INTRODUCTION TO MATHEMATICAL ECOLOGY WITH SPECIAL REFERENCE TO POPULATION DYNAMICS

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submitted by

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CERTIFICATE

This is to certify that the dissertation entitled, **"Introduction to Mathematical Ecology with Special Reference to Population Dynamics"** *being submitted by the students with the enrollments 21068120007, 21068120042, 21068120050, 21068120051 to the* ***Department of Mathematics, University of Kashmir, Srinagar,*** *for the award of Master's degree in Mathematics, is an original project work carried out by them under my guidance and supervision.*

The project dissertation meets the standard of fulfilling the requirements and regulations related to the award of the Master's degree in Mathematics. The material embodied in the project dissertation has not been submitted to any other institute, or to this university for the award of Master's degree in Mathematics or any other degree.

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Abstract

The project dissertation is comprising of three chapters that focuses on *"Basic Mathematical Ecology with Special Reference to Population Dynamics"*.

The first chapter provides a brief idea on mathematical ecology. The second chapter explains a survey work with reference to prey predator model.

In the last chapter we have gone through the basic mathematical modeling of Population dynamics.

CONTENTS

1

General Introduction to Mathematical Ecology

1.1 Introduction

The word 'ecology' has attracted attention of scientists and philosophers from the early ages of human civilization. It was first defined by the German biologist Ernst Haeckel in 1869. According to him ecology is the science of interrelation between living organisms and their environment. The word 'ecology' owes its origin to the Greek word 'Oikos' meaning 'house' or 'place to live' [1].

Ecology is essentially a mathematical subject which deals with the increase and fluctuations of population (e.g. plant population, animal population or other organic population). The mathematical study of the problems in ecology is not of recent origin in fact, Lotka (1924) and Voltera (1926) were early pioneer developing foundation work in this field.

In the field of ecology, scientists are confronted with the dynamics of nature, in terms of population growth or decline in a large number of plant and animal species. Given the strong influence of man and nature and the entire living world, it is necessary to apply mathematical models that will evaluate the impact of the environment on population of some species. Knowledge about the environmental impact on living organisms may contribute to their preservation [2]. Species are not similarly distributed across the Earth. The serious decrease in species number increased the urgency to understand species distribution in order to develop effective conservation strategies [3]. An understanding of the relationship between species and the environment is of great importance. The utilization of insect as bio indicators is limited to their habitat type. The odonates are freshwater invertebrates and are often used as ecological indicators of habitat quality [5]. Some studies have pointed out that odonates may serve as an indicator for changes in landscapes. But, their reaction to environmental conditions innumerous areas of the world is unknown. Knowledge about the environmental impact on odonates may contribute to their preservation. A substantial number of studies have focused on the impact of temperature and precipitation on distribution odonatas. Temperature increments may encourage the development of odonata species ranges and lead to increments in local biodiversity in northern latitudes. Latitudinal gradients in species richness are observed for a wide range of taxonomic groups. Spatial patterns in species richness can be described as the result of several mechanisms. Among the factors crucial for the impact on the species number of some area, the most dominant are: altitude, energy availability, climate, habitat heterogeneity, and disturbance. No previous studies have addressed the effects of climate and habitat parameters on odonates in Serbia. In this study, factors that influence number of species in Serbia were examined. The effects of temperature, precipitation and altitude on the number of odonates species were investigated. So, the potential of odonates to serve as indicators of climate effects on freshwater systems of this region was evaluated [6].

1.2 Interaction among two species

There may exist various types of interactions between the two different species living in the same habitat, these interactions are as follows:

(i) When none of the population is affected by the association with other, such type of interaction is called Neutralism.

(ii) When both the species actively inhabit the growth of each other, the type of interaction is Mutual Inhibition Competition.

(iii) When both the species compete for a common source of food, they adversely affect each other if the supply of that food is very limited. This type of interaction is Resource Use Competition.

(iv) When the growth of one species is inhibited by association with a second species and that of the second is not at all influenced by the first one, the type of interaction is called Amensalism.

(v) When one population adversely affects the other by direct attack and is dependent on the other, the type of interaction is Predation or Parasitism.

(vi) When one species is benefited by association with second species and second species remains unaffected, the type of interaction is called commensalism.

(vii) When both populations benefit by the association but the relations between them are not obligatory, the type of interactions is called Proto

Cooperation.

(viii) When growth and survival of both the species are strengthened and neither of them can survive under natural conditions without the other, the type of interactions is called Mutualism.

Remark 1:

a) If the growth rate of one population is decreased and the other increased, the population is said to be in predator-prey situation.
b) If the growth rate of each population is decreased, then it is competition.
c) If each population's growth rate is enhanced, then it is called mutualism or symbiosis.

Remark 2:

The best-known models of the phenomena involving predation and competition are given by Lotka-Voltera equations, which were derived independently by Lotka in 1925 in the United States and by Voltera in 1926 in Italy. Lotka an American biophysicist is remembered mainly for his formulation of the Lotka Voltera equations. Vit-Voltera gave his theory of interacting species at the time of First World War.

1.3 Competing Species

The term "competition" between two living organisms refers to the interaction between them when they strive for the same thing. There may be two types of competition indirect or resource competition and direct or interference competition. Resource competition occurs when a number of organisms (of the same or of different species) utilize common resources that are in short supply, interface occurs when the organisms seeking a resource harm one another in the process, even if the resource is not in short supply. These competitions or interactions between the populations of two or more species adversely affect their growth and survival. The tendency for competition to bring about an ecological separation is closely related. Otherwise similar species is known as the "Competitive exclusion principle".

Remark: The competition may be inter-specific (between two or more different species) or intra-specific (between members of the same species). Here we restrict ourselves to the study of populations of two species, which are competing for a common resource (food, space, light etc.). Competition occurs over resources, and a variety of resources may become the Centre of the competitive interactions. For plants, light, nutrients and water may be important resources, but plants may compete for pollinators or for attachment sites. Water, food and mates are possible sources for competition for animals. Competition for also occurs in some animals and may involve many types of specific requirements such as nesting sites, wintering sites, or sites that are safe from predators. Thus, resources are diverse and complex.

1.4 Mutualism or Symbiosis

There are many examples where the interaction of two or more species is positive and have favorable results for both species. The advantages of all mutualism or symbiosis often play the critical role in promoting and even maintaining such species. In other words, Symbiotic relationships are the close associations formed between pairs of species. They come in a variety of forms, such as parasitism (where one species benefits and the other is harmed) and commensalism (where one species benefits and the other is neither harmed nor helped).

Mutualism is a type of symbiotic relationship where all species involved benefit from their interactions. While mutualism is highly complex, it can be roughly broken down into two types of relationship. In some cases, the species are entirely dependent on each other (obligate mutualism) and in others, they derive benefits from their relationship but could survive without each other (facultative mutualism Plant and seed dispersal) is one example [7]. And other examples are given below: -

Woolly bats and pitcher plants

(Woolly bats are known to roost in Nepenthes hemsleyana). Pitcher plants are carnivores that use nectar at the rim of their tube-like structure to attract prey such as insects and small vertebrates. A slippery substance at the rim causes these animals to fall into the digestive juices contained in the plant's equivalent of a stomach. While you might think it would be prudent for animals to avoid these plants where possible, some bats voluntarily clamber inside them. Woolly bats are known to roost in Nepenthes hemsleyana, a tropical pitcher plant found in Borneo. While the bat gets a hidey-hole to rest in, the plant benefits by catching the guano (faces) that the little mammal produces. This provides the plant with the nutrients it needs to survive.

