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Topics on Algebra for INMO

- Basics on functions
  - Increasing & Decreasing functions
  - Even and Odd functions
- Polynomials
- Fundamental theorem of algebra
- Fundamental theorem of arithmetic's
- Problems and Solutions for INMO exam

Problem-1:

Let a, b, and c be real and positive parameters.

Solve the equation

$$\sqrt{a+bx}+\sqrt{b+cx}+\sqrt{c+ax}=\sqrt{b-ax}+\sqrt{c-bx}+\sqrt{a-cx}.$$

Solution 1

It is easy to see that $x = 0$ is a solution. Since the right hand side is a decreasing function of x and the left hand side is an increasing function of x, there is at most one solution.

Thus $x = 0$ is the only solution to the equation.

Problem 2

Find the general term of the sequence defined by $x_0 = 3$, $x_1 = 4$ and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all $n \in \mathbb{N}$.

Solution 2

We shall prove by induction that $x_n = n + 3$. The claim is evident for $n = 0, 1$.

For $k \geq 1$, if $x_{k-1} = k + 2$ and $x_k = k + 3$, then

$$x_{k+1} = x_{k-1}^2 - kx_k = (k+2)^2 - k(k+3) = k + 4,$$

as desired.

This completes the induction.

Problem 3 [AHSME 1999]

Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that

(i) $-1 \le x_i \le 2$, for $\imath = 1, 2, \ldots, n$;

(ii) $x_1 + x_2 + \cdots + x_n = 19$;

(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.

Determine the minimum and maximum possible values of

$$x_1^3 + x_2^3 + \cdots + x_n^3.$$

Solution 3

Let a, b, and c denote the number of -1s, 1s, and 2s in the sequence, respectively. We need not consider the zeros. Then a, b, c are nonnegative integers satisfying

$$-a + b + 2c = 19 \text{ and } a + b + 4c = 99.$$

It follows that $a = 40 - c$ and $b = 59 - 3c$, where $0 \le c \le 19$ (since $b \ge 0$), so

$$x_1^3 + x_2^3 + \cdots + x_n^3 = -a + b + 8c = 19 + 6c.$$

When $c = 0$ ($a = 40, b = 59$), the lower bound (19) is achieved.

When $c = 19$ ($a = 21, b = 2$), the upper bound (133) is achieved.

Problem 4 [AIME 1997]

The function f, defined by

$$f(x) = \frac{ax+b}{cx+d},$$

where a, b, c, and d are nonzero real numbers, has the properties

$$f(19) = 19, \quad f(97) = 97, \quad \text{and} \quad f(f(x)) = x,$$

for all values of x, except $-\frac{d}{c}$.

Find the range of f.

Solution 4, Alternative 1

For all x, $f(f(x)) = x$, i.e.,

$$\frac{a\left(\dfrac{ax+b}{cx+d}\right)+b}{c\left(\dfrac{ax+b}{cx+d}\right)+d} = x,$$

i.e.

$$\frac{(a^2+bc)x+b(a+d)}{c(a+d)x+bc+d^2}=x,$$

i.e.

$$c(a+d)x^2+(d^2-a^2)x-b(a+d)=0,$$

which implies that $c(a+d)=0$. Since $c\neq 0$, we must have $a=-d$.

The conditions $f(19)=19$ and $f(97)=97$ lead to the equations

$$19^2c=2\cdot 19a+b \qquad \text{and} \qquad 97^2c=2\cdot 97a+b.$$

Hence

$$(97^2-19^2)c=2(97-19)a.$$

It follows that $a=58c$, which in turn leads to $b=-1843c$. Therefore

$$f(x)=\frac{58x-1843}{x-58}=58+\frac{1521}{x-58},$$

which never has the value 58.

Thus the range of f is $\mathbb{R}-\{58\}$.

Solution 4, Alternative 2

The statement implies that f is its own inverse. The inverse may be found by solving the equation

$$x = \frac{ay+b}{cy+d}$$

for y. This yields

$$f^{-1}(x) = \frac{dx-b}{-cx+a}.$$

The nonzero numbers a, b, c, and d must therefore be proportional to d, $-b$, $-c$, and a, respectively; it follows that $a=-d$, and the rest is the same as in the first solution.

Problem 5

Prove that

$$\frac{(a-b)^2}{8a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{(a-b)^2}{8b}$$

for all $a \geq b > 0$.

Solution 5, Alternative 1

Note that

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{a}}\right)^2 \leq 1 \leq \left(\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{b}}\right)^2,$$

i.e.

$$\frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{a}-\sqrt{b})^2}{4a} \le (\sqrt{a}-\sqrt{b})^2 \le \frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{a}-\sqrt{b})^2}{4b},$$

i.e.

$$\frac{(a-b)^2}{8a} \le \frac{a-2\sqrt{ab}+b}{2} \le \frac{(a-b)^2}{8b},$$

from which the result follows.

Solution 5, Alternative 2

Note that

$$\frac{a+b}{2} - \sqrt{ab} = \frac{\left(\frac{a+b}{2}\right)^2 - ab}{\frac{a+b}{2} + \sqrt{ab}} = \frac{(a-b)^2}{2(a+b)+4\sqrt{ab}}.$$

Thus the desired inequality is equivalent to

$$4a \ge a+b+2\sqrt{ab} \ge 4b,$$

which is evident as $a \ge b > 0$ (which implies $a \ge \sqrt{ab} \ge b$).

Problem 6 [St. Petersburg 1989]

Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say a and b, and then write the numbers $a + \frac{b}{2}$ and $b - \frac{a}{2}$ instead.

Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

Solution 6

Let S be the sum of the squares of the numbers on the board. Note that S increases in the first operation and does not decrease in any successive operation, as

$$\left(a + \frac{b}{2}\right)^2 + \left(b - \frac{a}{2}\right)^2 = \frac{5}{4}(a^2 + b^2) \geq a^2 + b^2$$

with equality only if $a = b = 0$.

This completes the proof.

Problem 7 [AIME 1986]

The polynomial

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$

may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and a_is are constants. Find a_2.

