{dt} = \omega, \text{ at } t = 0$$

Solution using Non-Linear Model

We begin with rewriting equation (1.24) after multiplying by $d\theta/dt$, we get

$$\left(\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta\right)\frac{d\theta}{dt} = 0 \qquad \ldots(1.34)$$

which we can also rewrite as

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos\theta\right) = 0$$

This implies that

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos\theta = a \text{ (Constant)} \qquad \ldots(1.35)$$

If the initial condition is such that the pendulum is started at rest from an arbitrary angle θ_0, then at $t = 0$

$$\theta(t) = \theta_0, \ \frac{d\theta}{dt} = 0$$

Therefore, if we put $\theta = \theta_0$, and $\frac{d\theta}{dt} = 0$ in equation (1.35), we get that the constant is $-\frac{g}{\ell}\cos\theta_0$

i.e.,

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos\theta = -\frac{g}{\ell}\cos\theta_0$$

Therefore,

$$\left(\frac{d\theta}{dt}\right)^2 = 4\frac{g}{\ell}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \qquad \ldots(1.36)$$

Substituting equation (1.36) in equation (1.28) we get the value of the tension T in terms of θ. We can get the expression for T as

$$T = mg\left[1 + 4\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)\right]$$

Also, from (36), we have

$$\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} = 2\sqrt{\frac{g}{\ell}}dt \qquad \text{...(1.37)}$$

Integrating, we can get the position of the pendulum θ as a function of t.
Since the pendulum swings from $-\theta_0$ to $+\theta_0$ and back again, so using this we can find the limits of the integration.
Suppose we denote as T_0 the period of the pendulum, during the period. A quarter period would be time interval $0 \le t \le \frac{T_0}{4}$, say, from $\theta = \theta_0$ to $\theta = 0$ as shown in the Figure-1.6.

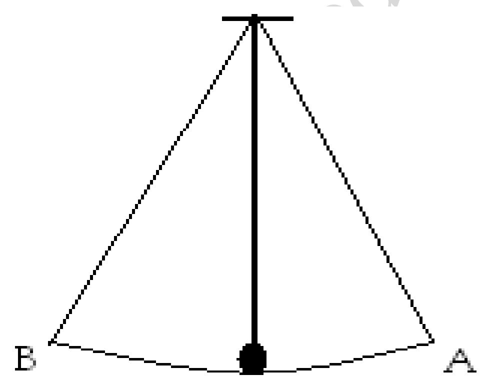

Figure -1.6: Time period of the Oscillations

Thus, equation (1.37) can be integrated as follows: of the total period

$$2\sqrt{\frac{g}{\ell}}\int_0^{\frac{T_0}{4}} dt = \int_{-\theta_0}^{0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Integrating the left hand side, we get

$$T_0 = 2\sqrt{\frac{\ell}{g}}\int_{-\theta_0}^{0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Put $\theta = -\phi$, we get

$$T_0 = 2\sqrt{\frac{\ell}{g}}\int_0^{\theta_0}\frac{d\phi}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\phi}{2}}} \qquad \ldots(1.38)$$

Let $\sin\frac{\phi}{2} = \sin\frac{\theta_0}{2}\sin\psi$.

Differentiating both sides we get

$$\frac{1}{2}\cos\frac{\phi}{2}\,d\phi = \sin\frac{\theta_0}{2}\cos\psi\,d\psi$$

Then,

$$d\phi = \frac{2\sin\frac{\theta_0}{2}\cos\psi}{\cos\frac{\phi}{2}}\,d\psi = \frac{2\sin\frac{\theta_0}{2}\cos\psi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\psi}}\,d\psi$$

Substituting for $d\phi$ in the integral on the RHS of equation (38), we have

$$T_0 = 2\sqrt{\frac{\ell}{g}}\int_0^{\frac{\pi}{2}}\frac{2\sin\frac{\theta_0}{2}\cos\psi\;d\psi}{\sin\frac{\theta_0}{2}\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\psi}\sqrt{1-\frac{\sin^2\frac{\theta_0}{2}\sin^2\psi}{\sin^2\frac{\theta_0}{2}}}}$$

$$T_0 = 4\sqrt{\frac{\ell}{g}}\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\psi}} \qquad \ldots(1.39)$$

The integral on the RHS of integral of (1.39) is a definite integral which gives T_0 as a function of θ_0 say $f(\theta_0)$. The integral is called an elliptical integral and the tables are available to find the value of the elliptical integrals.

Exercise: Find T_0 if $\theta_0 = 20^0$ given that $\ell = 20\,cm$ and g = 9.8 cm/sec^2.

Solution: Substituting for θ_0 and ℓ in equation (1.39) above, we have

$$T_0 = 4\sqrt{\frac{20}{980}}\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-\sin^2\frac{20}{2}\sin^2\psi}}\,\text{sec}.$$

Clearly, from the table of elliptical integrals

$$\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-\sin^2 10\sin^2\psi}}\,\text{sec} \approx 1.58284\ \text{sec}$$

Therefore,

$$T_0 \approx \frac{4}{7} \times 1.58284 \text{ sec} . \approx 1.00448 \text{ sec.}$$

Examples 1: Consider the free fall of a body in a vacuum. The fall must be related to the gravitational acceleration g and the height h from which the body is released. Use dimensional analysis to show that the velocity V of the falling body is determined by the dimensional equations $V/\sqrt{gh}$ = constant.

We start with a functional equation that expresses in general terms the dependence of V on g and h. i.e.,

$$V = V(g,h)$$

We know that $[V] = \frac{L}{T}$, $[g] = \frac{L}{T^2}$, $[h] = L$. Since the dimension of time appears only in the velocity and the gravitational acceleration. We can now write

$$V \propto \sqrt{g} V_1(h)$$

We repeat this process to make the left hand side dimension in length as well as time. We must then have

$$\frac{V}{\sqrt{gh}} = \text{Constant}$$

Alternative method: Another straightforward way of doing the above is to write

$$V \propto g^a h^b$$

This implies if the dimensions on both sides of this proportionality to be equal,

$$\frac{L}{T} = \left(\frac{L}{T^2}\right)^a (L)^b$$

$\Rightarrow$ $$a + b = 1, \; 2a = 1$$

Therefore,

$$a = \frac{1}{2}, \; b = \frac{1}{2}$$

i.e., $$V \propto \sqrt{gh}$$

Example 2: A string of length ℓ is connected to a fixed point at one end and to a stick of mass m at other. The stick is whirling in a circle at constant velocity v. Use dimensional analysis to show that the force in the string is determined from the dimensionless equation $\frac{F\ell}{mV^2} =$constant.

We can write the force in the string as

$$F \propto \ell^a m^b v^c$$

$\Rightarrow$ $$\frac{ML}{T^2} = L^a M^b \left(\frac{L}{T}\right)^c$$

Comparing the exponential of L, M and T, we get

$$b = 1, \; a + c = 1, \; c = 2$$

$$\Rightarrow \qquad a = -1$$

$$\Rightarrow \qquad F \propto \frac{mv^2}{\ell}$$

Exercise 1: Using equation (1.17) and show that the bob of the simple pendulum achieves its maximum angular velocity at $\theta = 0$. Why is this physically reasonable? Show that the results are applicable to both linear and non-linear problems.

Exercise 2: Using non-linear model of the pendulum, find the period of oscillation for $\theta_0 = 12\sec$ and $\ell = 4$.

Chapter 2

Simple Harmonic Motion

2.1 Introduction

Of all oscillatory motions, the most important is simple harmonic motion, because being a simplest motion to describe mathematically; it constitutes a rather accurate description of many oscillations found in nature.

Definition: If the force on a particle is proportional to its distance from a fixed point and is directed towards it, then the particle will execute a simple harmonic motion.

There are two types of simple harmonic motion (SHM):

(i). **Linear:** when the body moves in a linear path under the action of a constant force, for example up and down oscillations of the piston of a cylinder containing gas, when suddenly pressed and released and oscillations released, and oscillations of an elastic spring suspended vertically and loaded at its lower end etc.

(ii). **Angular:** when the body rotates about an axis under the action of a torque or couple; examples are pendulum oscillations and torsional oscillations etc.

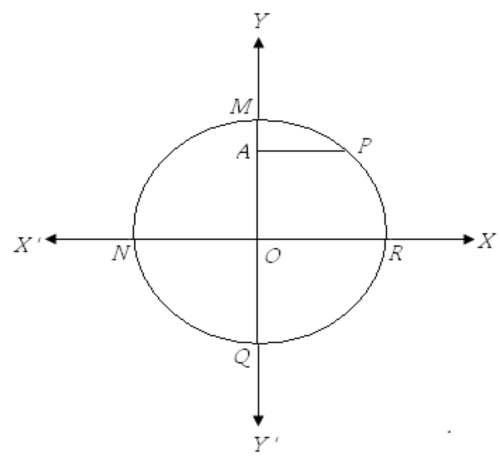

Figure-2.1: Linear simple harmonic motion

In order to visualize linear SHM consider a particle P moving uniformly in a circle as shown in Figure-2.1. From P draw a line PA perpendicular to the y-axis. As P moves toward M along the circle, A moves towards M along OY. When P starts moving along the arc MN, A moves from M to O as P traces arc NQ, A traces OQ and finally as P reaches R from Q, A reaches P from Q. Thus, the point A keeps moving about O, on either side along the y-axis. This particle thus, executes a linear SHM along YOY'.