A similar relationship occurs between tree shrews and another Bornean pitcher plant, Nepenthes lowii. The shrews climb onto the pitcher's rim to feed on the nectar. In return, with the plant's hollow body acting a bit like a toilet bowl, the shrews drop their nutritional faces into the plant's stomach [8].

1.5 Overview of Mathematical Modeling

1.5.1 Introduction to Models

If we were to ask several people for example of models, we would get a variety of responses which might include mathematical equations, toy, trains, photo type, cars, etc. What these very different objects have in common is that they are representation of reality. That is, the mathematical equation is a representation of growth of population, the toy train is the representation of a real train and the phototype is representation of a future car. Therefore, a model is a purposeful representation of reality.

In other words, mathematical model is a translation of a real life problem in the world into a mathematical description.

1.5.2 Simple Situations Requiring Mathematical Modelling

Consider the following problems:

(i) Find the height of a tower, say the Qutab Minar at New Delhi or the leaning tower at Pisa (without climbing it).

(ii) Find the width of a river or a canal (without crossing it).

(iii) Find the mass of the Earth (without using a balance).

(iv) Find the temperature at the surface or at the Centre of the Sun (without taking a thermometer there).

(v) Estimate the yield of wheat in India from the standing crop (without cutting and weighing the whole of it).

(vi) Find the volume of blood inside the body of a person (without bleeding him to death).

(vii) Estimate the population of China in the year 2000 A.D. (without waiting till then).

(viii) Find the time it takes a satellite at a height of 10,000Kms, above the Earth's surface to complete one orbit (without sending such a satellite into orbit).

(ix) Find the effect on the economy of 30 per cent reduction in income-tax (without actually reducing the rate).

(x) Find the gun with the best performance when the performance depends on ten parameters, each of which can take 10 values (without manufacturing 1010 guns).

(xi) Estimate the average life span of a light bulb manufactured in a factory (without lighting each bulb till it gets fused).

(xii) Estimate the total amount of insurance claims a company has to pay next year (without waiting till the end of that year) All these problems and thousands of similar problems can be and have been solved through mathematical modeling.

1.5.3 Mathematical Modelling

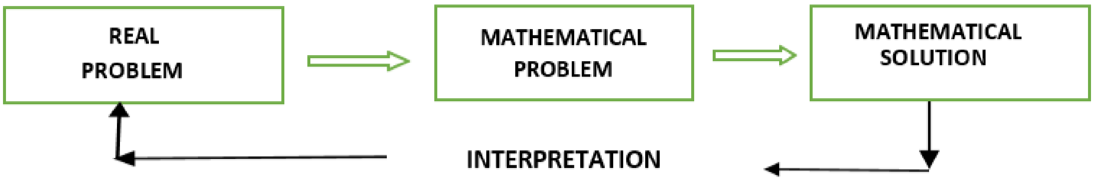
Mathematical modelling is a technique of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the language of real world.

Mathematical modelling is described as illustrating real-life problems in mathematical terms and expressions. It is usually simplified in the form of equations. Furthermore, it could easily identify answers to those problems by utilizing such equations. It will also help to discover a whole new variety of features concerning the problems.

While modeling, the observer’s perspective is very important. One should be able to see the models through their mind. Engineers and scientists use such techniques to model and design future technologies. Along with this process, prototypes are commonly used. A prototype is a compact model of an actual working model. Prototypes are used on all occasions where there is a need for testing or analyzing a model without affecting or damaging the actual one [1].

1.5.4 THE TECHNIQUES OF MATHEMATICAL MODELING

Mathematical modelling essentially consists of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the language of the real world as shown by the figure given below:



(Figure A)

This is expressed graphically by saying that catch hold a real world problem in teeth, dive into the mathematical ocean, swim there for some time and we come out to the surface with the solution of the real world problem. Alternatively, we may say that we soar high into the mathematical atmosphere along with the problem, fly there for some time and come down to the earth with the solution.

A real world problem, in all its generality can seldom be translated into a mathematical problem and even if it can be so translated, it may not be possible to solve the resulting mathematical problem. As such it is quite often necessary to 'idealize' or 'simplify' the problem or approximate it by another problem which is quite close to the original problem and yet it can be translated and solved mathematically. In this idealization, try to retain all the essential features of the problem giving up those features which are not very essential or relevant to the situation that is investigating.

Sometimes the idealization assumptions may look quite drastic. Thus for considering the motions of planets, consider the planets and sun as point masses and neglect their sizes and structures. Similarly, for considering the motion of a fluid, so treat it as a continuous medium and neglect its discrete nature in terms of its molecular structure. The justification for such assumptions is often to be found in terms of the closeness of the agreement between observations and predictions of the mathematical models [9].

1.5.5 Formulation of a Mathematical Model

The formation of mathematical model contains three main steps:

i. **Stating the problem**: First of all, we understand the natural phenomena of the problem and make a description which contains the content to the problem drawn from real world and then stating the problem within this content.
ii. **Identifying the relevant factors**: After stating the problem we first identify that which quantities and relations are important and which are not or can be ignored. If some factors having no effect on the process of the problem, then we neglect them. Therefore, take a relevant factor which give the better application to the problem.
iii. **Mathematical description**: After identifying the relevant factors, we assign a suitable mathematical entity to each factor, that is it should be a variable or a function etc. Each relation between the factors should be represented by an equation, inequality or other suitable mathematical assumption.

Solution of a Mathematical Model

In the formulation of the model, obtain an equation or a set of equations or inequalities. On solving these equations, therefore a mathematical solution will be obtained.

Interpretation of the solution

Since a model is a simplified representation of a real world problem. So that we find the theoretical conclusion by the solution of the model. If the obtained results are interpreted physically and the model gives reasonable answers, then we say that the model is good. If the model is not accurate enough, then we try to identify the shortcomings and remove them by forming new mathematical formulation.

1.5.6 Types of Models

According to the nature of the models, mathematical models can be classified into the four types:

i. Linear or Non –Linear
ii. Static or Dynamic

iii. Discrete or Continuous
iv. Deterministic or Stochastic

(i) **Linear and Non-Linear:** A model is said to be linear or non-linear according as the mathematical equation is linear or nonlinear.

If the model equation is of the type;

$$\frac{dN}{dt} = \lambda N, \quad \lambda > 0 \qquad \text{...... (1)}$$

then the model is linear and equation (1) gives a models for population growth.

If the model equation is

$$\frac{dN}{dt} = -\lambda N, \quad \lambda > 0 \qquad \text{....... (2)}$$

then the equation (2) gives the models for radioactive decay.

Similarly, if the model equation is of the type;

$$\frac{dN}{dt} = \lambda N(c - N), \quad \lambda > 0.\ C > 0$$

Where N is the size of the population and λ ,C are the constants of proportionality, then the model is said to be non-linear because the power of N is not all one.