Solution 7, Alternative 1

Let $f(x)$ denote the given expression. Then

$$xf(x) = x - x^2 + x^3 - \cdots - x^{18}$$

and

$$(1 + x)f(x) = 1 - x^{18}.$$

Hence

$$f(x) = f(y-1) = \frac{1 - (y-1)^{18}}{1 + (y-1)} = \frac{1 - (y-1)^{18}}{y}.$$

Therefore a_2 is equal to the coefficient of y^3 in the expansion of

$$1 - (y-1)^{18},$$

i.e.,

$$a_2 = \binom{18}{3} = 816.$$

Solution 7, Alternative 2

Let $f(x)$ denote the given expression. Then

$$\begin{aligned} f(x) = f(y-1) &= 1 - (y-1) + (y-1)^2 - \cdots - (y-1)^{17} \\ &= 1 + (1-y) + (1-y)^2 + \cdots + (1-y)^{17}. \end{aligned}$$

Thus

$$a_2 = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{17}{2} = \binom{18}{3}.$$

Here we used the formula

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

and the fact that

$$\binom{2}{2} = \binom{3}{3} = 1.$$

Problem 8

Let a, b, and c be distinct nonzero real numbers such that

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$

Prove that $|abc|=1$.

Solution 8

From the given conditions it follows that

$$a-b=\frac{b-c}{bc},\ b-c=\frac{c-a}{ca},\ \text{and } c-a=\frac{a-b}{ab}.$$

Multiplying the above equations gives $(abc)^2=1$, from which the desired result follows.

Problem 9 [Putnam 1999]

Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases} -1 & \text{if } x<-1 \\ 3x+2 & \text{if } -1\le x\le 0 \\ -2x+2 & \text{if } x>0. \end{cases}$$

Problem 10

Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

Solution 10

By setting $2^x = a$ and $3^x = b$, the equation becomes

$$\frac{a^3 + b^3}{a^2b + b^2a} = \frac{7}{6},$$

i.e.

$$\frac{a^2 - ab + b^2}{ab} = \frac{7}{6},$$

i.e.

$$6a^2 - 13ab + 6b^2 = 0,$$

i.e.

$$(2a - 3b)(3a - 2b) = 0.$$

Therefore $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies that $x = -1$ and $x = 1$.

It is easy to check that both $x = -1$ and $x = 1$ satisfy the given equation.

Problem 11 [Romania 1990]

Find the least positive integer m such that

$$\binom{2n}{n}^{\frac{1}{n}} < m$$

for all positive integers n.

Solution 11

Note that

$$\binom{2n}{n} < \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = (1+1)^{2n} = 4^n$$

and for $n = 5$,

$$\binom{10}{5} = 252 > 3^5.$$

Thus $m = 4$.

Problem 12

Let a, b, c, d, and e be positive integers such that

$$abcde = a + b + c + d + e.$$

Find the maximum possible value of $\max\{a, b, c, d, e\}$.

Solution 12, Alternative 1

Suppose that $a \le b \le c \le d \le e$. We need to find the maximum value of e. Since

$$e < a + b + c + d + e \le 5e,$$

then $e < abcde \le 5e$, i.e. $1 < abcd \le 5$.

Hence $(a, b, c, d) = (1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 1, 4)$, $(1, 1, 2, 2)$, or $(1, 1, 1, 5)$, which leads to $\max\{e\} = 5$.

Solution 12, Alternative 2

As before, suppose that $a \le b \le c \le d \le e$. Note that

$$1 = \frac{1}{bcde} + \frac{1}{cdea} + \frac{1}{deab} + \frac{1}{eabc} + \frac{1}{abcd}$$

$$\le \frac{1}{de} + \frac{1}{de} + \frac{1}{de} + \frac{1}{e} + \frac{1}{d} = \frac{3 + d + e}{de}.$$

Therefore, $de \le 3 + d + e$ or $(d - 1)(e - 1) \le 4$.

If $d = 1$, then $a = b = c = 1$ and $4 + e = e$, which is impossible.

Thus $d - 1 \ge 1$ and $e - 1 \le 4$ or $e \le 5$.

It is easy to see that $(1, 1, 1, 2, 5)$ is a solution.

Therefore $\max\{e\} = 5$.

Comment: The second solution can be used to determine the maximum value of $\{x_1, x_2, \ldots, x_n\}$, when $x_1, x_2, \ldots, x_n$ are positive integers such that

$$x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n.$$

Problem 13

Evaluate

$$\frac{3}{1! + 2! + 3!} + \frac{4}{2! + 3! + 4!} + \cdots + \frac{2001}{1999! + 2000! + 2001!}.$$

Solution 13

Note that

$$\begin{aligned}\frac{k+2}{k!+(k+1)!+(k+2)!} &= \frac{k+2}{k![1+k+1+(k+1)(k+2)]}\\ &= \frac{1}{k!(k+2)}\\ &= \frac{k+1}{(k+2)!}\\ &= \frac{(k+2)-1}{(k+2)!}\\ &= \frac{1}{(k+1)!}-\frac{1}{(k+2)!}.\end{aligned}$$

By telescoping sum, the desired value is equal to

$$\frac{1}{2}-\frac{1}{2001!}.$$

Problem 14

Let $x=\sqrt{a^2+a+1}-\sqrt{a^2-a+1}$, $a\in\mathbb{R}$.
Find all possible values of x.

Solution 14, Alternative 1

Since

$$\sqrt{a^2+|a|+1}>|a|$$

and

$$x=\frac{2a}{\sqrt{a^2+a+1}+\sqrt{a^2-a+1}},$$

we have

$$|x|<|2a/a|=2.$$

Squaring both sides of

$$x+\sqrt{a^2-a+1}=\sqrt{a^2+a+1}$$

yields

$$2x\sqrt{a^2-a+1}=2a-x^2.$$

Squaring both sides of the above equation gives

$$4(x^2-1)a^2 = x^2(x^2-4) \text{ or } a^2 = \frac{x^2(x^2-4)}{4(x^2-1)}.$$

Since $a^2 \geq 0$, we must have

$$x^2(x^2-4)(x^2-1) \geq 0,$$

Since $|x| < 2$, $x^2-4 < 0$ which forces $x^2-1 < 0$. Therefore, $-1 < x < 1$.

Conversely, for every $x \in (-1, 1)$ there exists a real number a such that

$$x = \sqrt{a^2+a+1} - \sqrt{a^2-a+1}.$$

Solution 14, Alternative 2

Let $A = (-1/2, \sqrt{3}/2)$, $B = (1/2, \sqrt{3}/2)$, and $P = (a, 0)$. Then P is a point on the x-axis and we are looking for all possible values of $d = PA - PB$.