We shall now model this SHM. Remember that SHM represents one-dimensional motion or motion in a straight line. As above we have formulated a model of a simple pendulum, which is a simple application of the SHM. Now, we shall again take up the motion of a simple pendulum and give two more formulations of it; (i) one when the motion is resisted by a force and (ii) another when the oscillation is induced by external force. But first consider the modeling of SHM.

Formulation 1: Suppose a particle of mass m at time t is at A, at a distance x from O. Also, let A moves under the influence of a force $F = \mu x$ towards O as shown in Figure-2.2

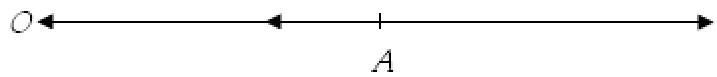

Figure-2.2: Motion of a particle in one dimension

Then Newton's second law of motion yield the following equation

$$m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} = m\frac{dv}{dt} = -\mu x \qquad \dots (2.1)$$

where $\mu > 0$.

The negative sign in equation (2.1) is because the force F is directed towards O. Therefore equation (2.1) can be written as

$$\frac{d^2x}{dt^2} = \frac{-\mu x}{m} = -\omega^2 x \qquad \dots (2.2)$$

where

$$\omega^2 = \frac{\mu}{m}.$$

Thus, when A is on the right side of O, the force is towards the left. This is the prototype equation for simple harmonic equation, since both μ and m are positive and ω is real. But it is interesting to see what does ω represent physically? Rewrite equation (2.2) in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Clearly, the solution of this equation is of the form

$$x(t) = A\cos\omega t + B\sin\omega t \qquad \dots (2.3)$$

where A and B are arbitrary constants of integration and their values can be obtained by using initial/boundary conditions. Equation (2.3) is the model equation for SHM.

Observe that the equation (2.3) can be written in the form

$$x(t) = \left[\frac{A\cos\omega t + B\sin\omega t}{\sqrt{A^2+B^2}}\right]\sqrt{A^2+B^2} \qquad \dots (2.4)$$

If we define

$$\frac{A}{\sqrt{A^2+B^2}} = \sin\phi,$$

$$\frac{B}{\sqrt{A^2+B^2}} = \cos\phi,$$

and $$R = \sqrt{A^2+B^2}$$

Then equation (2.4) reduces to

$$x(t) = R\sin(\omega t + \phi) \qquad \ldots(2.5)$$

where $\phi = \tan^{-1}(A/B)$

Since $\sin\theta = \sin(\theta + 2n\pi)$ for all integral values of n, we obtain from equation (2.5)

$$x(t) = x\left(t + \frac{2\pi}{\omega}\right) \qquad \ldots(2.6)$$

This shows that the position of the particle is same after intervals of time $\frac{2\pi}{\omega}$.

This value $\frac{2\pi}{\omega}$ is called the time period of the SHM, and ω is called the frequency of the oscillation as shown in the Figure-2.3 below.

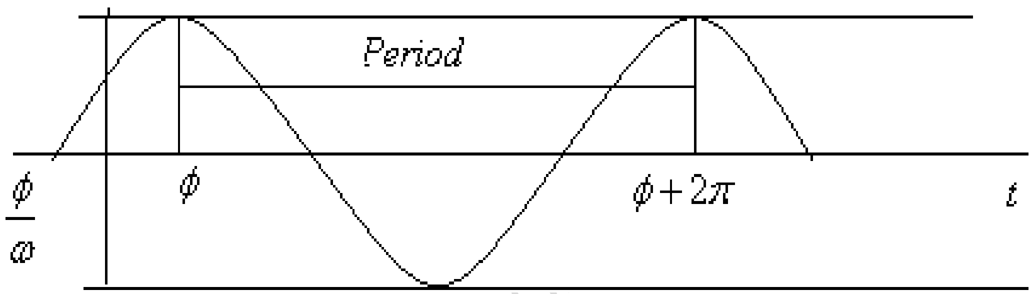

Figure-2.3: **Simple Harmonic Motion**

From equation (2.5), we find that the maximum displacement of the particle occurs when the peak value of sin ($\omega t + \phi$) is maximum, i.e., 1, and hence the maximum displacement is R. This maximum displacement is called the amplitude of the SHM. The quantity $\omega t + \phi$ is called the phase of the oscillation and thus ϕ is the initial phase i.e., its value for $t = 0$.

Exercise: Show that the model $x(t) = A\cos\omega t + B\sin\omega t$ for SHM can be written in the form $x = R\sin(\omega t + \phi_1)$.

Solution: Suitably define $\sin\phi_1$ and $\cos\phi_1$ and reduce $x(t) = A\cos\omega t + B\sin\omega t$ to the required form.

Example 1: A particle is executing SHM about the origin O. Find its displacement $x(t)$ at any time t if at $t = 0$, $x = 3$ and $\frac{dx}{dt} = 0$.

We know that

$$x(t) = R\sin(\omega t + \phi)$$

Now, using the given conditions, we have

$$3 = R\sin\phi$$
$$0 = R\omega\cos\phi$$

which implies that

$$R = 3 \text{ and } \phi = \frac{\pi}{2}.$$

Hence the displacement at any time t is given by

$$x(t) = 3\sin\left(\omega t + \frac{\pi}{2}\right) = 3\cos \omega t$$

Now if we want to obtain the velocity in terms of the displacement, we consider the model equation in the form

$$mv\frac{dv}{dx} = -\mu x$$

so that $$v\frac{dv}{dx} = -\frac{\mu}{m}x = -\omega^2 x$$

Integrating, this equation and assuming that $x = x_0$ when $v = 0$, i.e., the maximum displacement or amplitude is x_0, we obtain the relation

$$v^2 = \omega^2\left(x_0{}^2 - x^2\right) \qquad \ldots(2.7)$$

Since $v^2 \geq 0$, equation (2.7) gives

$$|x| \leq |x_0|$$

This means that the displacement is always less than x_0. But, we assumed x_0 to be the position at which the velocity is zero. Hence, we conclude that the velocity of the particle becomes zero when its displacement is maximum.

Exercise 1: When and where does a particle executing SHM in a straight line has maximum velocity?

Example 2: The maximum velocity of a particle moving in SHM is 10 meters/sec and its period is 5 sec. Find its amplitude.

Let the SHM be $x = A\sin(\omega t + \phi)$

Then the velocity is

$$\frac{dx}{dt} = A\omega\cos(\omega t + \phi)$$

Since maximum velocity is 10 meter/sec, at that time, acceleration has to be zero and hence

$$\frac{d^2x}{dt^2} = 0 = -A\omega^2\sin(\omega t + \phi)$$

This gives $\omega t + \phi = 0$ and this time $\frac{dx}{dt} = 10$

Hence $$A\omega = 10$$

Also, we are given that the period = $\frac{2\pi}{\omega} = 5$

Hence the amplitude of the SHM is

$$A = \frac{10}{\omega} = \frac{50}{2\pi} \text{ meters.}$$

Example 3: The amplitude of a particle executing SHM is b and its period is T . Show that the maximum velocity of the particle is $\frac{2\pi b}{T}$.

We know that

$$x(t) = b\sin(\omega t + \phi) = b\sin\left(\frac{2\pi}{T}t + \phi\right)$$

Therefore, $$\frac{dx}{dt} = \frac{2\pi b}{T}\cos\left(\frac{2\pi}{T}t + \phi\right)$$

Since the maximum value of $\cos\left(\frac{2\pi}{T}t + \phi\right)$ is at most 1.

Therefore, $$\left.\frac{dx}{dt}\right|_{\max} = \frac{2\pi b}{T}$$

Exercise 2: A particle executing SHM passes two points a and b with velocities v_1 and v_2. Show that the amplitude is

$$A = \sqrt{\left[\left({v_1}^2 b^2 - {v_1}^2 a^2\right)/\left({v_1}^2 - {v_2}^2\right)\right]}$$

Remark: As we have mentioned earlier simple harmonic motion is the most important of all the oscillatory motion. There are many applications of this motion in nature. One of its simplest applications is a simple pendulum.

2.2 Applications of a SHM-Simple Pendulum

A simple pendulum is a point mass tied to an inextensible string and suspended from a fixed point. The point mass is allowed to oscillate in its own plane with small amplitude. Earlier, using Newton's second law of motion we set up the model equation for the motion of this point mass and later on obtained its solution. We only recapitulate that the motion of a pendulum is a SHM with time period $T = 2\pi\sqrt{\frac{\ell}{g}}$.

However in real situation if we perform this experiment with a ball tied to a string and observe the oscillations, we will see that the oscillations die down after some time. This is because of the air resistance and the frictional forces at the point of suspension, which have been ignored in the previous model. Of course, the model can be improved to incorporate these effects, but we will not take up that part here. However, we shall give the model equations for damped SHM, i.e., when a force resists the motion.