(ii) **Static or Dynamic model:** If in a model, the mathematical equations are independent of time, then the model is said to be static. The fluid flowing through a rigid diverging tube is an example of static model.
On the other hand, if the time plays a very important role with the variables or relations describing the model changing with time, then the model is said to be **dynamic**.
Most of the real life problem, e.g. the population growth model, Bacterial growth, rocket launching model are the examples of dynamic models.

(iii) **Discrete or Continuous models:** A model is said to be discrete if the dependent variable takes the discrete value of the independent variables. In this model the mathematical equations are taken as difference equations.

On the other hand, if the model is based on continuous variable, then it is called continuous model. Most of the continuous models result in differential equations either ordinary or partial.

(iv) Deterministic or Stochastic Models: A model is said to be deterministic if the values assumed by the variable or the change to variable are predictable with certainly.

For example, the model for the motion of a simple pendulum is deterministic. On the other hand, a model is said to be stochastic, if the values assumed by the variables or changes to the variables are not predictable with certainly.

For example, if a rubber ball is dropped from a given height and measures the height of the bounce with sufficient accuracy it will be found that if the same process is repeated many times the height of the bounces is not the same every time [1].

1.5.7 SOME CHARACTERISTICS OF MATHEMATICAL MODELS:

a. **Realism of models**: We want a mathematical model to be as realistic as possible and to represent reality as closely as possible. However, if a model is very realistic, it may not be mathematically tractable. In making a mathematical model, there has to be a trade-off between tractability and reality.
b. **Hierarchy of models**: Mathematical modelling is not a one-shot affair. Models are constantly improved to make them more realistic. Thus for every situation, we get a hierarchy of models, each more realistic than the preceding and each likely to be followed by a better one.
c. **Relative precision of models**: Different models differ in their precision and their agreement with observations.
d. **Robustness of models**: A mathematical model is said to be robust if small changes in the parameters lead to small changes in the behavior of the model. The decision is made by using sensitivity analysis for the models.
e. **Self-consistency of models**: A mathematical model involves equations and inequations and these must be consistent, e.g. a model cannot have both $x + y > a$ and $x + y < a$. Sometimes the inconsistency results from inconsistency of basic

assumptions. Since mathematical inconsistency is relatively easier to find out, this gives a method of finding inconsistency in requirements which social or biological scientists may require of their models. A well-known example of this is provided by Arrow's Impossibility Theorem.

f. **Oversimplified and overambitious models**: It has been said that mathematics that is certain does not refer to reality and mathematics that refers to reality is not certain. A model may not represent reality because it is oversimplified. A model may also be overambitious in the sense that it may involve too many complications and may give results accurate to ten decimal places whereas the observations may be correct to two decimal places only.

g. **Complexity of models**: This can be increased by subdividing variables, by taking more variables and by considering more details. Increase of complexity need not always lead to increase of insight as after a stage, diminishing returns begin to set in. The art of mathematical modelling consists in stopping before this stage.

h. **Models can lead to new experiments, new concepts and new mathematics**: Comparison of predictions with observations reveals the need for new experiments to collect needed data. Mathematical models can also lead to development of new concepts. If known mathematical techniques are not adequate to deduce results from the mathematical model, new mathematical techniques have to be developed.

i. **A model may be good, adequate, similar to reality for one purpose and not for another**: So different models are needed for explaining different aspects of the same situation or even for different ranges of the variables. Of course in this case, search for a unified model continues.

j. **Models may lead to expected or unexpected predictions or even to non- sense**: Usually models give predictions expected on common sense considerations, but the model predictions are more quantitative in nature. Sometimes they give unexpected predictions and then they may lead to break through or deep thinking about assumptions. Sometimes models give prediction completely at variance with observations and then these models have to be drastically revised.

k. **A model is not good or bad; it does or does not fit**: Models may lead to nice and elegant mathematical results, but only those models are acceptable which can explain, predict or control situations. A model may also fit one situation very well and may give a hopeless fit for another situation.

l. **Modeling forces us to think clearly**: Before making a mathematical model, one has to be clear about the structure and essentials of the situation [9].

1.5.8 Limitation of Models

Mathematical modeling is a Multi-stage process in which there requires a variety of concepts and techniques, therefore an ultimate caution is required in formulating the model otherwise an absurd model gives an absurd result which comes far away from practical results. This happens, when the basic formulation of a model is wrong because the model is only an amplification of real world problem. During the formulation of any model some assumptions are made, therefore the model is only as good as the assumptions made while formulating it and which violates the assumptions may be dangerous.

Consider a population model in which model equation is

$$\frac{dN}{dt} = \pm\lambda N$$

where N represents the population at any time t.

Take,

$$\frac{dN}{dt} = \lambda N$$

Then its solution is $N(t) = N_o e^{\lambda t}$, which gives $N(t) \to \infty$ as $t \to \infty$ this means population grows exponentially without any bound, whereas, the equation

$$\frac{dN}{dt} = -\lambda N$$

gives the solution as which $N(t)= N_o e^{-\lambda t}$ gives $N(t) \to 0$ as $t \to \infty$, this means the population is ultimately driven to extinction.

Both these results are not found to occur in the nature, thus, there is a need to modify the model. The models are improved by improving the assumptions.

1.5.9 Areas of Modelling

(a) Mathematical Models may be classified according to the subject of the models e.g.

- Mathematical models in Physics
- Mathematical models in Chemistry
- Mathematical models in Biology
- Mathematical models in Medicine
- Mathematical models in Sociology
- Mathematical models in Economy
- Mathematical models in Psychology and so many other models

(b) Mathematical models may also be classified according to the mathematical technique used in showing them e. g.

- Mathematical models through classical algebra.
- Mathematical models through linear algebra.
- Mathematical models through ordinary and partial differential equations.
- Mathematical model through integral equation.
- Mathematical models through integro-differential equations.
- Mathematical models through differential difference equations.
- Mathematical models through functional equations.
- Models through graph.
- Mathematical models through calculus of variations.
- Mathematical models through maximum principle and so on.

(c) Mathematical models may also be classified according to their nature. e.g.

- Mathematical models may be linear or non-linear.
- Mathematical models may be static or Dynamic.
- Mathematical models may be Deterministic or Stochastic.
- Mathematical models may be discrete or continuous.

(d) Mathematical models may also be classified according to the purpose we have for model. e.g.

- Mathematical models for Description.
- Mathematical models for Insight.
- Mathematical models for prediction.
- Mathematical models for Control.
- Mathematical models for Action.

In general, non-linear dynamic and stochastic models are more realistic but linear, static and deterministic models are easier to handle and give good approximate answers.

Some Simple Mathematical Models

Some mathematical models are given below:

1. To find the height of a tower without climbing it.

Here in this model the height of tower is expressed in terms of some distances and angles which can be measured on the ground.

2. To find the mass of moon.

In this model the mass of moon is expressed in terms of some known masses and distances.

3. To estimate the yield of rice in India from the standing crop.

This model requires to find the area under rice and average yield per acre by cutting and weighing the crop from some representative plots.

4. To estimate the average life span of a light bulb manufactured in a factory.

In order to understand the above model, take a random sample of bulbs into consideration, find their life span and use statistical inference models to estimate the life span for the population of bulbs.

5. To find the volume of blood inside the body of a person.

In this model some glucose is injected into the blood stream of a person and then find the increase in the concentration of sugar in blood.