By the **Triangle Inequality**, $|PA - PB| < |AB| = 1$. And it is clear that all the values $-1 < d < 1$ are indeed obtainable. In fact, for such a d, a half hyperbola of all points Q such that $QA - QB = d$ is well defined. (Points A and B are foci of the hyperbola.)

Since line AB is parallel to the x-axis, this half hyperbola intersects the x- axis, i.e., P is well defined.

Problem 15

Find all real numbers x for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

Solution 15

It is easy to check that $x = 2$ is a solution. We claim that it is the only one. In fact, dividing by 13^x on both sides gives

$$\left(\frac{10}{13}\right)^x + \left(\frac{11}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1 + \left(\frac{14}{13}\right)^x.$$

The left hand side is a decreasing function of x and the right hand side is an increasing function of x.

Therefore their graphs can have at most one point of intersection.

Comment: More generally,

$$
\begin{aligned}
&a^2 + (a+1)^2 + \cdots + (a+k)^2 \\
&\quad = (a+k+1)^2 + (a+k+2)^2 + \cdots + (a+2k)^2
\end{aligned}
$$

for $a = k(2k+1)$, $k \in \mathbb{N}$.

Problem 16 [Korean Mathematics Competition 2001]

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that $f(1,1) = 2$,

$$f(m+1,n) = f(m,n) + m \text{ and } f(m,n+1) = f(m,n) - n$$

for all $m, n \in \mathbb{N}$.

Find all pairs (p,q) such that $f(p,q) = 2001$.

Solution 16

We have

$$
\begin{aligned}
f(p,q) &= f(p-1,q) + p - 1 \\
&= f(p-2,q) + (p-2) + (p-1) \\
&= \cdots \\
&= f(1,q) + \frac{p(p-1)}{2} \\
&= f(1,q-1) - (q-1) + \frac{p(p-1)}{2} \\
&= \cdots \\
&= f(1,1) - \frac{q(q-1)}{2} + \frac{p(p-1)}{2} \\
&= 2001.
\end{aligned}
$$

Therefore

$$\frac{p(p-1)}{2} - \frac{q(q-1)}{2} = 1999,$$

i.e.

$$(p-q)(p+q-1) = 2 \cdot 1999.$$

Note that 1999 is a prime number and that $p-q < p+q-1$ for $p, q \in \mathbb{N}$. We have the following two cases:

1. $p - q = 1$ and $p + q - 1 = 3998$. Hence $p = 2000$ and $q = 1999$.

2. $p - q = 2$ and $p + q - 1 = 1999$. Hence $p = 1001$ and $q = 999$.

Therefore $(p, q) = (2000, 1999)$ or $(1001, 999)$.

Problem 17 [China 1983]
Let f be a function defined on $[0,1]$ such that

$$f(0) = f(1) = 1 \text{ and } |f(a) - f(b)| < |a - b|,$$

for all $a \neq b$ in the interval $[0,1]$.
Prove that

$$|f(a) - f(b)| < \frac{1}{2}.$$

Solution 17
We consider the following cases.

1. $|a - b| \leq 1/2$. Then $|f(a) - f(b)| < |a - b| \leq \frac{1}{2}$, as desired.
2. $|a - b| > 1/2$. By symmetry, we may assume that $a > b$. Then

$$\begin{aligned} |f(a) - f(b)| &= |f(a) - f(1) + f(0) - f(b)| \\ &\leq |f(a) - f(1)| + |f(0) - f(b)| \\ &< |a - 1| + |0 - b| \\ &= 1 - a + b - 0 \\ &= 1 - (a - b) \\ &< \frac{1}{2}, \end{aligned}$$

as desired.

Problem 18
Find all pairs of integers (x, y) such that

$$x^3 + y^3 = (x + y)^2.$$

Solution 18
Since $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$, all pairs of integers $(n, -n)$, $n \in \mathbb{Z}$, are solutions.
Suppose that $x + y \neq 0$. Then the equation becomes

$$x^2 - xy + y^2 = x + y,$$

i.e.

$$x^2 - (y + 1)x + y^2 - y = 0.$$

Treated as a quadratic equation in x, we calculate the discriminant

$$\Delta = y^2 + 2y + 1 - 4y^2 + 4y = -3y^2 + 6y + 1.$$

Solving for $\Delta \geq 0$ yields

$$\frac{3-2\sqrt{3}}{3} \leq y \leq \frac{3+2\sqrt{3}}{3}.$$

Thus the possible values for y are 0, 1, and 2, which lead to the solutions $(1,0)$, $(0,1)$, $(1,2)$, $(2,1)$, and $(2,2)$.

Therefore, the integer solutions of the equation are $(x,y) = (1,0)$, $(0,1)$, $(1,2)$, $(2,1)$, $(2,2)$, and $(n,-n)$, for all $n \in \mathbb{Z}$.

Problem 19 [Korean Mathematics Competition 2001]

Let

$$f(x) = \frac{2}{4^x + 2}$$

for real numbers x. Evaluate

$$f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + \cdots + f\left(\frac{2000}{2001}\right).$$

Solution 19

Note that f has a half-turn symmetry about point $(1/2, 1/2)$. Indeed,

$$f(1-x) = \frac{2}{4^{1-x}+2} = \frac{2 \cdot 4^x}{4 + 2 \cdot 4^x} = \frac{4^x}{4^x + 2},$$

from which it follows that $f(x) + f(1-x) = 1$.

Thus the desired sum is equal to 1000.

Problem 20

Prove that for $n \geq 6$ the equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1$$

has integer solutions.

Solution 20

Note that

$$\frac{1}{a^2} = \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2},$$

from which it follows that if $(x_1, x_2, \cdots, x_n) = (a_1, a_2, \cdots, a_n)$ is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1,$$

then

$$\begin{aligned}&(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3})\\ &= (a_1, a_2, \cdots, a_{n-1}, 2a_n, 2a_n, 2a_n, 2a_n)\end{aligned}$$

is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n+3}^2} = 1.$$

Therefore we can construct the solutions inductively if there are solutions for $n = 6, 7$, and 8.

Since $x_1 = 1$ is a solution for $n = 1$, $(2, 2, 2, 2)$ is a solution for $n = 4$, and $(2, 2, 2, 4, 4, 4, 4)$ is a solution for $n = 7$.

It is easy to check that $(2, 2, 2, 3, 3, 6)$ and $(2, 2, 2, 3, 4, 4, 12, 12)$ are solutions for $n = 6$ and $n = 8$, respectively. This completes the proof.