Formulation 2: Let the body be attached to a spring at the point O as shown in the Figure-2.4

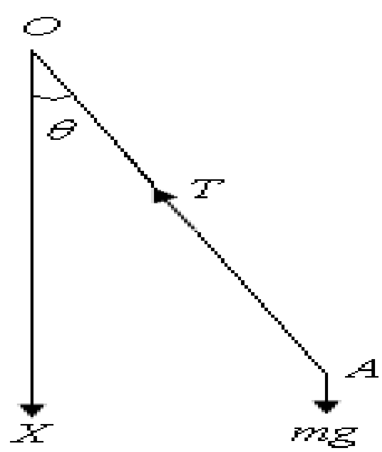

Figure-2.4: Resistive force in the Oscillations

Then the forces acting on the mass are

i) the tension T along the string from A towards O.
ii) the weight mg of the body acting vertically downwards.

Let a resistance force $R = -f\left|\frac{dx}{dt}\right|$, proportional to velocity acts on the body apart from the tension $T = -\mu x$, where both f and μ are positive constants.

Using Newton's law of motion we obtain

$$m\frac{d^2x}{dt^2} = -\mu x - f\,\frac{dx}{dt}$$

Let $\alpha = \dfrac{f}{2m}$ and $\omega^2 = \dfrac{\mu}{m}$,

Then certainly $\alpha \geq 0$ and for this α and ω the above equation reduces to

$$\frac{d^2x}{dt^2} + 2\alpha\omega\frac{dx}{dt} + \omega^2 x = 0 \qquad \ldots(2.8)$$

Comparing equation (2.8) with equation (2.2), we would notice that $2\alpha\omega\dfrac{dx}{dt}$ is the additional term in equation (2.8).

To solve the second order linear differential equation (2.8), we substitute a trial solution

$$x = Ce^{pt}$$

where C and p are the constants to be determined.

With the choice of x, equation (2.8) reduces to

$$p^2Ce^{pt} + 2\alpha\omega\, pCe^{pt} + \omega^2Ce^{pt} = 0$$

or $$Ce^{pt}\left(p^2 + 2\alpha\omega\, p + \omega^2\right) = 0 \qquad \ldots(2.9)$$

Now for $x \neq 0$, we obtain the quadratic equation in p,

i.e.,

$$p^2 + 2\alpha\omega\, p + \omega^2 = 0 \qquad \ldots(2.10)$$

which yields

$$p = \left[-\alpha\omega \pm \omega\sqrt{\left(\alpha^2 - 1\right)}\right] \qquad \ldots(2.11)$$

and hence the solution of equation (2.8) will be

$$x = A\exp\left\{-\alpha\omega t + t\omega\sqrt{\left(\alpha^2 - 1\right)}\right\} + B\exp\left\{-\alpha\omega t - t\omega\sqrt{\left(\alpha^2 - 1\right)}\right\} \qquad \ldots(2.12)$$

where A and B are constants of integration. Now in equation (2.12), if we put $\alpha = 0$, we obtain

$$x = A\exp(it\omega) + B\exp(-it\omega)$$
$$= A_1\cos(\omega t) + B_1\sin(\omega t)$$

2.3 Projectile Motion

The free motion of a body that is projected in a non-vertical direction under gravity is called Projectile motion. Numerous examples of projectile motion can be had from our everyday experiments. Throwing a ball from the boundary in a cricket ground, the path of the ball is a projectile. Flight of a football when kicked represents a projectile motion, etc. Now suppose that a ball is thrown from a point O with a velocity u_0 at an angle α with the horizontal.

Then the following observations are of interest for the projectile motion:

i) How far does the ball reach before it touches the ground?
ii) How high does it rise?
iii) What is the maximum distance it can cover for a given velocity, and what is the angle of projection to achieve the distance?
iv) What is the total time of flight?

To know all these things, we have to model the projectile motion. But, before we do that we shall describe a small experiment to show the independence of horizontal and vertical motions.

Experiment 1: Place three marbles A, B and C at the edge of a table as shown in Figure-2.5 below. At the same time, hit B and C horizontally with different forces, so that their horizontal velocities are different, while allow A to fall vertically downwards. The following are the observations:

i) Horizontal distance traversed by A, B and C are different. They depend on the horizontal axis.
ii) They all hit the ground at the same instance.
iii) Vertices distances covered by all the particles are same, equal to the height of the table.

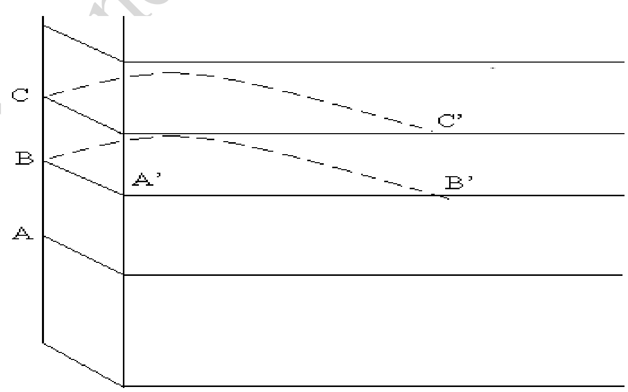

Figure-2.5: Three marbles at the edge of a table

We conclude from this experiment that

a) the effect of gravity is not altered by horizontal motion
b) when a particle with an initial horizontal velocity moves under gravity the motion can be studied by considering the horizontal and vertical motions independently.

Having got this much information from the experiment about the way the two components of motion are effected by gravity, we are now in a position to set up a model equation for the motion of a projectile.

Formulation 1: we shall consider the projectile without any resistance. Let a particle from O with an initial velocity u_0 making an angle θ with the horizontal. Since, we do not consider any force other than gravity the particle will be confined to the plane in which the particle is thrown. We call this plane XOY, with OX and OY the two axes as shown in the Figure-2.6.

Suppose at time t, the particle is at P(x, y), as observed through the experiment above that the motion of the particle can be studied in the vertical and horizontal directions independently.

Thus, writing down the Newton's laws of motion, we have the following equations of motion.

$$m\frac{d^2x}{dt^2} = 0, \text{ since there is no force in x- direction} \qquad \ldots(2.13)$$

$$m\frac{d^2y}{dt^2} = -mg \qquad \ldots(2.14)$$

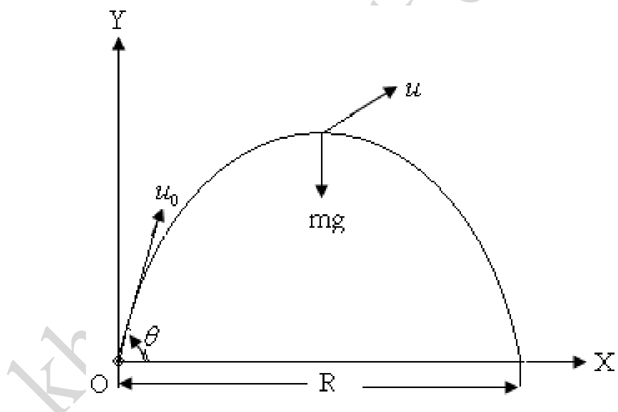

Figure-2.6: Projectile motion without any resistance

where d^2x/dt^2 and d^2y/dt^2 are the components of the acceleration in x and y directions respectively.

The initial conditions are that at $t = 0$, x = y = 0 and $\frac{dx}{dt} = u_0 \cos\theta$, while $\frac{dy}{dt} = u_0 \sin\theta$. With these conditions, the solution of equation (2.13) and (2.14) is

$$x = u_0 \cos\theta t \qquad \ldots(2.15)$$

$$y = u_0 \sin\theta t - \frac{gt^2}{2} \qquad \ldots(2.16)$$

These equations describe the motion of a projectile.
Eliminating t between these equations, we obtain the equation to the trajectory as

$$y = x\tan\theta - \frac{g\sec^2\theta}{2{u_0}^2}x^2 \qquad \ldots(2.17)$$

This is clearly the equation of the parabola.