6. To find the population of India in the year 2050 A.D.

In this case a model is developed by expressing the population as a function of time, taking into consideration the data from previous census. In order to express population as a function of time, some assumptions are required. Let us assume that the increase in population in a unit time is equal to the excess of births in that time over the number of deaths in that time and the number of births and deaths are proportional to the size of the population. These assumptions will give us a mathematical model whose solution gives population size as a function of time. Compare the results of this model with the actual size of population in the past. If the results are good and no significant changes are taken place in birth and death rates, the results of the models can be used for estimating future populations. On the other hand, if the results are not good then it is necessary to modify the hypothesis until the good results are obtained.

Mathematical Modelling in Biology or Biomathematics

Biomathematics is an interdisciplinary subject. In this we study the applications of mathematical modelling and mathematical techniques to get an inside into the problems of biosciences.

This study deals with some aspects of population dynamics, mathematical biology, mathematical bioeconomics, mathematical epidemiology, pollution control, optimization models in biology and medicine and many other fields of biosciences. It mainly concerned with mathematical modelling in biology and medicine and deal with the above areas of bioscience which have already been anathematized.

Since situation in life sciences are quite complex, we should have some insight into a situation before formulating a new mathematical model. Once a model is formulated, its consequence can be deduced by using mathematical techniques and the result can be compared with observations. If there are any discrepancies between theoretical conclusions and observations, the process is repeated till a really satisfactory model is obtained [1].

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2

Survey Work

1. **Simaika and Samways** have established a model on **"Comparative assessment of indices of freshwater habitat conditions using different invertebrate taxon sets"** published in **2011.** In which they discussed on that Dragonflies (Insecta: Odonata) are a valuable tool for assessing aquatic systems and have been used as indicators of ecological health, ecological integrity, and environmental change.

2. **Krishna Pada Das,Kusumika Kunda, J.Chatto Padhay** have established a model on "**A Predator-Prey with both the Populations affected by Diseases**" which was published in ecological complexity in **2011.** In this paper they analyzed that force of infection and prediction rate are the key parameters on the dynamics of the system the oscillatory coexistence of the species which is very common in nature is observed for predator diseases free system.

3. **S.Chakra,S.Paul,N.Bairagi** have established a model on **"Predator Prey interaction with Harvesting: Mathematical study with Biological Ramifications"** which was published in "Applied Mathematical Modeling" in **2012.** In which they observed that the results have been tested with the parameter values of Paramesion Aurelia and its predator Didirium Nasutum.

4. **Soovoojeet Jana (Ramsaday Collage Amta Howrah) and Tapan Kumar Kar (Indian Institute of Engineering Science and Technology, Shibpur)** have established a model **"A Mathematical Study of a Prey–Predator Model in Relevance to Pest Control"** which was published in Nonlinear Dynamics in **2013.** In which they propose and analyze an ecological system consisting of pest and its natural enemy as predator.

5. **Ravi Pratap Gupta,Peeyush Chandra and Malay Banaejee** have established a model on "**Dynamical complexity of a prey-predator model with nonlinear predator harvesting**" published in Discrete and

Continuous Dynamical Systems in **2015.** In which they study systematically the dynamical properties of a predator-prey model with nonlinear predator harvesting. They observed that the model has atmost two interior equilibria and can exhibit numerous kinds of bifurcations.

6. **Ravi Pratap Gupta & Peeyush Chandra** have established a model on "**Dynamical Properties of a Prey-Predator-Scavenger Model with Quadratic Harvesting"** published in Communication in Nonlinear Science and Numerical Simulation, Volume 49 in **2017.**They analyze an extended model for the prey-predator-scavenger in presence of harvesting to study the effects of harvesting of predator as well as scavenger.

7. **Maitri Verma and A.K Misra** have established a model on "**Modeling the Effect of Prey Refuge on a Ratio-Dependent Predator–Prey System with the Allee Effect**" published in Bulletin of Mathematics Biology in **2018**. Studied that the impact of a constant prey refuge on the dynamics of a ratio-dependent predator-prey system with strong Allee effect in prey growth.

8. **Sonia Akter, Md.Sirajul Islam(Bangubandhu Sheikh Mujibur Rahman**, Science and Technology University, Department of Mathematics M.Sc. in Applied Mathematics) have established a model on "**A Mathematical Model applied to Understand the Dynamical Behavior of Predator Prey Model**" which was published in Communication in Mathematical Modelling and Applications in **2019**. In which they discussed about shark and fish Lotka-Volterra modified predator prey model in differential equation. They also analyze about the steady state and stability criteria using Jacobian matrix method.

9. **Robinet et al, Franzese et al** have established a model on "**Application of mathematical modeling in ecology**" which was published in Economics of Sustainable Development in **2019.** In which they analyzed the occurrence of the species was highest at the Sava-Danube site, followed by the Tisa. Odonates occurrence was lowest in the Golija site. The results of the study showed that environmental variables are significantly associated with odonates distribution.

10. **Apurav Agarwal, Bianchi S.Sangma,Devasri Lal** have established a model on "**Mathematical Modelling for Circular Prey-Predator Model**" which

was established in **2020**. The paper intends to highlight the possibility of an organism to shoulder both roles as well as a prey predator with respect to same organism and environment, hence forming a cyclical relationship between all stakeholders in a given prey-predator ecosystem.

11. **Erika Diz-Pita And M.Victoria Otero-Espinar** have established a model "**Predator–Prey Models: A Review of some recent Advances**" which was published in Numerical Methods in Mathematical Ecology in **2021**. In which they gave a state-of-the-art review of recent predator–prey models which include some interesting characteristics such as Allee effect, fear effect, cannibalism, and immigration.

12. **Basaznew Belew , Dawit Melese & Qiakun Song** have established a model on **"Modeling and Analysis of Predator-Prey Model with Fear effect in Prey and Hunting Cooperation among Predators and Harvesting"** which was published in the Journal of Applied Mathematics in **2022**. In this paper, they analyze predator-prey system where prey population is linearly harvested and affected by fear and the prey population has grown logistically in the absence of predators.

13. **M.Mukherjee, D.Pal,S.K.Mahato, Ebenezer Bonyah** have established a model on "**Prey–Predator optimal Harvesting Mathematical Model in the Presence of Toxic Prey under interval Uncertainty**" which was published in the journal of Scientific African volume 21 in **2023**. This paper explores a multispecies prey–predator harvesting system based on Lotka–Volterra model with two preys (palatable and toxic prey) and one predator with interval biological parameters.

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3

Basic Mathematical Modeling Of Population Dynamics

3.1 Introduction

A *population* is a set of organisms of the same kind, some long time living in one territory (occupying a particular area) and are completely isolated from other same groups. The term “population” is used in various sections of biology, ecology, demography, medicine and psychometrics. In the life sciences the *population dynamics* branch studies the size and age composition of populations as dynamical systems. The environmental and biological processes driving populations can be as birth and death rates, and immigration and emigration. Examples are population growth, ageing populations, or population decline. The population dynamics is a well-developed branch of mathematical biology, which has a history of more than two hundred years.