Problem 21 [AIME 1988]

Find all pairs of integers (a, b) such that the polynomial

$$ax^{17} + bx^{16} + 1$$

is divisible by $x^2 - x - 1$.

Solution 21, Alternative 1

Let p and q be the roots of $x^2 - x - 1 = 0$. By **Vieta's theorem**, $p + q = 1$ and $pq = -1$. Note that p and q must also be the roots of $ax^{17} + bx^{16} + 1 = 0$. Thus

$$ap^{17} + bp^{16} = -1 \text{ and } aq^{17} + bq^{16} = -1.$$

Multiplying the first of these equations by q^{16}, the second one by p^{16}, and using the fact that $pq = -1$, we find

$$ap + b = -q^{16} \text{ and } aq + b = -p^{16}. \tag{1}$$

Thus

$$a = \frac{p^{16} - q^{16}}{p - q} = (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q).$$

Since

$$\begin{aligned} p+q &= 1, \\ p^2+q^2 &= (p+q)^2 - 2pq = 1+2 = 3, \\ p^4+q^4 &= (p^2+q^2)^2 - 2p^2q^2 = 9-2 = 7, \\ p^8+q^8 &= (p^4+q^4)^2 - 2p^4q^4 = 49-2 = 47, \end{aligned}$$

it follows that $a = 1 \cdot 3 \cdot 7 \cdot 47 = 987$.

Likewise, eliminating a in (1) gives

$$\begin{aligned} -b &= \frac{p^{17}-q^{17}}{p-q} \\ &= p^{16} + p^{15}q + p^{14}q^2 + \cdots + q^{16} \\ &= (p^{16}+q^{16}) + pq(p^{14}+q^{14}) + p^2q^2(p^{12}+q^{12}) \\ &\quad + \cdots + p^7q^7(p^2+q^2) + p^8q^8 \\ &= (p^{16}+q^{16}) - (p^{14}+q^{14}) + \cdots - (p^2+q^2) + 1. \end{aligned}$$

For $n \geq 1$, let $k_{2n} = p^{2n} + q^{2n}$. Then $k_2 = 3$ and $k_4 = 7$, and

$$\begin{aligned} k_{2n+4} &= p^{2n+4} + q^{2n+4} \\ &= (p^{2n+2} + q^{2n+2})(p^2+q^2) - p^2q^2(p^{2n}+q^{2n}) \\ &= 3k_{2n+2} - k_{2n} \end{aligned}$$

for $n \geq 3$. Then $k_6 = 18$, $k_8 = 47$, $k_{10} = 123$, $k_{12} = 322$, $k_{14} = 843$, $k_{16} = 2207$.

Hence

$$-b = 2207 - 843 + 322 - 123 + 47 - 18 + 7 - 3 + 1 = 1597$$

or

$$(a,b) = (987, -1597).$$

Solution 21, Alternative 2

The other factor is of degree 15 and we write

$$(c_{15}x^{15} - c_{14}x^{14} + \cdots + c_1 x - c_0)(x^2 - x - 1) = ax^{17} + bx^{16} + 1.$$

Comparing coefficients:

$$\begin{aligned} x^0: &\quad c_0 = 1, \\ x^1: &\quad c_0 - c_1 = 0, c_1 = 1 \\ x^2: &\quad -c_0 - c_1 + c_2 = 0, c_2 = 2, \\ \text{and for } 3 \leq k \leq 15, \quad x^k: &\quad -c_{k-2} - c_{k-1} + c_k = 0. \end{aligned}$$

It follows that for $k \le 15$, $c_k = F_{k+1}$ **(the Fibonacci number)**.

Thus $a = c_{15} = F_{16} = 987$ and $b = -c_{14} - c_{15} = -F_{17} = -1597$ or $(a, b) = (987, -1597)$.

Comment: Combining the two methods, we obtain some interesting facts about sequences k_{2n} and F_{2n-1}. Since

$$3F_{2n+3} - F_{2n+5} = 2F_{2n+3} - F_{2n+4} = F_{2n+3} - F_{2n+2} = F_{2n+1},$$

it follows that F_{2n-1} and k_{2n} satisfy the same recursive relation. It is easy to check that $k_2 = F_1 + F_3$ and $k_4 = F_3 + F_5$.

Therefore $k_{2n} = F_{2n-1} + F_{2n+1}$ and

$$F_{2n+1} = k_{2n} - k_{2n-2} + k_{2n-4} - \cdots + (-1)^{n-1}k_2 + (-1)^n.$$

Problem 22 [AIME 1994]

Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of S?

Solution 22

Consider each positive integer less than 1000 to be a three-digit number by prefixing 0s to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is

$$\begin{aligned}&(0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 1 + \cdots + 9 \cdot 9 \cdot 9) - 0 \cdot 0 \cdot 0\\ &= (0 + 1 + \cdots + 9)^3 - 0.\end{aligned}$$

However, $p(n)$ is the product of non-zero digits of n. The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent to thinking of them as 1's in the products. (Note that the final 0 in the above expression becomes a 1 and compensates for the contribution of 000 after it is changed to 111.)

Hence

$$S = 46^3 - 1 = (46 - 1)(46^2 + 46 + 1) = 3^3 \cdot 5 \cdot 7 \cdot 103,$$

and the largest prime factor is 103.

2. For positive real numbers a, b, c, which of the following statements necessarily implies $a = b = c$: (I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$, (II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.

Solution: We show that (I) need not imply that $a = b = c$ where as (II) always implies $a = b = c$.

Observe that $a(b^3 + c^3) = b(c^3 + a^3)$ gives $c^3(a - b) = ab(a^2 - b^2)$. This gives either $a = b$ or $ab(a + b) = c^3$. Similarly, $b = c$ or $bc(b + c) = a^3$. If $a \neq b$ and $b \neq c$, we obtain

$$ab(a + b) = c^3, \quad bc(b + c) = a^3.$$

Therefore

$$b(a^2 - c^2) + b^2(a - c) = c^3 - a^3.$$

This gives $(a - c)(a^2 + b^2 + c^2 + ab + bc + ca) = 0$. Since a, b, c are positive, the only possibility is $a = c$. We have therefore 4 possibilities: $a = b = c$; $a \neq b$, $b \neq c$ and $c = a$; $b \neq c$, $c \neq a$ and $a = b$; $c \neq a$, $a \neq b$ and $b = c$.