To determine distance the particle reaches, that is, the range R, we substitute $y = 0$ in the above equation, to obtain

$$0 = y = x\tan\theta - \frac{g\sec^2\theta}{2{u_0}^2}x^2$$

$$\Rightarrow \qquad x\left(\tan\theta - \frac{g\sec^2\theta}{2{u_0}^2}x\right) = 0 \qquad \ldots(2.18)$$

This equation gives two values of x, one $x = 0$, that is the point from where the particle was thrown, and the other point where it hits the ground.
Thus, the range R of the particle is

$$R = \left(\frac{2{u_0}^2}{g}\right)\sin\theta\cos\theta = \left(\frac{{u_0}^2}{g}\right)\sin 2\theta \qquad \ldots(2.19)$$

Using the value of R, equation (2.17) to the trajectory can be written in a neat form as:

$$y = x\tan\theta\left(1 - \frac{x}{R}\right) \qquad \ldots(2.20)$$

Thus, using equation (2.15), the total time of the flight is given by

$$T = \frac{R}{u_0\cos\theta} \qquad \ldots(2.21)$$

$$= \frac{2u_0}{g}\sin\theta \qquad \ldots(2.22)$$

But this is the total time of a particle moving under gravity with an initial velocity $u_0\sin\theta$.
Now in order to obtain the maximum height attached by the particle we proceed as follows:
For y to be maximum, we know that $dy/dx = 0$, and hence from equation (2.20).

$$0 = \tan\theta - 2\frac{x}{R}\tan\theta$$

giving the value of x for which y is maximum as $x = R/2$. To make sure that this corresponds to a maximum, take the second derivative of y and ensure that d^2y/dx^2 is negative for this value of x.
Thus, the maximum y is obtained from equation (2.20) as

$$y_{\max} = \frac{R}{4}\tan\theta \qquad \ldots(2.23)$$

Also, from equation (2.21), we see that for a given velocity of projection, the maximum value of R, i.e., the maximum range is given by

$$R_{\max} = \frac{u_0^{\ 2}}{g} \qquad \ldots(2.24)$$

and the angles of projection for this range is

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{Rg}{u_0^{\ 2}}\right)$$

and

$$\theta = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{Rg}{u_0^{\ 2}}\right) \qquad \ldots(2.25)$$

Thus, for a given velocity of projection, to reach a certain range within the maximum possible with that velocity, there are two possible angles of projection.

Example 1: Find the velocity and direction of projection of a ball, which moves in the horizontal direction just over the top of the wall 100 meters high at a distance of 100 meters.

Suppose that the velocity of the projection is u_0 and angle of projection is θ. Let PA be the wall and O the point of projection as shown in Figure-2.7 below
Then OA = PA = 100 meters.
Suppose at time t, the ball is at P. Then,

$$x = OA = 100$$
$$= u_0 \cos\theta \; t \quad \text{and}$$
$$y = PA = 100$$
$$= u_0 \sin\theta \; t - \frac{1}{2}gt^2$$

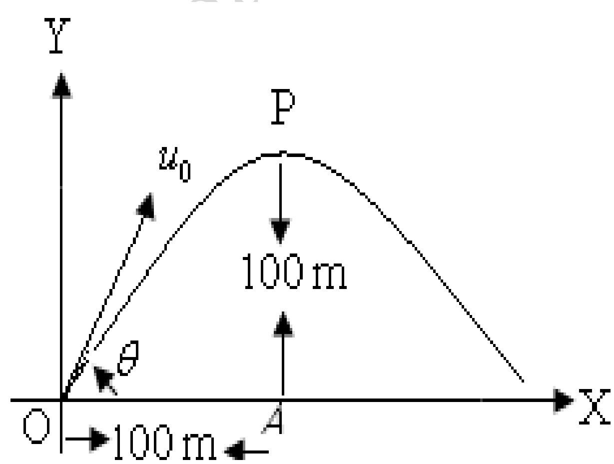

Figure-2.7 Representation of a Projectile Motion

Also, since the ball crosses the wall horizontally, the vertical velocity of the ball at P.

i.e. $$\frac{dy}{dt} = 0.$$

Hence $$0 = u_0 \sin\theta - gt$$

Thus, giving $$t = \frac{u_0}{g}\sin\theta.$$

Substituting the value of t in the above equations, we can obtain the values of θ and u_0.

Exercise: A projectile, when thrown at an angle $\tan^{-1}(3/4)$ falls 40 meters short of the target. When it is fired at an angle of 45^0, it falls 50 meters beyond the target. Find the distance of the target from the point of projection.

Remark: We do observe from the above discussion as follows:
From equation (2.17) of the trajectory,

i.e.,

$$y = x\tan\theta - \frac{g\sec^2\theta}{2u_0^{\ 2}}x^2$$

or $$y = x\tan\theta - \frac{gx^2}{2u_0^{\ 2}}\left(1+\tan^2\theta\right)$$

or $$\tan^2\theta - \left(\frac{2u_0^{\ 2}}{gx}\right)\tan\theta + \left(1 + \frac{2u_0^{\ 2}y}{gx^2}\right) = 0 \qquad \ldots(2.26)$$

This being quadratic in $\tan\theta$ will give for a given x, y and u_0, two roots for θ, provided

$$\frac{4u_0^{\ 2}}{gx^2} \geq 4 + 8\frac{u_0^{\ 2}y}{gx^2}$$

If this condition is not satisfied, then we can't reach the point (x, y), with the velocity of projection u_0. When this condition is satisfied, we can reach the given point through two trajectories - one when the particle is moving upwards and the other when it is moving downwards. In this formulation, the air resistance has been completely ignored, so formulation-1 can be improved by incorporating the effect of air resistance on the motion of a projectile.

Formulation 2: Let us consider the projectile motion under gravity with air resistance R proportional to velocity. Equation (2.13) and (2.14) of motion then take the form

$$m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \qquad \ldots(2.27)$$

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt} \qquad \ldots(2.28)$$

where $k > 0$ is a constant of proportionality. Using the initial conditions $t = 0$, x = y = 0 and $\frac{dx}{dt} = u_0\cos\theta$, while $\frac{dy}{dt} = u_0\sin\theta$, the solution of equations (2.27) and (2.28) can be written as

$$x = \frac{m}{k} u_0 \cos\theta \left(1 - e^{-\frac{k}{m}t}\right) \qquad \ldots(2.29)$$

$$y + \frac{mg}{k} t = \frac{m}{k}\left(u_0 \sin\theta + \frac{mg}{k}\right)\left(1 - e^{-\frac{k}{m}t}\right) \qquad \ldots(2.30)$$

Equations (2.29) and (2.30) describe the motion of a projectile in a resisting medium. Eliminating t between these equations, we obtain the equation of the trajectory as

$$y = \frac{m^2}{k^2} g \,\ln\left(1 - \frac{kx}{mu_0 \cos\theta}\right) + \frac{x}{u_0 \cos\theta}\left(u_0 \sin\theta + \frac{mg}{k}\right) \qquad \ldots(2.31)$$

For y to be maximum, we know $\frac{dy}{dx} = 0$, this gives

$$x = \frac{mmu_0^{\,2} \sin\theta \cos\theta}{ku_0 \sin\theta + mg} \qquad \ldots(2.32)$$

and for this value of x we get from equation (2.31), maximum y as

$$y_{\max} = \frac{mu_0 \sin\theta}{k} - \frac{m^2 g}{k^2} \ln\left(1 + \frac{ku_0 \sin\theta}{mg}\right) \qquad \ldots(2.33)$$

It is clear from equation (2.29) and (2.30) that when $t \to \infty$, $x = \frac{mu_0 \cos\theta}{k}$ and $y \to -\infty$.

Hence the path has a vertical asymptote at a horizontal distance $\frac{mu_0 \cos\theta}{k}$ from the point of projection as shown in the Figure-2.8 below.

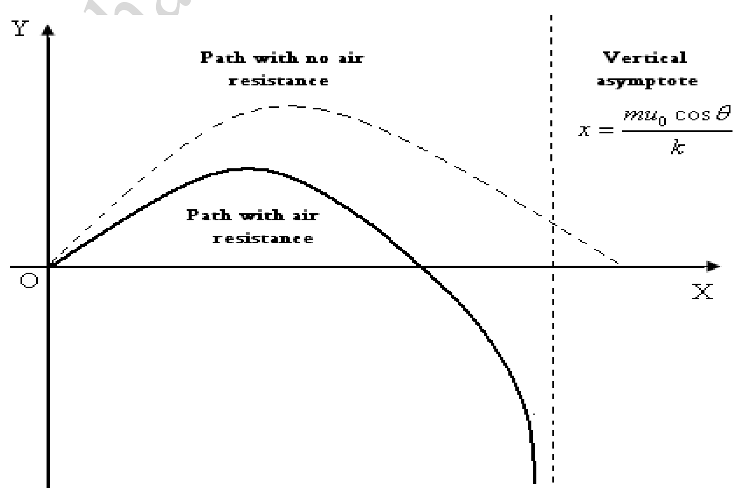

Figure-2.8: Effect of air resistance on projectile motion

2.4 Planetary Motion

In the case of projectiles, the path was a parabola, a two dimensional curve. But all these motions had one thing in common– the only forces causing motions were in a fixed direction. Gravity acts towards the centre of the earth, but for motion under gravity, the displacements are so small compared to the earth's radius, that the direction of the force could be considered to be constant. Now, here we introduce motion under a force directed towards a fixed point– that is called motion under central forces. Planetary motion is one of the examples of this type. In this case motion is in the same plane; however the direction of the force is no longer a constant.

Historically, celestial bodies have been of interest to mankind from time immemorial. There are records of Chinese, Indian and Greek astronomers, observations of celestial motion. Since, to us, it appears as if the sun moves round the earth, early men thought that earth was the centre of the universe, and all planets moved round the earth. However, in 1543, Copernicus, a Polish monk, proposed the 'heliocentric theory' in his book RevolutionibusOrbiumCoelestium. According to this theory, Sun was the centre of the universe, and that all planets moved round the sun in circular orbits.