Theoretically a population may have unbounded growth, but in the real life, populations cannot grow without bound and that it is competition between individuals for resources which restricts growth. The first principle of population dynamics is a Malthusian growth model based on the idea of the function being proportional to the speed to which the function grows. This model also known as a simple exponential growth model. Another famous examples are the Lotka Volterra predator prey equations, as well as the alternative Arditi-Ginzburg equations. The computer games (as SimCity, Sim Earth and the MMORPG Ultima Online) also tried to simulate some of these population dynamics. Last developments of population dynamics have been complemented by evolutionary game theory, developed first by John Maynard-Smith. In these dynamics, evolutionary biology concepts can be given as a deterministic mathematical form. Population dynamics also overlap with following branches in mathematical biology: mathematical epidemiology, the study of infectious disease affecting populations. Several models of viral spread in a population may be applied to health policy decisions. Population dynamics theory is important to a proper understanding of living populations at all levels.

Thus the analysis of gene frequency arrays assuredly has a part to play in anthropological comparisons and observations of animal behavior patterns. Mathematics, as the language of quantitative measurement, is clearly central to these pursuits. The relevant mathematics undoubtedly requires hybridization of nonlinear analysis, nonlinear differential and difference equations, probability theory, compounded stochastic processes modeling, algebra, dynamical systems, innovative statistical analysis of complex data, and the creative implementation of the gigantic computer methodology and all its ramification.

3.2 Single Species Models

The models which considers a single species are called single species models. The word "single" means only one type of species, it may either be human or birds, rats or fishes etc.

Consider the single species as human and in this case human population model is developed as follows:

Let x (t) be the number of human present at time t and a constant α is known as natural growth rate of the species. Thus, when the number of species present is sufficiently small, they multiply and their numbers increases as if

$$\frac{dx}{dt} = (\text{birth rate} - \text{Death rate})\, x$$

$$\frac{dx}{dt} = \alpha x \qquad \text{..... (1)}$$

So that, $$x(t) = Ke^{\alpha t} \qquad \text{..... (2)}$$

where K is constant of integration.

Which implies an exponential growth of population. The equation (2) has also been called the law of **Malthus.** Obviously, this result cannot be true for all time

t. Eventually there will be an over population of the species x and there will be insufficient food to provide for the entire population, resulting in death through starvation. Hence, the term $-\beta x^2$ is included to represent a self-limiting growth of the species. Therefore, the equation governing a single species habitat is given as:

$$\frac{dx}{dt} = \alpha x - \beta x^2 \quad .... (3)$$

This equation is known as the logistic equation [1].

3.3 Two Species Population Model

Any ecosystem is consisting of several species, which are interrelated to each other in the system. It is therefore necessary to study multi-species population models to understand the nature and diversity of natural ecosystem. Here, the study is confined upto two species only and a simple mathematical model is developed for the growth of two populations having interrelations in the form of prey-predator or competition [7].

Definitions

1. **Population:** It is well known fact that an individual living organism of any species does not live alone in nature, they live in a groups. These groups are called populations.

   Remark: The term population means that a group of individuals contains any one kind of living organism.
2. **Density:** The density of population is defined as the number of individuals per unit area or volume. Density is the basic characteristic of a population which gives its size in relation to some unit of space.
3. **Growth rate:** It is well known that a population changes over time, therefore, the rate of change of its density is called the growth-rate of a population. It is determined by the birth-rate and by the death-rate.
4. **Birth-rate:** The maximum production of new individuals per unit of time under certain ideal conditions is called the birth-rate of a population.
5. **Death-rate:** It is defined as the number of individuals dying per unit of time [1].

<u>Some Thoughts About Mathematical Biology: Phenomena in Nature and Society for Modelling</u>

- All species (living or nonliving including humans) on this planet earth, their functions, their interactions among themselves, with their environment in the habitat and outside their habitat form a part of

Biology. This of course includes all Natural as well as Social phenomena on this planet. Biology, when studied using Mathematics is known as Mathematical Biology**.**

- Nature is highly complex. Society is equally complex as the human mind is involved which is highly dynamic. Mathematical modeling is a very important tool to predict the behavior of various phenomena in Nature and Society. It can predict behavior of such systems which cannot be experimented upon. According to the famous mathematician Ramanujam "An equation for me has no meaning until it represents the thought of God" That thought, is Nature and Society.
- So it is important to understand Nature and try to solve social problems relevant to survival of species including human beings, plants, animals, etc.

Carrying Capacity of the Earth

- Human population will reach 10 billion in a few decades. It is not enough to feed these people but also provide facilities for education, health, clean water, housing etc. Since our planet is limited, how to increase its carrying capacity.
- One way to do this is to go in for multilayer cropping. This can be easily done even in deserts areas by growing crops where creepers are involved, such as grapes. Also, by using the straightforward terrain of rivers in the summer and winter seasons, the time when rivers become very narrow leaving aside fertile soil where vegetables of all kinds can be grown.
- Roof of houses can be used for of farming of vegetables.
- In India during monsoon lots of fresh water goes waste as there are no adequate facilities for storage. One way to store it could be by making reservoirs all over the country. Also, the rivers can be joined.
- In a desert state like Rajasthan, during monsoon, the floods are usual. By constructing a large cemented reservoir new concept of ecology can be used to store water in the reservoir.
- The sea water can be desalinated by using ecological methods which are much cheaper for details.

The following papers provides us much more information regarding Carrying Capacity of Earth;

- J B Shukla, Rashmi Singh, Ashish Goyal and A.K Misra, Modeling the desalination of saline water by using bacteria and marsh plants Desalination, Vol 277, pp 113-120 ,2011.

Darwin's Theory of Evolution, Wild Life Protection and Biodiversity

- According to Darwin, in a given closed environment (say an isolated island) only the fittest species will survive and eventually the weaker species will be eliminated by the stronger species one- by-one.
- This preposition is true and valid only for closed environment. If the species has a chance for convective or diffusive migration all species can survive if all other facilities are available for their survival.
- There are several models which can prove this concept in an open environment.

The following papers provides us much more information regarding Wild Life Protection and Biodiversity;

- J.B.Shukla and S.Verma, Effects of convective and dispersive interactions on the stability of two species.
- Bulletin of Mathematical Biology, Vol 43, No 5, pp 593-610, 1981.
- J. B. Shukla and V.P. Shukla, Multispecies food webs with diffusion, J. Math Biology, Vol 13, pp 339-344, ,1982.

Ecological Hydrodynamics

- This is another important area where modelling is needed. It is the combined study of Ecology and Hydrodynamics. This knowledge is needed to control landslides during monsoon rain, flooding, increased water seepage by using plantations, survival of fish populations in rivers maintenance of fresh water lakes, etc. It can also be used to study environmental and ecological impacts while joining rivers.

Global warming-01

- It is well known that due to greenhouse gases like carbon dioxide, methane, etc., the average temperature is increasing slowly but surely affecting all species on the planet earth including agriculture, forests, seawater, etc. It can spread infectious diseases such as TB, Malaria, Dengue, etc., in the Northern Hemisphere, where, at present, they do not exist. It can affect the polar ice caps, cause glacial melting, sea level rise, migration of population in coastal areas, etc. A modeling study is needed for all such type of problems. Researchers are involved with working on such type of problems for modeling.