Suppose $a = b$ and $b, a \neq c$. Then $b(c^3 + a^3) = c(a^3 + b^3)$ gives $ac^3 + a^4 = 2ca^3$. This implies that $a(a - c)(a^2 - ac - c^2) = 0$. Therefore $a^2 - ac - c^2 = 0$. Putting $a/c = x$, we get the quadratic equation $x^2 - x - 1 = 0$. Hence $x = (1 + \sqrt{5})/2$. Thus we get

$$a = b = \left(\frac{1 + \sqrt{5}}{2}\right) c, \quad c \text{ arbitrary positive real number.}$$

Similarly, we get other two cases:

$$b = c = \left(\frac{1 + \sqrt{5}}{2}\right) a, \quad a \text{ arbitrary positive real number;}$$

$$c = a = \left(\frac{1 + \sqrt{5}}{2}\right) b, \quad b \text{ arbitrary positive real number.}$$

And $a = b = c$ is the fourth possibility.

Consider (II): $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$. Suppose a, b, c are mutually distinct. We may assume $a = \max\{a, b, c\}$. Hence $a > b$ and $a > c$. Using $a > b$, we get from the first relation that $a^3 + b^3 < b^3 + c^3$. Therefore $a^3 < c^3$ forcing $a < c$. This contradicts $a > c$. We conclude that a, b, c cannot be mutually distinct. This means some two must be equal. If $a = b$, the equality of the first two expressions give $a^3 + b^3 = b^3 + c^3$ so that $a = c$. Similarly, we can show that $b = c$ implies $b = a$ and $c = a$ gives $c = b$.

Alternate for (II) by a contestant: We can write

$$\begin{aligned}\frac{a^3}{c}+\frac{b^3}{c} &= \frac{c^3}{a}+a^2,\\ \frac{b^3}{a}+\frac{c^3}{a} &= \frac{a^3}{b}+b^2,\\ \frac{c^3}{b}+\frac{a^3}{b} &= \frac{b^3}{c}+c^2.\end{aligned}$$

Adding, we get

$$\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}=a^2+b^2+c^2.$$

Using C-S inequality, we have

$$\begin{aligned}(a^2+b^2+c^2)^2 &= \left(\frac{\sqrt{a^3}}{\sqrt{c}}\cdot\sqrt{ac}+\frac{\sqrt{b^3}}{\sqrt{a}}\cdot\sqrt{ba}+\frac{\sqrt{c^3}}{\sqrt{b}}\cdot\sqrt{cb}\right)^2\\ &\le \left(\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\right)\left(ac+ba+cb\right)\\ &= (a^2+b^2+c^2)(ab+bc+ca).\end{aligned}$$

Thus we obtain

$$a^2+b^2+c^2\le ab+bc+ca.$$

However this implies $(a-b)^2+(b-c)^2+(c-a)^2\le 0$ and hence $a=b=c$.

3. Let $\mathbb{N}$ denote the set of all natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k+1) = 2k+2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

(i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

(ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

Solution:

(i) For $n = 1$, we have $T(1) = 2$ and $T^2(1) = T(2) = 1$. Hence we may assume that $n > 1$.

Suppose $n > 1$ is even. Then $T(n) = n/2$. We observe that $(n/2) \leq n - 1$ for $n > 1$.

Suppose $n > 1$ is odd so that $n \geq 3$. Then $T(n) = n+1$ and $T^2(n) = (n+1)/2$. Again we see that $(n+1)/2 \leq (n-1)$ for $n \geq 3$.

Thus we see that in at most $2(n-1)$ steps T sends n to 1. Hence $k \leq 2(n-1)$. (Here $2(n-1)$ is only a bound. In reality, less number of steps will do.)

(ii) We show that $c_n = f_{n+1}$, where f_n is the n-th Fibonacci number.

Let $n \in \mathbb{N}$ and let $k \in \mathbb{N}$ be such that $T^k(n) = 1$. Here n can be odd or even. If n is even, it can be either of the form $4d+2$ or of the form $4d$.

If n is odd, then $1 = T^k(n) = T^{k-1}(n+1)$. (Observe that $k > 1$; otherwise we get $n + 1 = 1$ which is impossible since $n \in \mathbb{N}$.) Here $n+1$ is even.

If $n = 4d+2$, then again $1 = T^k(4d+2) = T^{k-1}(2d+1)$. Here $2d+1 = n/2$ is odd.

Thus each solution of $T^{k-1}(m) = 1$ produces exactly one solution of $T^k(n) = 1$ and n is either odd or of the form $4d+2$.

If $n = 4d$, we see that $1 = T^k(4d) = T^{k-1}(2d) = T^{k-2}(d)$. This shows that each solution of $T^{k-2}(m) = 1$ produces exactly one solution of $T^k(n) = 1$ of the form $4d$.

Thus the number of solutions of $T^k(n) = 1$ is equal to the number of solutions of $T^{k-1}(m) = 1$ and the number of solutions of $T^{k-2}(l) = 1$ for $k > 2$. This shows that $c_k = c_{k-1} + c_{k-2}$ for $k > 2$. We also observe that 2 is the only number which goes to 1 in one step and 4 is the only number which goes to 1 in two steps. Hence $c_1 = 1$ and $c_2 = 2$. This proves that $c_n = f_{n+1}$ for all $n \in \mathbb{N}$.

Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\cdots\left[\frac{n}{n}\right]+[\sqrt{n}]$$

is **even**. (Here $[x]$ denotes the largest integer smaller than or equal to x.)

Solution. Let $f(n)$ denote the given equation. Then $f(1)=2$ which is even. Now suppose that $f(n)$ is even for some $n \geq 1$. Then

$$\begin{aligned} f(n+1) &= \left[\frac{n+1}{1}\right]+\left[\frac{n+1}{2}\right]+\left[\frac{n+1}{3}\right]+\cdots\left[\frac{n+1}{n+1}\right]+\left[\sqrt{n+1}\right] \\ &= \left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\cdots\left[\frac{n}{n}\right]+\left[\sqrt{n+1}\right]+\sigma(n+1)\,, \end{aligned}$$

where $\sigma(n+1)$ denotes the number of positive divisors of $n+1$. This follows from $\left[\frac{n+1}{k}\right]=\left[\frac{n}{k}\right]+1$ if k divides $n+1$, and $\left[\frac{n+1}{k}\right]=\left[\frac{n}{k}\right]$ otherwise. Note that $\left[\sqrt{n+1}\right]=[\sqrt{n}]$ unless $n+1$ is a square, in which case $\left[\sqrt{n+1}\right]=[\sqrt{n}]+1$. On the other hand $\sigma(n+1)$ is odd if and only if $n+1$ is a square. Therefore it follows that $f(n+1)=f(n)+2l$ for some integer l. This proves that $f(n+1)$ is even.