Tycho Brahe, a Danish astronomer, made systematic observations of the heavenly bodies for about 25 years. These observations were so accurate that they used by navigators for a long time. Johannes Kepler, TychoBrache's assistant tried to find a circular orbit for Mars based on these observations, because the orbit of Mars was the most troublesome one. However, he could not fit a circular orbit for the data. In a flash of genius, he found he could fit an elliptical orbit. He also formulated the following laws:

Law 1: Every planet moves around the sun in an elliptical orbit with sun at one of its foci.

Law 2: The area swept out by the radius vector joining any planet to the sun, in given period of time, is always a constant.

Law 3: The square of the time Period in a planet is proportional to the cube of its semi-major axis.

Later on Sir Isaac Newton, the great English Mathematician used these empirical laws, and enunciated the universal law of gravitation. This states that the force of attraction between two bodies is inversely proportional to the square of the distance between them. He also showed that Kepler's laws can be deduced from his laws.

2.5 Newton's Laws of Gravitation

The law gives the forces of attraction or repulsion between any two bodies. Newton deduced this from Kepler's third law, first for the planets. He later realized that the same law holds for all bodies. This law can be stated as follows:

Everybody attracts every other body with a force, which is proportional to the masses of the bodies and inversely proportional to the square of the distances between the centres of the two bodies.

Suppose the two bodies having masses m_1 and m_2 are separated by distance r. then the force F of attraction between the bodies is given by

$$F \propto \frac{m_1 m_2}{r^2}$$

Consequently,

$$F = \frac{Gm_1m_2}{r^2} \qquad \ldots(2.34)$$

G is called the universal gravitational constant. This law has been verified experimentally– both in the laboratory and from other observations of heavenly bodies. Since $G = \frac{Fr^2}{m_1m_2}$, we see that G is the force of attraction between two unit masses separated by unit distance. Since the dimensions of force are MLT^{-2}, the dimensions of G are $M^{-1}\ L^3\ T^{-2}$ and its value is $6.67 \times 10^{-11}\ m^3/kg\ sec^2$ in SI systems of units.

Using Newton's law of gravitation, we can find the acceleration of anybody due to gravitational attraction of the earth. As we usually use the value 9.8 m/sec^2 for g, the acceleration due to gravity. This value can be actually derived by using this law.

Let m be the mass of a particle and M that of the earth. Let r be the radius of the earth and h, the height of the particle above earth's surface. Then, from Newton's laws of motion and the law of gravitation we get the following equation

$$\frac{GMm}{(r+h)^2} = mf \qquad \ldots(2.35)$$

where f is the acceleration induced on the body by the gravitational forces. If the particle is very close to earth, say less than 100 km, we can replace r + h by r and equation (2.35) reduces to

$$\frac{GM}{r^2} = f \qquad \ldots(2.36)$$

The acceleration f is called acceleration due to gravity and is denoted by g. Substituting the values of $G = 6.67 \times 10^{-11}\ m^3/kg\ sec^2$, $M = 5.97 \times 10^{24}$ kg and $r = 6.37 \times 10^6$ m, we can check that value of g is 9.8 m/sec^2.

The force mg acting on a body of mass m due to attraction of the earth and acting towards the centre of the earth is called the weight of the body.

Induce if m is the mass of any planet or satellite and r is its radius, and then equation (2.36) gives the acceleration due to gravity.

Exercise: Find the acceleration due to gravity on the Moon and Saturn with the given following data.

	Radius	Mass
Moon	1.738×10^6 m	7.35×10^{22} kg
Saturn	0.6×10^7 m	5.6782×10^{26} kg

Remark: As we have observed from equation (2.34) that the two bodies exert the same force on each other. Now, let us see how the apple falls towards the earth and not vice versa.

Let the masses of the earth and apple be m_e and m_a respectively, and let the distance between their centers be r. Then the force exerted by the earth on the apple is

$$F_{ea} = \frac{Gm_em_a}{r^2} \qquad \ldots(2.37)$$

If the acceleration induced in the two cases are f_e and f_a respectively, then by Newton's laws of motion,

$$F_{ea} = m_a f_a \quad \text{...(2.38)}$$

and

$$F_{ae} = m_e f_e \quad \text{...(2.39)}$$

From equations (2.37) and (2.38), we see that

$$F_{ea} = F_{ae} \quad \text{...(2.40)}$$

Therefore, from equations (2.39) and (2.40), $m_e f_e = m_a f_a$ and so

$$f_a = \frac{m_e}{m_a} f_e$$

Since the mass of the earth is very large compared to that of the apple, the value of $\frac{m_e}{m_a}$ is very large. So f_a is large compared to f_e. This means that the acceleration of the apple is very large compared to that of the earth. This is the reason for the apple falling towards the earth and not vice versa.

2.6 Escape Velocity

So far, we modeled the upward motion of a body under gravity by assuming that the gravity is constant. There also we saw that the body came down to the earth with same velocity with which it was projected. But what should be the velocity of projection if we want body to neither come back to the earth nor orbit the earth? To find this we shall model the problem again in this section. But, this time we will assume that the gravity is based on Newton's law. So, it will not be a constant. Using this model, we will find the escape velocity of a particle projected upwards.

Escape Velocity: The minimum velocity, which an object should have in order to overcome the earth's gravity and enter into space, is called escape velocity.

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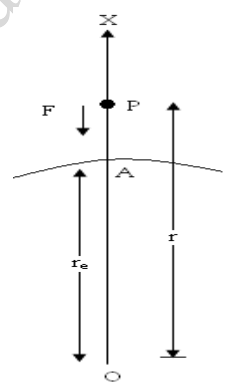

Figure-2.9: Projection for escape velocity

Formulation: Let us consider the motion of a particle of mass m projected vertically upwards. Let us begin by choosing an appropriate coordinate system. We choose the centre of the earth O as the origin and the liner joining O and the point of intersection A as the x-axis as shown in the Figure-2.9. Also, Let OA = r_e, and F stands for the force of gravity given by Newton's inverse square law.

Suppose at time t, the particle is at position P at a distance r from the centre of the earth. Let M be the mass of the earth, then by Newton's second law of motion,

$$F = m\,a$$
$$= m\frac{d^2r}{dt^2} = m\frac{d}{dt}\left(\frac{dr}{dt}\right) = m\frac{d}{dr}\left(\frac{dr}{dt}\right)\frac{dr}{dt}$$
$$= m\frac{dv}{dr}\frac{dr}{dt}$$

So,

$$F = mv\frac{dv}{dr}, \qquad \ldots(2.41)$$

Since $v = \dfrac{dr}{dt}$

On the other hand

$$F = -\frac{GMm}{r^2} \qquad \ldots(2.42)$$

By Newton's law of gravitation, the negative sign in the RHS of equation (2.42) is because we have chosen OX direction as the positive direction and F acts in the opposite direction. Substituting for F from equation (2.41), we have

$$mv\frac{dv}{dr} = -\frac{GMm}{r^2} \qquad \ldots(2.43)$$

which reduces to

$$v\frac{dv}{dr} = -\frac{GM}{r^2} \qquad \ldots(2.44)$$

which is the required model equation.
This is an ordinary, linear first order differential equation, which is in separable form. Solving this, we get

$$v^2 = \frac{2GM}{r} + const. \qquad \ldots(2.45)$$

Let the particles initial velocity (at t = 0) be v_i on the surface of the earth and let $r = r_e$ be the radius of the earth. Using these initial conditions in equation (2.45), we get

$$v_i^{\,2} = \frac{2GM}{r_e} + const.$$

Thus, $$v^2 = v_i^{\,2} + 2\left(\frac{GM}{r} - \frac{GM}{r_e}\right) \qquad \ldots(2.46)$$

Now, since $v^2 \geq 0$, equation (2.46) gives

$$v_i^{\,2} + 2\left(\frac{GM}{r} - \frac{GM}{r_e}\right) \geq 0 \qquad \ldots(2.47)$$

We know that, if the particle returns to the earth or goes into the orbit around the earth, r will remain bounded for all values of t. But, since we want neither, of this to happen, r must tend to infinity as t tends to infinity. Thus, letting t tends to infinity (and hence r tends to infinity) in equation (2.47), we have

$$v_i^{\ 2} - 2\frac{GM}{r_e} \geq 0$$

or

$$v_i \geq \sqrt{\frac{2GM}{r_e}} \qquad \ldots(2.48)$$

From equation (2.48), we see that, if the particle is to escape from the earth, its initial velocity, i.e. the velocity of projection, must be at least $\sqrt{\frac{2GM}{r_e}}$, so by definition

$$v_{escape} = \sqrt{\frac{2GM}{r_e}} \qquad \ldots(2.49)$$

If we substitute the values of G = 6.67 x 10^{-11} m^3/kg sec^2, M = 5.97 x 10^{24} kg and r_e = 6.37 x 10^6 m, then the escape velocity comes to be 11.18 km/sec. To get an idea how big it is, we can compare this velocity with the velocity of sound, which is 0.330 km/sec.

The formulation in equation (2.49) can be used to find the escape velocity on any other celestial body. We have to just plug in the mass and radius of that body in the formula for the escape velocity.

Exercise: Find out the escape velocity on the Moon and Saturn. (Hint, use the data already given above).