Aerosols and artificial rain

- Rainfall is an important as well as complex phenomenon in nature. Aerosols are fine particles and can increase condensation of water vapours as cloud droplets and thus rain is formed.

- Aerosols of calcium chloride and calcium oxide are used for artificial rain making.

  The following papers provides us much more information regarding Aerosols and artificial rain;

- J.B. Shukla, A.K. Misra, R Naresh and P. Chandra, how artificial rain can be produced.

  J. B Shukla, S.S Misra, A.K. Misra, and R Naresh, Modeling the removal of gaseous pollutants and particulate matters from the atmosphere of a city by rain: Effects of cloud density.

- Environment. Model. Asses, Vol 13, pp255-263.2008

- A.K. Misra, A. Tripathi, R Naresh and J.B Shukla, Modeling and analysis of the effects of aerosols in making artificial rain

Sustainable Development

- It is well known that the carrying capacity of earth is limited and therefore the need for a concept or some criteria arises for an objective assessment of industrial, economic, social development of our society. Sustainable development, a multifaceted concept, has been defined as the development that meets the needs of the present generation without compromising the ability of the future generations to meet their own needs.
- Although it has been widely endorsed and accepted at national and international fora, in both developing and developed nations, the concept is still wide open to interpretation, criticism and revision in terms of measurement and quantification. All industrial and economic development must take into account all factors related to environment, ecology and wildlife, including the protection of biodiversity, flora and fauna in both terrestrial and aquatic systems of a region under consideration.
- All developing countries should design their own models of sustainable development as environmental, ecological, economic, social, political and cultural conditions are different in each region/country. In order to achieve the goal of sustainable development and improvement in the quality of life and social welfare, the need of times is to adopt both short term and long term measures for population growth, resource conservation, environmental protection and equitable distribution of the benefits of development.
- A vital step in this direction would be to understand not only what is being done in terms of utilization of resources and their conservation at the local/regional levels but also to be able to predict their impact on the development process for future generation.

3.4 Prey-Predator Model

In this system, the two species: the predator-feeds on the other species- the prey- which in turn feeds on some third food items readily available in the environment. For example, population of foxes and rabbits in a woodland; the foxes (predator eat rabbits (the Prey), while the rabbits eat certain vegetation in the woodland. Other examples are sharks (predator) and food fish (prey), bass (predator) and sunfish (prey), ladybugs (predator) and aphids (prey), beetles (predator) and scale insects (prey) etc.

3.4.1 Formulation of the model

To construct a mathematical model, let the first species, the number of prey (or host) at any time *t* be taken as *N(t)* and the second species, the size of predator (or Parasite) be taken as *P(t).*
The following assumptions are to be made:

1. In the absence of the predators, the prey population would grow at a natural rate, with

$$\frac{dN}{dt} = aN \ ; \ (a > 0)$$

2. In the absence of prey, the predator population would decline at a natural rate with

$$\frac{dP}{dt} = -cP \ ; \ c > 0$$

3. When both predator and prey are present, there occurs, in combination with these natural rates of growth and decline, a decline in the prey population and growth in the predator population, each at a rate proportional to the frequency of encounters between individuals of two species.
   Assume further that the frequency of such encounters is proportional to the product *NP*, reasoning that doubling either population alone should double the frequency of encounters, while doubling both populations ought to quadruple the frequency of encounters. Consequently, the effect of predators eating prey is an interaction rate of decline *bNP* in the prey population *N(t)* and an interaction rate of growth *mNP* of the predator population *P(t),* with *b* and *m* being positive constants.

On adding the natural and interaction rates described above, the predator prey equation is obtained as;

$$\left.\begin{aligned} \frac{dN}{dt} &= N(a - bP) \\ \frac{dP}{dt} &= P(mN - n) \end{aligned}\right\} \qquad \text{...... (1)}$$

where *a, b, m* and *n* are positive constants; *a* and *n* are the growth rate of prey and death rate of the predator respectively, and *b* and *m* are measures of the effect of the interactions between the two species.

Equation (1) along with the initial conditions $N(0) = N_0 \ and \ P(0) = P_0$ are known as **Lotka-Voltera** equations.

3.4.2 Solution of The Model

Voltera argued that if the size of the prey population N (food fish) be sufficiently large, the predator population *P* (Selachians) has an abundant supply of food and hence P increases. As *P* goes on increasing, more and more of the prey *N* is consumed as food and this leads to a rapid decrease of *N*. As the prey *N* becomes scarce, *P* stops increasing due to lack of food, thus allowing the remaining *N* to increase again. This cycle of phenomena is repeated over and over again.

When *P(t)=0* and *N(t)>0* the first equation of system (1) becomes

$$\frac{dN}{dt} = aN$$

Which corresponds to the exponential growth model, the solution is therefore

$$N(t) = N_0 e^{at} \ where \ N(0) = N_0$$

This shows that in the absence of the predators the prey grows exponentially according to Malthus law of population growth.

On the other hand, when *N(t)=0 and P(t) > 0,* second equation of system (1) reduces to

$$\frac{dP}{dt} = \ -nP$$

From the above equation it is clear that instead of an exponentially increasing solution, we get an exponentially decaying solution of the form

$$P(t) = P_0 e^{-nt} \qquad \text{where } P(0)=P_0$$

This relation implies in the absence of the prey, the predator population dies out exponentially (due to lack of food).

Again, consider the system of equations in (1)

$$\frac{dN}{dt} = 0 \ \Rightarrow \ N = 0 \ or \ P = a / b$$

and

$$\frac{dP}{dt} = 0 \Rightarrow N = n/m \ or \ P = 0$$

Hence the critical or equilibrium points of the system given by:

$$\frac{dN}{dt} = 0 = \frac{dP}{dt} \ are \ X = (0,0) \ and \ Y = (n/m, a/b)$$

Note: A critical point of the system of equations;

$$\frac{dN}{dt} = F(N, P), \ \frac{dP}{dt} = G(N, P)$$ is a point (N^*, P^*), such that

$$F(N^*, P^*) = G(N^*, P^*) = 0$$

Also, the constant valued function $N(t) = N^*$ and $P(t) = P^*$ satisfying the system is called the equilibrium solution.

Here *X* is a trivial steady state and p is a non-trivial one. The critical point of the interest, it satisfies a constant population *n/m* of prey and *a/b* of predator that can coexist with one another in the environment.

3.4.3 Geometrical Interpretation of the Prey-Predator Model

If *P*=0 and *N*>0 at some instant, then $\frac{dP}{dt} = 0 \ and \ \frac{dN}{dt} = aN$. This means that the predator population continues to remain at the zero level, while the prey population goes on increasing. Geometrically this means that the positive x-axis (y=*0*) is an orbit of the system. On the other hand if $N = 0 \ and \ P > 0$ at any time, we have $\frac{dN}{dt} = 0$ while $\frac{dP}{dt} = -nP < 0$.

This implies that the prey population continues to remain at zero level while the predator goes on decreasing. Geometrically, this means that the negative y-axis (*x*=0) is an orbit of the system. This analysis reconfirms our previous observations that:

(i) The prey grows exponentially in the absence of the predator and
(ii) The predator dies out exponentially in the absence of the prey.

Thus, the equilibrium solution *N(t)*=0=*P(t)* corresponds to the critical point (0,0) describes simultaneous extinction of both species.