Thus it follows by induction that $f(n)$ is even for all natural number n.

Problem 3. *Let a, b, c, d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.*

Solution. Suppose that m is an integer root of $x^4 - ax^3 - bx^2 - cx - d = 0$. As $d \neq 0$, we have $m \neq 0$. Suppose now that $m > 0$. Then $m^4 - am^3 = bm^2 + cm + d > 0$ and hence $m > a \geq d$. On the other hand $d = m(m^3 - am^2 - bm - c)$ and hence m divides d, so $m \leq d$, a contradiction. If $m < 0$, then writing $n = -m > 0$ we have $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$, a contradiction. This proves that the given polynomial has no integer roots. □

Problem 4. *Let n be a positive integer. Call a nonempty subset S of $\{1, 2, \ldots, n\}$* good *if the arithmetic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets of $\{1, 2, \ldots, n\}$. Prove that t_n and n are both odd or both even.*

Solution. We show that $T_n - n$ is even. Note that the subsets $\{1\}, \{2\}, \cdots, \{n\}$ are good. Among the other good subsets, let A be the collection of subsets with an integer average which belongs to the subset, and let B be the collection of subsets with an integer average which is not a member of the subset. Then there is a bijection between A and B, because removing the average takes a member of A to a member of B; and including the average in a member of B takes it to its inverse. So $T_n - n = |A| + |B|$ is even. □

Alternate solution. Let $S = \{1, 2, \ldots, n\}$. For a subset A of S, let $\overline{A} = \{n + 1 - a | a \in A\}$. We call a subset A symmetric if $\overline{A} = A$. Note that the arithmetic mean of a symmetric subset is $(n+1)/2$. Therefore, if n is even, then there are no symmetric good subsets, while if n is odd then every symmetric subset is good.

If A is a proper good subset of S, then so is $\overline{A}$. Therefore, all the good subsets that are not symmetric can be paired. If n is even then this proves that t_n is even. If n is odd, we have to show that there are odd number of symmetric subsets. For this, we note that a symmetric subset contains the element $(n + 1)/2$ if and only if it has odd number of elements. Therefore, for any natural number k, the number of symmetric subsets of size $2k$ equals the number of symmetric subsets of size $2k+1$. The result now follows since there is exactly one symmetric subset with only one element. □

Problem 3. Let $\mathbb{N}$ denote the set of all positive integers. Find all real numbers c for which there exists a function $f : \mathbb{N} \to \mathbb{N}$ satisfying:

(a) for any $x, a \in \mathbb{N}$, the quantity $\frac{f(x+a)-f(x)}{a}$ is an integer if and only if $a = 1$;

(b) for all $x \in \mathbb{N}$, we have $|f(x) - cx| < 2023$.

Solution 1. We claim that the only possible values of c are $k + \frac{1}{2}$ for some non-negative integer k. The fact that these values are possible is seen from the function $f(x) = \left\lfloor \left(k + \frac{1}{2}\right) x \right\rfloor + 1 = kx + \left\lfloor \frac{x}{2} \right\rfloor + 1$. Indeed, if you have any $x, a \in \mathbb{N}$, then

$$\frac{f(x+a)-f(x)}{a} = \frac{1}{a}\left(ka + \left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor\right) = k + \frac{1}{a}\left(\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor\right).$$

This is clearly an integer for $a = 1$. But for $a \geq 2$, we have

$$\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor \geq \left\lfloor \frac{x+2}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor = 1.$$

If $a = 2k$, then

$$\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor = k < 2k = a,$$

and if $a = 2k + 1$ for $k \geq 1$, then

$$\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor \leq \left\lfloor \frac{x+2k+2}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor = k + 1 < 2k + 1 = a.$$

So in either case, the quantity $\left\lfloor \frac{x+a}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor$ is strictly between 0 and a, and thus cannot be divisible by a. Thus condition (a) holds; condition (b) is obviously true.

Now let us show these are the only possible values, under the weaker assumption that there exists some $d \in \mathbb{N}$ so that $|f(x) - cx| < d$. It is clear that $c \geq 0$: if $-d < f(x) - cx < d$ and $c < 0$, then for large x the range $[cx - d, cx + d]$ consists only of negative numbers and cannot contain $f(x)$.

Now we claim that $c \geq \frac{1}{2}$. Indeed, suppose that $0 \leq c < \frac{1}{2}$, and that $d > 0$ is such that $|f(x) - cx| \leq d$. Pick $N > \frac{2d}{1-2c}$ so that $2(cN + d) < N$. Then the N values $\{f(1), \ldots, f(N)\}$ must be all be in the range $\{1, \ldots, cN + d\}$, and by pigeonhole principle, some three values $f(i), f(j), f(k)$ must be equal. Some two of i, j, k are not consecutive: suppose WLOG $i > j + 1$. Then $\frac{f(i)-f(j)}{i-j} = 0$, which contradicts condition (a) for $x = j$ and $a = i - j$.

Now for the general case, suppose $c = k + \lambda$, where $k \in \mathbb{Z}$ and $\lambda \in [0, 1)$. Let $d \in \mathbb{N}$ be such that $-d \leq f(x) - cx \leq d$. Consider the functions

$$g_1(x) = f(x) - kx + d + 1, g_2(x) = x - f(x) + kx + d + 1.$$

Note that

$$g_1(x) \geq cx - d - kx + d + 1 = \lambda x + 1 \geq 1,$$
$$g_2(x) \geq x - (cx + d) + kx + d + 1 = (1 - \lambda)x + 1 \geq 1$$

so that these are also functions from $\mathbb{N}$ to $\mathbb{N}$. They also satisfy condition (a) for f:

$$\frac{g_1(x+a) - g_1(x)}{a} = \frac{f(x+a) - k(x+a) + d - f(x) + kx - d}{a} = \frac{f(x+a) - f(x)}{a} - k$$

is an integer if and only if $\frac{f(x+a)-f(x)}{a}$ is, which happens if and only if $a = 1$. A similar argument holds for g_2.