2.7 Central Forces – Basic Concepts

As we know, the planets move under the force of gravity, which is directed towards the sun, is an example of motion under a central force. Before, we proceed to model the planetary motion; we should equip ourselves with the velocities and accelerations of a particle moving under the influence of a central force.

A particle moving in a curve under a central force which is always directed towards a fixed point O, since the force is towards a fixed point, we will work in polar coordinates, with O as origin and OX as the initial line. The force diagram is given in the Figure-2.10. In this figure, F is the force acting on the particle towards O.

Let at time t, the particle be at P. Suppose the coordinates of P are (r, θ), that is $OP = r$ and $\angle XOP = \theta$. The only forces acting on P is towards PO.

Hence, while modeling the motion of planets and satellites, we shall write down the equations of motions along OP and in a direction perpendicular to OP. For this, we shall derive the acceleration along OP and perpendicular to OP, which are respectively called radial and transverse acceleration.

While studying projectile motion, we wrote down the equation of motion of two perpendicular directions OX and OY. We use the same principle here also with the only difference that the coordinate system used is the polar coordinate system. For determining the radial and transverse accelerations, we need radial and transverse velocities.

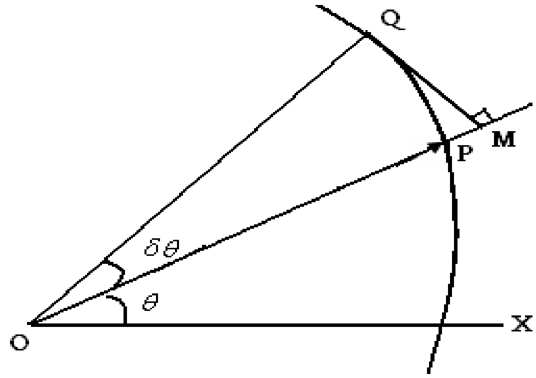

Figure -2.10: Foot of perpendicular drawn from Q to OP

2.8 Radial Velocity: Let the particle be at the points P and Q at times t and $t+\delta t$ respectively. Let M be the foot of the perpendicular drawn from Q to OP, as shown in above Figure-2.10. Then,

Average radial velocity =
$$\frac{\text{Displacement in radial direction in time } \delta t}{\delta t}$$

The displacement along OP direction when the particle is at P is OP. The displacement along OP direction when the particle is at Q is OM. Therefore,

Average radial velocity = $\dfrac{OM-OP}{\delta t}$

$$=\frac{OQ\cos\delta\theta-OP}{\delta t}$$

The radial velocity at P is

$$v_{radial}=\lim_{\delta t\to 0}\frac{OQ\cos\delta\theta-OP}{\delta t}$$
$$=\lim_{\delta t\to 0}\frac{r(t+\delta t)\cos\delta\theta-r(t)}{\delta t}$$

Expanding $r(t+\delta t)$ in a Taylor's series and neglecting the terms of the order $O(\delta t^2)$ and higher, we get

$$r(t+\delta t)=r(t)+\frac{dr(t)}{dt}\delta t$$

Therefore,

$$v_{radial}=\lim_{\delta t\to 0}\frac{1}{\delta t}\left[\left\{r(t)+\frac{dr(t)}{dt}\delta t\right\}\cos\delta\theta-r(t)\right]$$
$$=\lim_{\delta t\to 0}\frac{1}{\delta t}\left[r(t)(\cos\delta\theta-1)+\frac{dr(t)}{dt}\delta t\cos\delta\theta\right]$$
$$=r(t)\lim_{\delta t\to 0}\frac{(\cos\delta\theta-1)}{\delta t}+\frac{dr(t)}{dt}\lim_{\delta t\to 0}\cos\delta\theta \qquad \text{…(2.50)}$$

Note that $\delta\theta \to 0$ and $\frac{\delta\theta}{\delta t} \to \frac{d\theta}{dt}$ as $\delta t \to 0$. So,

$$\lim_{\delta t \to 0} \frac{(\cos\delta\theta - 1)}{\delta t} = \lim_{\delta\theta \to 0} \frac{\delta\theta}{\delta t} \lim_{\delta\theta \to 0} \frac{(\cos\delta\theta - 1)}{\delta\theta}$$

$$= \frac{d\theta}{dt} \lim_{\delta\theta \to 0} \frac{-\sin\delta\theta}{1} = 0 \qquad \ldots(2.51)$$

(By using L ' Hospital's rule)

Since $\cos\delta\theta \to 1$ as $\delta\theta \to 0$. So, from equation (2.50) and (2.51), we have

$$v_{radial} = \frac{dr}{dt} \qquad \ldots(2.52)$$

2.9 Transverse Velocity

Transverse velocity at P is the velocity in a direction perpendicular to OP.
Average transverse velocity = (Displacement in δt times in the transverse direction)/ δt

$$\frac{M}{\delta t} = (OQ sin\delta\theta)/\delta t$$
$$= r(t + \delta t)\, sin\delta\theta/\delta t$$

Hence the transverse velocity is given by

$$V_{tranverse} = \lim_{\delta t \sim 0} r(t+\delta t)\, sin\delta\theta/\delta t$$
$$= \lim_{\delta t \sim 0} r(t+\delta t) \sin\delta\theta/\delta\theta(\delta\theta/\delta t)$$
$$= \lim_{\delta t \to 0} r(t+\delta t) \lim_{\delta t \to 0} \frac{\sin\delta\theta}{\delta\theta} \lim_{\delta \to 0} \frac{\delta\theta}{\delta t}$$
$$= r\, d\theta/dt$$

Therefore,

$$v_{transverse} = r\frac{d\theta}{dt} \qquad \ldots(2.53)$$

Example 1: Find the radial and transverse velocities of a body moving in a circle.

Let the radius of the circle be 'a'. The equation of a circle of radius 'a', in polar coordinates, r = a. Since 'a' is constant,

$$v_{radial} = \frac{dr}{dt} = 0$$

$$v_{transverse} = r\frac{d\theta}{dt} = a\,\omega \qquad \ldots(2.54)$$

where ω is the angular velocity $\frac{d\theta}{dt}$ of the particle.

Radial Acceleration: Radial acceleration at P is given by
$(ACC)_{radial}$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \begin{bmatrix} \text{velocity in the OP direction at time } (t + \delta t) \\ - \text{ velocity in the OP direction at time } t \end{bmatrix}$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \begin{bmatrix} \text{velocity at Q in the OP direction} \\ - \text{ velocity at P in the OP direction} \end{bmatrix} \qquad \ldots(2.55)$$

Transverse Acceleration: Transverse acceleration at P is given by

$$(ACC)_{transvers} = \lim_{\delta t \to 0} \frac{1}{\delta t} \begin{bmatrix} \text{velocity along PP' at time } (t + \delta t) \\ - \text{velocity along PP' at time } t \end{bmatrix}$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \begin{bmatrix} \text{velocity along PP' at Q} \\ - \text{velocity along PP' at P} \end{bmatrix} \qquad \ldots(2.56)$$

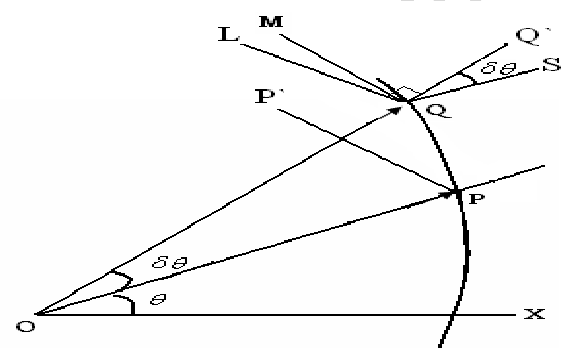

Figure-2.11: Transverse velocity of the particle

From equation (2.52) and (2.53), velocity of the particle at P in the radial, i.e. in the OP direction is $\frac{dr}{dt}$ and the velocity of the particle at P in the transverse, i.e. PP′ direction $r\frac{d\theta}{dt}$.

Similarly, we can write the velocity of the particle at Q in the radial, i.e., along QQ' and transverse i.e., along QL directions as

$$v_1 = \frac{dr(t + \delta t)}{dt}, \qquad \ldots(2.57)$$

and

$$v_2 = r(t + \delta t)\frac{d\,\theta(t + \delta t)}{d\,t} \qquad \ldots(2.58)$$

respectively. But, from equation (2.55), for calculating the radial acceleration at P, we need the velocity at Q in the OP direction, i.e., in the QS direction. (Here we have drawn QS parallel to OP.) Similarly, from equation (2.56), for the transverse acceleration we need the velocity at Q in the QM direction (Here we have drawn QM parallel to PP'.) As we can see, the QS and QM directions are got by rotating the QQ' and QL directions by $\delta\theta$ clockwise.