Since, $N(t) \geq 0$ *and* $P(t) \geq 0$ for all times, all other orbits of the system lie entirely in the first quadrant of the x-y plane.

To get an idea of the other orbits, let *P=a/b and N=n/m* divide the first quadrant into four regularly shaped regions, therefore

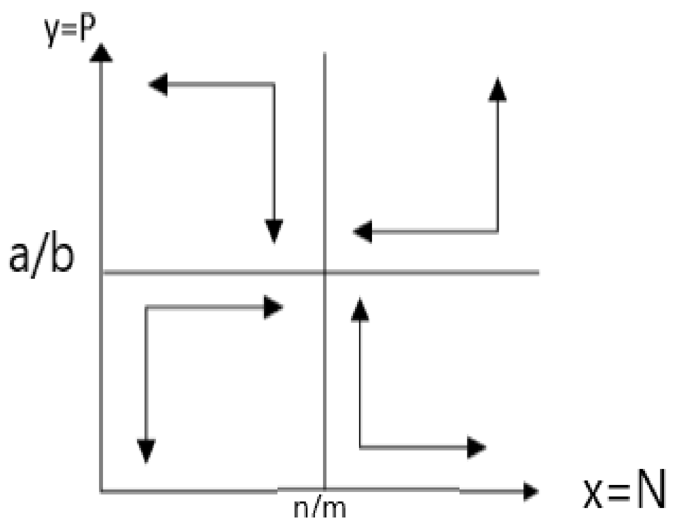
$$\frac{dN}{dt} > 0 \text{ if } P < a/b \text{ while } \quad \frac{dN}{dt} < 0 \text{ if } P > a/b.$$

This means that N increases in the region III and IV and decreases in the region I *and* II as shown in Figure 1.1.

Moreover,

$$\frac{dP}{dt} > 0 \; if \; N > n/m \text{ and } \frac{dP}{dt} < 0 \; if N < n/m,$$

this implies that P increases in the region I and IV while decreases II and III as shown in the figure.1.1.



Region	$\frac{dN}{dt}$	$\frac{dP}{dt}$
I	< 0	> 0
II	< 0	< 0
III	> 0	< 0
IV	> 0	> 0

Figure 1.1: Oscillation of population in prey-predator system

It is clear from the Figure that the orbit will follow a counter clockwise direction about critical point (*n/m, a/b*) whatever be the initial size of the populations. For example, if there are small numbers of prey and predator populations initially, i.e., if the orbit begins in the region III, then the prey increases and the predators decrease. This is what is expected in reality. For, a small number of foxes pose little threat to the rabbits so that the rabbits go on increasing in number. On the other hand, scarcity of the rabbits forces the fox population to decline. When the size of the rabbit population exceeds the critical value *n/m*, the orbit is in region IV and then the fox population also begins to increase due to availability of sufficient food (rabbits).

When the fox population exceeds the critical value *a/b*, the orbit enters the Region-I. Now foxes being plenty in number to endanger the rabbits, the rabbit population begins to decrease. Ultimately when the rabbit population declines below the critical level *n/m*, the orbit enters the region II. As a result of declining rabbit population, now the fox population also begins to decline due to shortage in food supply. When the fox population declines below the critical value *a/b*, there is a small number of foxes to endanger the lives of rabbits existing at that point of time. As a result, the rabbits start growing and we are again in region IV, this cycle of phenomenon continues to repeat again and again. Thus the fluctuations of the population, flows some kind of cylindrical pattern about the critical (*n/m, a/b*), let us denote the critical point as (N*, P*).

3.4.4 Analytical Solution of the Prey-Predator Model

To find the solution of the system of equations (1) for $N(t)>0$, $P(t)>0$ with initial conditions $N(0)=N_0$ *and* $P(0)=P_o$,

$$\frac{dP}{dN} = \frac{P(mN-n)}{N(a-bP)}$$

or

$$\frac{(a-bP)}{P}\, dP = \frac{(mN-n)}{N}\, dN$$

Which on integration gives,

$$a\, logP - bP - mN + nlogN = logK_1$$

Where K_1 is the constant of integration to be determined using the initial conditions. We have,

$$\log P^a - \log e^{bP} - \log e^{mN} + \log N^n = \log K_1$$

$$or \qquad \frac{P^a}{e^{bP}} \cdot \frac{N^n}{e^{mN}} = K_1$$

Which represents a family of closed curves gives the solution of system of equation (1). The above equation can be written in the form

$$\left(\frac{P}{e^{(b/a)P}}\right)^a \cdot \left(\frac{N}{e^{(m/n)N}}\right)^n = K_1$$

$$or \qquad \left(\frac{P}{e^{P/P^*}}\right)^a \cdot \left(\frac{N}{e^{N/N^*}}\right)^n = K_1$$

$$where \qquad N^* = n/m \ and \ P^* = a/b$$

Using the transformation, $X = N/N^*$ and $Y = P/P^*$ we have,

$$\left(\frac{YP^*}{e^Y}\right)^a \cdot \left(\frac{XN^*}{e^X}\right)^n = K_1$$

$$\left(\frac{e^X}{X}\right)^n \cdot \left(\frac{e^Y}{Y}\right)^a = \frac{1}{K_1} P^{*a} N^{*n} = C \qquad \text{.... (2)}$$

where the constant C has to be determined using the initial conditions $N(0) = N_0$ and $P(0) = P_0$ we get

$$C = \frac{P^{*a} N^{*n}}{K_1} = \left(\frac{e^{N_0/N^*}}{N_0/N^*}\right)^n \left(\frac{e^{P_0/P^*}}{P_0/P^*}\right)^a \qquad \text{....... (3)}$$

Thus, for a given (N_0, P_0) value of C is known. So the final solution

$$\left(\frac{e^N}{X}\right)^n \cdot \left(\frac{e^P}{Y}\right)^a = C \qquad \text{....... (4)}$$

can be obtained. The quantitative pictures of this solution are already given through geometrical considerations as shown in figure 1.1 and hence it is obvious what to expect.

Definition: For a linear system, critical point ***Q*** is stable if for initial population (N_0 , P_0) close to ***Q*** , the populations (*N(t), P(t)*) remain near it for all *t*>0.

3.5 Stability Analysis of Prey Predator Interaction

To check the stability of the critical point, *Q(n/m , a/b*) and get an idea of the pattern of the orbits near the critical point i.e., whether the orbits are moving towards the critical point or moving away from it or exhibiting some other type of behavior, here use the perturbation technique. The basic idea of the technique is to perturb or disturb the equilibrium slightly and then to see whether the system remains in the neighborhood of the equilibrium or deviates far away from it. Mathematically, change the equilibrium values of *N* and *P* slightly by adding to them very small quantities.

Let

$$N = \frac{n(1+u)}{m} \ , \ P = \frac{a(1+v)}{b} \qquad \text{..... (5)}$$

where u, v are very small quantities, the transformation indicates small departure from the equilibrium point (*n/m , a/b*)

We have from equation (1), that

$$\frac{du}{dt} = -av - auv$$

$$\frac{dv}{dt} = nu + nuv \qquad \text{.... (6)}$$

Clearly, the system of equations in (6) is almost linear system and has (0, 0) as the critical point (*n/m , a/b*), that of the system of equation (1).