Now note that $g_1(x) - \lambda x = f(x) - cx + d + 1$ is bounded, and so is $g_2(x) - (1 - \lambda)x = cx - f(x) + d + 1$. So they satisfy the weaker form of condition (b) as well. Thus applying the reasoning in the second paragraph, we see that $\lambda \geq \frac{1}{2}$ and $1 - \lambda \geq \frac{1}{2}$. This forces $\lambda = \frac{1}{2}$, which finishes our proof. □

Solution 2. We will show that for any such c, we have $c > 0$ and $\{c\} = \frac{1}{2}$. Also 2023 can be replaced by any fixed $d \geq 1$ in condition (b) which we assume now.

Clearly $c \geq 0$ else for $c < 0$ and $x > \frac{d}{|c|}$, $cx - d < f(x) < cx + d < 0$ which is a contradiction. Suppose $\{c\} \neq \frac{1}{2}$. Put $r = \lfloor c \rfloor$ and $\lambda = \min(\{c\}, 1 - \{c\})$ and define

$$g(x) = \begin{cases} f(x) - rx & \text{if } \{c\} < \frac{1}{2} \\ x + rx - f(x) & \text{if } \{c\} > \frac{1}{2} \end{cases}$$

so that $|f(x) - cx| = |g(x) - \lambda x|$ and $g(x) \in \mathbb{Z}$ for all $x \in \mathbb{N}$. Here $0 \leq \lambda < \frac{1}{2}$. Take $N > 2(\lambda N + 2d)$. Then from $|g(x) - \lambda x| = |f(x) - cx| < d$, we get

$$-d \leq \lambda n - d < g(n) < \lambda n + d \leq \lambda N + d$$

for all $1 \leq n \leq N$. That is N integers $g(n), 1 \leq n \leq N$ can take at most $\lambda N + 2d$ values. Since $N > 2(\lambda N + 2d)$, by pigeonhole principle, there are 3 positive integers $i < j < k$ such that $g(i) = g(j) = g(k)$. Then $k - i \geq 2$ and

$$f(k) - f(i) = \begin{cases} g(k) + rk - (g(i) + ri) = r(k - i) & \text{if } \{c\} < \frac{1}{2} \\ (1 + r)k - g(k) - ((1 + r)i - g(i)) = (1 + r)(k - i) & \text{if } \{c\} > \frac{1}{2} \end{cases}$$

so that $\frac{f(k) - f(i)}{k - i}$ is an integer. This contradicts the condition (a). Also for each $c = k + \frac{1}{2}$, the function $f(x) = \lfloor (k + \frac{1}{2})x \rfloor$ satisfy the conditions (a) and (b). $\square$

Solution 3. We give a different proof that $\{c\} = 1/2$. Let us first prove a claim:

Claim. For any $k \geq 1$ and any x, $f(x + 2^k) - f(x)$ is divisible by 2^{k-1} but not 2^k.

Proof. We prove this via induction on k. For $k = 1$, the claim is trivial. Now assume the statement is true for some k, and note that $f(x + 2^k) - f(x) = 2^{k-1} y_1$ and $f(x + 2^k + 2^k) - f(x + 2^k) = 2^{k-1} y_2$ for some odd integers y_1, y_2. Adding these, we see that

$$f(x + 2^{k+1}) - f(x) = 2^{k-1}(y_1 + y_2)$$

which is divisible by 2^k because $y_1 + y_2$ is even. The fact that this is not divisible by 2^{k+1} follows from the condition on f. □

Now using this claim, we see that for any $k \geq 1$, $f(1 + 2^k) = f(1) + 2^{k-1}(2y_k + 1)$ for some integer y_k, which means

$$f(1 + 2^k) - c(1 + 2^k) = f(1) - c + 2^k \left(y_k + \frac{1}{2} - c \right).$$

Thus $2^k(y_k + \frac{1}{2} - c)$ is bounded. But $y_k + \frac{1}{2} - c$ has the same fractional part as $\frac{1}{2} - c$, so if this quantity is never zero, its absolute value must be at least $m = \min\left(\left\{\frac{1}{2} - c\right\}, \left\{c - \frac{1}{2}\right\}\right)$ and thus we have

$$2^k \left| y_k + \frac{1}{2} - c \right| \geq 2^k m,$$

contradicting boundedness. Thus we must have $y_k + \frac{1}{2} - c = 0$ for some k. Since y_k is an integer, so that $\{c\} = \frac{1}{2}$. □

A more rigorous treatment is given below.

Obtain

$$f(1 + 2^k) - c(1 + 2^k) = f(1) - c - 2^k \left(y_k + \frac{1}{2} - c \right)$$

as before. We obtain that $2^k |y_k + \frac{1}{2} - c| \leq M$ for some $M > 0$ by condition (b). Suppose that $\{c\} \neq \frac{1}{2}$. Writing $y_k + \frac{1}{2} - c = m_k + \delta$ with $m_k \in \mathbb{Z}$ and $0 \leq \delta < 1$, we have $0 < \delta < 1$. Then there exists $\ell > 1$ such that $\min(\delta, 1 - \delta) \geq \frac{1}{2^\ell}$. Hence

$$|y_k + \frac{1}{2} - c| = |m_k + \delta| \geq \begin{cases} \delta \geq \frac{1}{2^\ell} & \text{if } m_k \geq 0 \\ -m_k - \delta \geq 1 - \delta \geq \frac{1}{2^\ell} & \text{if } m_k < 0 \end{cases}$$

implying $M \geq 2^k |y_k + \frac{1}{2} - c| \geq 2^{k-\ell}$ which is a contradiction for large k. Thus $\{c\} = \frac{1}{2}$.

Problem 2. Find all pairs of integers (a, b) so that each of the two cubic polynomials

$$x^3 + ax + b \text{ and } x^3 + bx + a$$

has all the roots to be integers.

Solution. The only such pair is $(0, 0)$, which clearly works. To prove this is the only one, let us prove an auxiliary result first.