Let us write $v_1^{\prime}$ and $v_2^{\prime}$ for the velocities in QS and QM directions, i.e., the directions of OP and PP'. $v_1^{\prime}$ and $v_2^{\prime}$ can be obtained from the equations

$$v_1^{\prime} = v_1 \cos\alpha + v_2 \sin\alpha$$
$$v_2^{\prime} = -v_1 \sin\alpha + v_2 \cos\alpha$$

Put $\alpha = -\delta\theta$, since we are rotating the coordinates in the clockwise direction. Substituting the values v_1 and v_2 from equation (2.57) and equation (2.58) in the above equations, we get

$$v_1^{\prime} = \frac{dr(t+\delta t)}{dt}\cos(-\delta\theta) + r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\sin(-\delta\theta)$$
$$= \frac{dr(t+\delta t)}{dt}\cos(\delta\theta) - r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\sin(\delta\theta) \qquad \ldots(2.59)$$

Similarly, substituting the values from equation (2.57) and (2.58), to the above equations, we have

$$v_2^{\prime} = \frac{dr(t+\delta t)}{dt}\sin(\delta\theta) + r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\cos(\delta\theta) \qquad \ldots(2.60)$$

From equation (2.59), the component of the velocity of the particle at Q in the OP direction is

$$\frac{dr(t+\delta t)}{dt}\cos(\delta\theta) - r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\sin(\delta\theta)$$

From equation (2.60), the velocity at Q in the PP' direction is

$$\frac{dr(t+\delta t)}{dt}\sin(\delta\theta) + r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\cos(\delta\theta) \qquad \ldots(2.61)$$

We now have all necessary information for calculating the acceleration in the radial and transverse directions. Let us now calculate them from one by one; first we want to calculate radial acceleration.

$$(ACC)_{radial} = \lim_{\delta t \to 0}\frac{1}{\delta t}\left[\frac{dr(t+\delta t)}{dt}\cos(\delta\theta) - r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\sin(\delta\theta) - \frac{dr(t)}{dt}\right]$$

Expanding $\frac{dr(t+\delta t)}{dt}$ (only in the $\frac{dr(t+\delta t)}{dt}(\cos\delta\theta)$) using Taylor series, we get

$$(ACC)_{radial} = \lim_{\delta t \to 0}\frac{1}{\delta}\left[\left\{\frac{dr(t)}{dt} + \frac{d^2 r(t)}{dt^2}\delta t\right\}\cos\delta\theta - r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\sin\delta\theta - \frac{dr(t)}{dt}\right]$$
$$= \frac{d^2 r(t)}{dt^2}\lim_{\delta t \to 0}\cos\delta\theta - \lim_{\delta t \to 0} r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\frac{\sin\delta\theta}{\delta\theta}\frac{\delta\theta}{\delta t} + \frac{dr(t)}{dt}\lim_{\delta t \to 0}\frac{(\cos\delta\theta - 1)}{\delta t}$$
$$\ldots(2.62)$$

From equation (2.51), it follows that the last term in equation (2.62) is zero. The first term tends to $\frac{d^2 r(t)}{dt^2}$ and the second term tends to $r(t)\left(\frac{d\theta(t)}{dt}\right)^2$ as $\delta t \to 0$.

Therefore,

$$(ACC)_{radial} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad \ldots(2.63)$$

Next, we calculate transverse acceleration:

The velocity at P in the PP' direction (i.e. the transverse direction at P) is $r(t)\frac{d\theta(t)}{dt}$. So using equation (2.57) and (2.62), we get

$$(ACC)_{transverse} = \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\frac{dr(t+\delta t)}{dt}\sin(\delta\theta) + \left(r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}\right)\cos(\delta\theta) - r(t)\frac{d\theta(t)}{dt}\right]$$

Expanding $r(t+\delta t)\frac{d\theta(t+\delta t)}{dt}$ in Taylor series, we get

$$(ACC)_{transverse} = \lim_{\delta t\to 0}\frac{1}{\delta t}\frac{dr(t+\delta t)}{dt}\frac{\sin\delta\theta}{\delta\theta}\frac{\delta\theta}{\delta t}$$

$$+\left[\lim_{\delta t\to 0}\frac{1}{\delta t}\left(r(t)\frac{d\theta(t)}{dt} + \frac{d}{dt}\left(r(t)\frac{d\theta}{dt}\right)\delta t\right)\cos(\delta\theta) - \left(r(t)\frac{d\theta}{dt}\right)\right]$$

Regrouping the terms, we get

$$(ACC)_{transverse} = \lim_{\delta t\to 0}\frac{1}{\delta t}\frac{dr(t+\delta t)}{dt}\frac{\sin\delta\theta}{\delta\theta}\frac{\delta\theta}{\delta t}$$

$$+\frac{d}{dt}\left(r(t)\frac{d\theta}{dt}\right)\lim_{\delta t\to 0}\cos(\delta\theta) + r\frac{d\theta(t)}{dt}\lim_{\delta t\to 0}\frac{(\cos\delta\theta - 1)}{\delta t}$$

$$= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{d}{dt}\left(r(t)\frac{d\theta}{dt}\right)$$

$$= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= \frac{1}{r}\left\{r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\right\}$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

So, $$(ACC)_{transverse} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \qquad \ldots(2.64)$$

Example 1: Find the radial and transverse accelerations of a particle moving in a circle.

We know that for the circle with constant radius 'a', we have

$$(ACC)_{radial} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -a\omega^2 \qquad \ldots(2.65)$$

and

$$(ACC)_{transverse} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = a\frac{d^2\theta}{dt^2} = a\frac{d\omega}{dt} \qquad \ldots(2.66)$$

Also, if we write v for the transverse velocity, from equation (2.54), we have

$$v = -a\frac{d\theta}{dt} = -a\omega \quad \text{or} \quad \frac{d\theta}{dt} = \frac{v}{a}.$$

So from equation (2.65)

$$(ACC)_{radial} = -\frac{v^2}{a} \qquad \ldots(2.67)$$

We now have all the information necessary for writing down the equation of a particle moving under a central force.

2.10 Equation of motion of a particle moving under a central force

Consider a particle of mass m, moving under a central force F that acts always along the line OP as shown in Figure-2.11. The acceleration along the radial direction is given by equation (2.63), writing down Newton's laws of motion in the radial direction, we obtain

$$m\frac{d^2r}{dt^2} - mr\left(\frac{d\theta}{dt}\right)^2 = F \qquad \ldots(2.68)$$

Using the expression for transverse acceleration given in equation (2.64) and writing down the Newton's laws of motion in the transverse direction, we have

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

i.e., $$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \qquad \ldots(2.69)$$

From equation (2.69), we get

$$r^2\frac{d\theta}{dt} = const. \qquad \ldots(2.70)$$

The later equation lends itself to physical interpretation. Consider the particle at P and Q at times t and $t + \delta t$ respectively as shown in Figure-2.12.

Then, Area OPQ = δA = Area swept by t in time δt

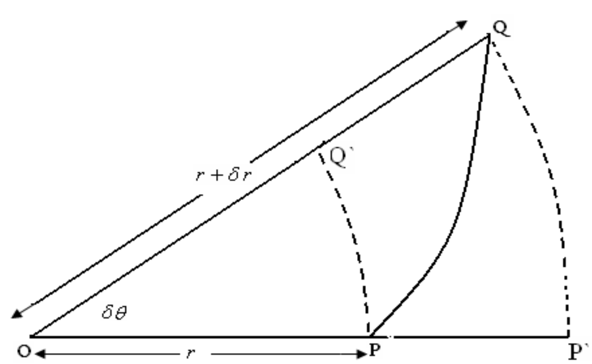

Figure-2.12: Transverse Acceleration

Also, we have

$$Area OPQ' \le \delta A \le Area OP'Q$$

But we know that area of a sector of a circle with angle θ and radius r is $\frac{1}{2}r^2\theta$. Hence

$$\frac{1}{2}r^2\delta\theta \le \delta A \le \frac{1}{2}(r+\delta r)^2\delta\theta$$

i.e.,

$$\frac{1}{2}r^2\frac{\delta\theta}{\delta t} \le \frac{\delta A}{\delta t} \le \frac{1}{2}(r+\delta r)^2\frac{\delta\theta}{\delta t} \qquad \dots (2.71)$$

As δt tends to zero, this inequality reduces to the following equality

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \qquad \dots (2.72)$$

From equation (2.70) and (2.72), it follows that

$$\frac{dA}{dt} = constant \qquad \dots (2.73)$$

Equation (2.73) shows that the rate at which the radius vector joining O and P sweeps out area is a constant. Let us call the rate of change of area with respect to time as areal velocity. So equation (2.73) says that P has a constant areal velocity. This means that the particle sweeps out equal areas in equal intervals of time.

2.11 Modeling Planetary Motion (Keplar's Laws)

Formulation: Consider a body of mass m moving around a body of mass M under gravitational attraction. (In the case of planetary motion, M and m are the masses of the sun and planet respectively). We will make the following assumptions:

i) The two bodies are homogeneous spheres. The assumptions will enable us to replace the planets by point masses.

ii) The inverse square law gives the force of attraction between the two bodies.

iii) The gravitational attraction of the other planets and satellites is negligible. This will simplify our model considerably and we can treat the problem as a central force problem.