In order to check the nature and stability of the critical point of system (3), consider the related linear system,

$$\frac{du}{dt} = -av$$

$$\frac{du}{dt} = nu \qquad \text{..... (7)}$$

The Eigen-values of (7) are given by the equations

$$|A - \lambda I| = 0, where\ A = \begin{pmatrix} 0 & -a \\ n & 0 \end{pmatrix} \qquad \text{.... (8)}$$

We obtain, from equation (7), that

$$\lambda = \pm i\sqrt{an} \qquad \text{.... (9)}$$

Thus the Eigen-values of the system (8) are purely imaginary. Thus conclude that critical point (0,0) of the system (6) is a centre. Further differentiating the two equations of the system (6) with respect to *t*, we obtain

$$\frac{d^2u}{dt^2} = -a\frac{dv}{dt} = -anu \qquad \text{....... (10)}$$

$$\frac{d^2v}{dt^2} = n\frac{du}{dt} = -anv \qquad \text{....... (11)}$$

Clearly both equations (10) and (11) represent a simple harmonic motion of a periodic time T=2π $/(an)^{1/2}$.

Thus the trajectories of the system (7) are closed curves exhibiting periodic oscillations of period 2π $/(an)^{1/2}$ in the neighborhood of point (0, 0), it can further show that these closed curves are ellipse in this case.

Multiply the first equation of system (7) by nu and the second by av and then adding together, we have

$$nudu + avdv = 0$$

This gives on integration

$$nu^2 + av^2 = \lambda$$

or

$$\frac{u^2}{\lambda/n} + \frac{v^2}{\lambda/a} = 1 \qquad \text{..... (12)}$$

where λ is an arbitrary non-negative constant of integration, thus the trajectories of the system are ellipse around the critical point (0,0). Some of these ellipses are shown in figure-1.2

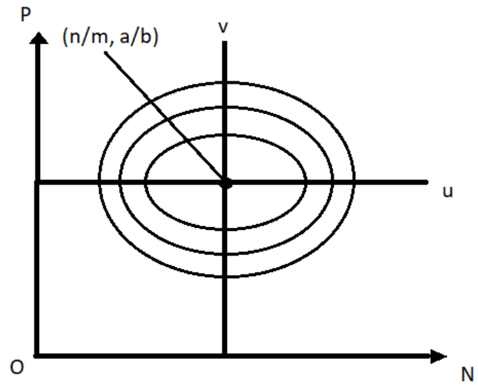


figure-1.2: Behavior along the critical points

It can be shown that the critical point (0,0) is a stable centre of the linear system (7). The effect of the non-linear terms may be to change the centre into a stable spiral point, or into unstable spiral point, or it may remain as a stable centre. Fortunately, in this case, it was actually solved the nonlinear equations (1) and see what happens. Also the graph of this equation for a fixed value of C in equation (5) is a closed curve (not an ellipse but deformed ellipse) enclosing the critical point (*n/m, a/b*). Thus the predator and prey have a cyclic variation about the critical point (*n/m, a/b*) and the critical point is also a centre of the system (1).

The following system of equations provides us an idea regarding the stability analysis of prey predator interaction

(i)

$$\frac{dx}{dt} = x - y + xy$$

x = prey and y = predator

$$\frac{dy}{dt} = 3x - 2y - xy \quad \text{.... (13)}$$

Clearly (0,0) is a critical point of the system (13)

It can be written in the form;

$$\frac{dx}{dt} = x - y + f(x, y)$$

$$\frac{dy}{dt} = 3x - 2y + g(x,y)$$

where, $$f(x,y) = xy \text{ } and \text{ } g(x,y) = -xy$$

For checking the condition for almost linear system, it is convenient to use polar coordinates by taking $\mathrm{x = rcos}\,\theta$ and $\mathrm{y = rsin}\,\theta$

Now $$\frac{f(x,y)}{r} = \frac{r^2\, cos\theta\, sin\theta}{r} = r\, cos\theta sin\theta \to 0 \text{ } as \text{ } r \to 0$$

and $$\frac{g(x,y)}{r} = -\frac{r^2 cos\theta sin\theta}{r} = -r\, cos\theta sin\theta \to 0 \text{ } as \text{ } r \to 0$$

Thus equation (13) is almost linear. The related linear system is in the neighborhood of (0,0) is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ .... (14)

Eigen-values of (14) are the roots of the equation

$$\lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

Since the Eigen-values are conjugate complex numbers, therefore the critical point (0,0) is a spiral. Also since λ < 0, it is asymptotically stable point. Since the system (13) is almost linear, critical point (0,0) of the system is also asymptotically stable spiral point.

(ii)

$$\frac{dx}{dt} = x \text{ };$$ x =prey and y =prdetor

$$\frac{dy}{dt} = -x + 2y$$ .... (15)

Clearly *(0, 0)* is the critical point of the system (15)

The Eigen-values of the system (15) are λ_1 = 1 and λ_2 = 2

Clearly Eigen-values are real distinct and are of the same sign. So the critical point is a node.

Also $\lambda_1 > 0$ *and* $\lambda_2 > 0$,it is unstable.

To find the general solution of the above system, it is necessary to find the eigenvectors corresponding to the eigenvalues λ_1 = 1 and λ_2 = 2.
Eigen-vector corresponding to the Eigenvalue λ_1= 1 is the solution of the equation

$$\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

We see that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is one possible Eigen-vector.

Similarly, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is one possible eigenvector corresponding to the Eigen-value $\lambda_2 = 2$.

Therefore, the general solution of the system can be written as;

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

$$x = c_1 e^t \text{ and } y = c_1 e^t + c_2 e^{2t} \qquad \text{......... (16)}$$

where c_1 *and* c_2 are arbitrary constants.
For $c_1 = 0, x = 0$ *and* $y = c_2 e^{2t}$. In this case the trajectory is positive y-axis when $c_2 > 0$ and it is negative y- axis when $c_2 < 0$ and also since $y \to \infty$ as t $\to\infty$, each path approaches to infinity as t$\to\infty$.

For $c_2 = 0, x = 0$ *and* $y = c_1 e^t$, the trajectory is a half line $y = x, x > 0$ when $c_1 > 0$

and the half line $y = x$, $x < 0$ when $c_1 < 0$ and again both paths approaches to infinity as t $\rightarrow\infty$.

When both c_1, c_2 are not equal to zero ,

the trajectories are parabolas $y = x + \left({c_2}/{c_1^2} \right) x^2$, which passes through the origin with slope 1.Each of these trajectories also approach to infnity as t $\rightarrow\infty$ [7].

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Discussion and Conclusion

The aim of our project work is to put light on the topic introduction to mathematical ecology with special reference to population dynamics. This project emphasizes on the main ideas related to given topics.

The first section of our project includes the introduction to mathematical ecology, discussion on two species followed by some definitions of competing species, mutualism or symbiosis and an overview of mathematical modeling.

In the second section we have incooperated some survey work including some recent research papers related to the given topic.

Finally, in the last section we have discussed interaction between two species with reference to prey predator model along with stability analysis.

In the proposed work, we tried our best to highlight various issues in ecology with relevant possible solutions through mathematical modeling.

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