Lemma If α, β, γ are reals so that $\alpha + \beta + \gamma = 0$ and $|\alpha|, |\beta|, |\gamma| \geq 2$, then

$$|\alpha\beta + \beta\gamma + \gamma\alpha| < |\alpha\beta\gamma|.$$

Proof. Some two of these reals have the same sign; WLOG, suppose $\alpha\beta > 0$. Then $\gamma = -(\alpha + \beta)$, so by substituting this,

$$|\alpha\beta + \beta\gamma + \gamma\alpha| = |\alpha^2 + \beta^2 + \alpha\beta|, \ |\alpha\beta\gamma| = |\alpha\beta(\alpha + \beta)|.$$

So we simply need to show $|\alpha\beta(\alpha + \beta)| > |\alpha^2 + \beta^2 + \alpha\beta|$. Since $|\alpha| \geq 2$ and $|\beta| \geq 2$, we have

$$\begin{aligned} |\alpha\beta(\alpha + \beta)| &= |\alpha||\beta(\alpha + \beta)| \geq 2|\beta(\alpha + \beta)|, \\ |\alpha\beta(\alpha + \beta)| &= |\beta||\alpha(\alpha + \beta)| \geq 2|\alpha(\alpha + \beta)|. \end{aligned}$$

Adding these and using triangle inequality,

$$\begin{aligned} 2|\alpha\beta(\alpha + \beta)| &\geq 2|\beta(\alpha + \beta)| + 2|\alpha(\alpha + \beta)| \geq 2|\beta(\alpha + \beta) + \alpha(\alpha + \beta)| \\ &\geq 2(\alpha^2 + \beta^2 + 2\alpha\beta) > 2(\alpha^2 + \beta^2 + \alpha\beta) \\ &= 2|\alpha^2 + \beta^2 + \alpha\beta|. \end{aligned}$$

Here we have used the fact that $\alpha^2 + \beta^2 + 2\alpha\beta = (\alpha + \beta)^2$ and $\alpha^2 + \beta^2 + \alpha\beta = \left(\alpha + \frac{\beta}{2}\right)^2 + \frac{3\beta^2}{4}$ are both nonnegative. This proves our claim. □

For our main problem, suppose the roots of $x^3 + ax + b$ are the integers r_1, r_2, r_3 and the roots of $x^3 + bx + a$ are the integers s_1, s_2, s_3. By Vieta's relations, we have

But $x_1+y_1+z_1$ is odd, and hence non-zero, so this cannot happen.

Thus we can assume WLOG that $\nu_2(x) > \nu_2(y)$. Then the third root is $-(x+y)$. Similarly, the three roots of x^3+bx+a can be written as $p, q, -(p+q)$ where $\nu_2(p) > \nu_2(q)$. By Vieta's relations,

$$xy - x(x+y) - y(x+y) = -(x^2+xy+y^2) = a = pq(p+q)$$
$$pq - p(p+q) - q(p+q) = -(p^2+pq+q^2) = b = xy(x+y)$$

Suppose $x = 2^k x_1$ and $y = 2^\ell y_1$ for odd x_1, y_1 and $k > \ell$; in particular $k > 0$. Then

$$xy(x+y) = 2^k x_1 \cdot 2^\ell y_1 \cdot (2^k x_1 + 2^\ell y_1) = 2^{k+2\ell} x_1 y_1 (2^{k-\ell} x_1 + y_1).$$

Here $x_1 y_1 (2^{k-\ell} x_1 + y_1)$ is clearly odd, so $\nu_2(xy(x+y)) = k + 2\ell$.

Also,

$$x^2+xy+y^2 = 2^{2k} x_1^2 + 2^k x_1 \cdot 2^\ell y_1 + 2^{2\ell} y_1^2 = 2^{2\ell}\left(2^{2k-2\ell} x_1^2 + 2^{k-\ell} x_1 y_1 + y_1^2\right).$$

Again, all the terms in the second factor are even except y_1^2, so the entire factor is odd. This means $\nu_2(x^2+xy+y^2) = 2\ell$. Therefore

$$\nu_2(xy(x+y)) > \nu_2(x^2+xy+y^2).$$

Similarly, one may show

$$\nu_2(pq(p+q)) > \nu_2(p^2+pq+q^2).$$

But then

$$\nu_2(b) = \nu_2(xy(x+y)) > \nu_2(x^2+xy+y^2) = \nu_2(pq(p+q)) > \nu_2(p^2+pq+q^2) = \nu_2(b).$$

Here we have used the fact that $\nu_2(n) = \nu_2(-n)$ for any integer n. But this is a contradiction, proving our claim. $\square$

$$r_1 + r_2 + r_3 = 0 = s_1 + s_2 + s_3$$
$$r_1r_2 + r_2r_3 + r_3r_1 = a = -s_1s_2s_3$$
$$s_1s_2 + s_2s_3 + s_3s_1 = b = -r_1r_2r_3$$

If all six of these roots had an absolute value of at least 2, by our lemma, we would have

$$|b| = |s_1s_2 + s_2s_3 + s_3s_1| < |s_1s_2s_3| = |r_1r_2 + r_2r_3 + r_3r_1| < |r_1r_2r_3| = |b|,$$

which is absurd. Thus at least one of them is in the set $\{0, 1, -1\}$; WLOG, suppose it's r_1.

1. If $r_1 = 0$, then $r_2 = -r_3$, so $b = 0$. Then the roots of $x^3 + bx + a = x^3 + a$ are precisely the cube roots of $-a$, and these are all real only for $a = 0$. Thus $(a, b) = (0, 0)$, which is a solution.

2. If $r_1 = \pm 1$, then $\pm 1 \pm a + b = 0$, so a and b can't both be even. If $a = -s_1s_2s_3$ is odd, then s_1, s_2, s_3 are all odd, so $s_1 + s_2 + s_3$ cannot be zero. Similarly, if b is odd, we get a contradiction.

The proof is now complete. □

Alternate Solution. The only such pair is $(0, 0)$, which clearly works. Let us prove this is the only one. In what follows, we use $\nu_2(n)$ to denote the largest integer k so that $2^k | n$ for any non-zero $n \in \mathbb{Z}$.

If one of the cubics has 0 as a root, say the first one, then $0^3 + 0 \cdot a + b = 0$, so $b = 0$. Then the roots of $x^3 + bx + a = x^3 + a$ are precisely the cube roots of $-a$, and these are all real only for $a = 0$. Thus $(a, b) = (0, 0)$.

So suppose none of the roots are zero. Take the cubic $x^3 + ax + b$, and suppose its roots are x, y, z. We cannot have $\nu_2(x) = \nu_2(y) = \nu_2(z)$; indeed, if we had $x = 2^k x_1, y = 2^k y_1, z = 2^k z_1$ for odd x_1, y_1, z_1, then

$$0 = x + y + z = 2^k(x_1 + y_1 + z_1).$$

THANK YOU