Equation (2.68) and (2.70) gives us equation of motion of a particle moving under a central force F. Since gravitational force is a central force, we can write down the equations in the radial and transverse directions as

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{GMm}{r^2} = -\frac{m\mu}{r^2} \qquad \dots (2.74)$$

$$m\frac{1}{r}\frac{d}{dt}\left(\frac{1}{r}\frac{d\theta}{dt}\right) = 0 \qquad \dots (2.75)$$

In equation (2.75), r is the distance between their centres and G is the universal gravitational constant. We have written μ for GM.

From equation (2.75), we get

$$r^2 \frac{d\theta}{dt} = constant = h \text{ (say)}$$

Hence

$$\frac{d\theta}{dt} = \frac{h}{r^2} \qquad \ldots(2.76)$$

Let $u = \frac{1}{r}$ or $r = \frac{1}{u}$

Hence

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{1}{u}\right).\frac{h}{r^2}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}hu^2$$

$$= -h\frac{du}{d\theta} \qquad \ldots(2.77)$$

Also,

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right)$$

$$= \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$= -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}\frac{h}{r^2}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} \qquad \ldots(2.78)$$

Substituting, the values of $\frac{d\theta}{dt}$ and $\frac{d^2r}{dt^2}$ from equation (2.76) and equation (2.78) respectively in equation (2.74), we obtain

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - r\frac{h^2}{r^4}\right) = -\frac{\mu m}{r^2}$$

i.e.,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \qquad \ldots(2.79)$$

This is the model equation of motion of two bodies.

Solution and Interpretation

Note that equation (2.79) is a linear second order, non-homogeneous, ordinary differential equation with constant coefficients. The complete solution of equation (2.79) is

$$\frac{1}{r} = u = A\cos(\theta - \alpha) + \frac{\mu}{h^2} \qquad \ldots(2.80)$$

On rearranging the terms,

$$\frac{h^2}{\mu r} = A\frac{h^2}{\mu}\cos(\theta - \alpha) + 1 \qquad \ldots(2.81)$$

Let us write

$$\frac{h^2}{\mu} = \ell \quad \text{and} \quad e = \frac{Ah^2}{\mu} \qquad \ldots(2.82)$$

We get

$$\frac{\ell}{r} = 1 + e\cos(\theta - \alpha) \qquad \ldots(2.83)$$

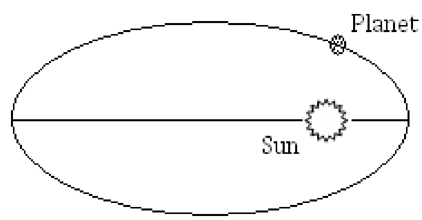

Figure-2.13: Geometrical Interpretation of Keplars first law

This is the equation of the conic with semi-latus rectum ℓ and eccentricity e. So, we have shown that that the orbit must be a conic section. Recall that the conic given by equation (2.83) is an ellipse, parabola or a hyperbola, according as e < 1, e = 1, e > 1 respectively.

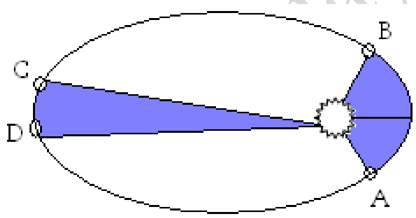

Figure-2.14: Keplars Second Law

Remark: For a planet to remain in the solar system, the orbit has to be ellipse.

If the orbit is a parabola or the hyperbola it passes out the solar system, never to come back. So, orbit has to be an ellipse in the case of planets. Thus, we have now obtained Kepler's first law as shown in Figure-2.13. The exact nature of the orbit depends on the values of the constants and the constants can be obtained from observations. Table 1 gives the eccentricities of the orbits of the various planets.

Table-1

Planet	e
Mercury	0.2056
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0483
Saturn	0.0560
Uranus	0.0461
Neptune	0.0100
Pluto	0.2484
Moon	0.0550

From equation (2.72) and (2.73), we have

$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$ = Constant. This is Kepler's second law, which is illustrated in the Figure-2.14 above. So even though r and $\frac{d\theta}{dt}$ vary with time, $\frac{1}{2} r^2 \frac{d\theta}{dt}$ is constant. So, when r is small, $\frac{d\theta}{dt}$ is large and when r is large $\frac{d\theta}{dt}$ is small. This means that the planets move faster when they are close to the sun (r is small.) and slower when they are far from the sun (r is large).

Now area of an ellipse of semi-major and semi-minor axes a and b is $\pi a b$. Also, we have shown above that the areal velocity is a constant = h/2. Let us call the time taken by the planet to complete one orbit as time period of the planet, or simply period of a planet, and is denoted by T.

Therefore,

$$\frac{h}{2} = \text{Areal Velocity} = \frac{\text{Area of the ellipse}}{\text{time period}}$$

$$= \frac{\pi a b}{T} \qquad \ldots(2.84)$$

From equation (2.84), we have

$$T = \frac{\pi a b}{h/2} \qquad \ldots(2.85)$$

Recall that $\ell = \frac{b^2}{a}$. So from equation (2.82), we get $h = \sqrt{\frac{\mu}{a}}$. Substituting this value for h in the expression for T given in equation (2.82), we get

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

or

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\mu} \qquad \ldots(2.86)$$

Since we used μ for GM, where G is the gravitational constant and M is the mass of the sun. So, the right hand side of equation (2.86) is constant, and therefore independent of the planet under consideration. This is precisely the third law of Kepler.

Exercise: Find the semi-major axis, semi-minor axis and apsidal distance of the earth, assuming its period to be 365.25 days.

(Hint the eccentricity of the earth's orbit to be 0.0167 and the mass of the sun to be 2.599 x 10^{38} kg.)

Keplar's law's lead to Newton's law of gravitation

Suppose the planet is moving in an ellipse and is at P at time t. At this time it has the following acceleration:

$$Radial \ \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = f_{radial} \qquad \text{…(2.87)}$$

$$Transverse \ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = f_{transverse} \qquad \text{…(2.88)}$$

But $\frac{1}{2}r^2\frac{d\theta}{dt}$ is the areal velocity which is constant = h, say, according to Kepler's second law. Hence, the transverse acceleration is zero. This means that the planet has only a radial acceleration S. but this along with the Newton's laws of motion implies that the only force on the planet is a radial force. Also, from the equation

$$r^2\frac{d\theta}{dt} = h$$

we get

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2 \qquad \text{…(2.89)}$$

Now,

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -h\frac{du}{d\theta} \qquad \text{…(2.90)}$$

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{…(2.91)}$$

Using this in equation (2.87), we get

$$f_r = -h^2u^2\frac{d^2u}{d\theta^2} - u^3 \qquad \text{…(2.92)}$$

$$= -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \qquad \text{…(2.93)}$$

Suppose the equation to the orbit is

$$\frac{\ell}{r} = 1 + e\cos\theta$$

i.e., $\ell u = 1 + e\cos\theta$ …(2.94)

Differentiating with respect to θ,

$$\ell\frac{du}{d\theta} = -e\sin\theta$$

and

$$\ell\frac{d^2u}{d\theta^2} = -e\cos\theta \qquad \text{…(2.95)}$$

Equation (2.94) and (2.95) gives

$$\ell\left(\frac{d^2u}{d\theta^2} + u\right) = 1 \qquad \text{…(2.96)}$$

Combining equations (2.93) and (2.95), we get

$$f_r = -\frac{h^2 u^2}{\ell} = -\frac{h^2}{\ell}\frac{1}{r^2} \qquad \ldots(2.97)$$

Which says that the radial acceleration is inversely proportional to the square of the distance.

We shall have to prove that $\frac{h^2}{\ell}$ is same for all planets.

Now, $$T = \pi \frac{ab}{h/2}$$

So that $$\frac{T}{a^3} = \frac{4\pi^2 b^2}{h^2 a} = \frac{4\pi^2 \ell}{h^2}$$

But by equation (2.86) [Keplar's third law]; $\frac{T}{a^3}$ is a constant, which implies that $\frac{h^2}{\ell}$ is a constant. Thus, Newton's law of gravitation has been deduced from Keplar's laws of planetary motion.

Exercise: The moon describes a circular orbit of radius 3.8 x 10^5 km about the earth in 27 days and the earth describes a circular orbit of radius 1.5 x 10^8 km round the sun in 365 days. Compare the mass of the sun in terms of the mass of the earth.

Solution: Let M_m , V_m be the mass and the velocity of the moon, while R_m be the distance of the moon from the earth. Then equating the centripetal force acting on the moon with the gravitational force due to the earth, we obtain

$$\frac{M_m V_m{}^2}{R_m} = \frac{GM_e M_m}{R_m{}^2}$$

or $$V_m^2 = \frac{GM_e}{R_m}$$

Similarly for the earth round the sun, we obtain

$$V_e^2 = \frac{GM_S}{R_e}$$

From the above two equations, on eliminating G, we obtain

$$M_S = \frac{M_m R_e V_e{}^2}{R_m V_m}$$

Substituting the values, we obtain

$$M_S = 33.66 \times 10^5 M_e$$

Exercise: Estimate the mass of the sun, assuming the orbit of the earth around the Sun to be a circle. The distance between the Sun and the earth is $1.49 \times 10^{11} m$, and $G = 6.67 \times 10^{-11} m^3 kg^{-1}$.