BASIC MATHEMATICAL MODELLING OF HEAT AND BLOOD TRANSPORT IN HUMAN EYE

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by

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CERTIFICATE

This is to certify that the dissertation entitled, **"BASIC MATHEMATICAL MODELLING OF HEAT AND BLOOD TRANSPORT IN HUMAN EYE"** *being submitted by the students with the enrollments* ***21068120005, 21068120013, 21068120031, 21068120034, 21068120039*** *to the* ***Department of Mathematics, University of Kashmir, Srinagar,*** *for the award of Master's degree in Mathematics, is an original project work carried out by them under my guidance and supervision.*

The project dissertation meets the standard of fulfilling the requirements of regulations related to the award of the Master's degree in Mathematics. The material embodied in the project dissertation has not been submitted to any other institute, or to this university for the award of Master's degree in Mathematics or any other degree.

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Abstract

The project dissertation comprises four chapters that focuses on basic mathematical modelling of heat and blood transport in human eye. The first chapter provides an introductory overview of the subject and anatomy of human eye.

The second chapter explains the heat transfer mechanism in human eye along with the background of the topic.

In the third chapter, we have gone through the one dimentional heat distribution in human eye using variational finite element method.

In the last chapter, we have reviewed the paper entitled "Thermal behaviour of human eye in relation with change in bloood perfusion, porosity, evaporation and ambient temperature" published in *Journal of Thermal Biology, 62(2016), 138-142* by Aasma Rafiq and M.A.Khanday. In the last section of this chapter, we have also discussed how to model blood transport in human eye.

Contents

List of Figures

Chapter 1

General introduction

Section 1.1

Introduction to Mathematical biology

Mathematical biology is an interdisciplinary field that uses mathematical models, techniques, and tools to study and understand biological phenomena, processes, and systems. It seeks to describe, analyze, and predict various biological phenomena and their underlying mechanisms using mathematical equations, computer simulations, and quantitative methods. Mathematical biology bridges the gap between biology and mathematics. It draws upon concepts from various mathematical disciplines such as calculus, differential equations, probability theory, and statistics to address complex biological questions[48]. Mathematical biologists create mathematical models to represent biological systems. Models can describe the behavior of individual organisms, populations, ecosystems, and even cellular processes. Mathematical biology has a wide range of applications, including epidemiology (the study of disease spread)[1], population dynamics (e.g., predator-prey interactions)[2], ecology (e.g., species interactions and ecosystem modelling), genetics (e.g., gene expression and inheritance), and physiology (e.g., modelling physiological processes). In general , mathematical biology provides a powerful framework for gaining insights into biological systems, making predictions, and advancing our understanding of the living world by harnessing the power of mathematics and computational techniques. It is a dynamic field at the intersection of mathematics and biology, contributing to advancements in various areas of science and medicine.

Section 1.2

Brief Concept of Human Physiology

Human physiology is the branch of biology that focuses on understanding how the human body functions[4]. It involves the study of various systems, organs, and processes that work together to maintain the body's internal balance and allow it to carry out essential functions. It includes the study of nervous, endocrine, cardiovascular, respiratory, and urinary systems as well as cellular and exercise physiology. Understanding human physiology is essential for diagnosing and treating health conditions and promoting wellbeing. One of the central principles of human physiology is homeostasis. The concept of homeostasis was given by French Scientist Claude Bernard in 1865[5]. The word homeostasis has been derived from two Greek words *homoios* meaning same or resembling and *stasis* means posture or to stand[6]. This concept refers to the body's ability to maintain a stable and balanced internal environment despite external changes.

Section 1.3

Anatomy of Human Eye

The eye is a complex optical system. A relatively small organ in the human body, the eye is a passageway to understanding and emotion. Not only does the eye allows us to see and interpret the shapes, colors, and dimensions of objects in the world by processing the

light they reflect or emit, but it also enables us to see and interpret unspoken words and unexplainable environments. It acts as a transducer as it changes light rays into electrical signals and transmits them to the brain, which interprets these electrical signals as visual images. It is protected by the cone-shaped cavity in the skull called the orbit or socket which protects them against any injury[4]. The eye is a spherical organ which measures approximately one inch or 2.5cm in diameter. It is composed of three main layers:
(i) Fibrous layer (ii) Vascular layer (iii) Retina.
In addition to these layers, there are other parts like lens, aqueous humour and vitreous humour as shown in figure(1.1) . Each of these layers have different function to perform. The fibrous layer of eye tissue allows light to enter into the eye, nourishes the eye and controls the amount of light to enter into the eye. The vascular layer helps in protecting the different portions of eye. The retina is the sensitive portion of eye which converts the image into electrical impulse to be interpreted by brain.The detailed description of different layers of eye is given below:

1.3.1 Fibrous Layer

Fibrous layer is the thick and tough layer which protects the eyeball. It also helps in maintaining the shape and form of eyeball. This layer has two distinct and unequal regions viz.,
(a) sclera (b) cornea.
(a) Sclera: It forms the posterior five-sixth of the fibrous layer. It is commonly known as "the outer wall of the eye". It is tough, opaque and bluish-white. It is largely hidden in the orbit. The sclera serves to support and protects the inner parts of the eye. It contains about 68% of water[7].
(b) Cornea: It forms the anterior one-sixth of the fibrous layer. The cornea is the transparent, dome-shaped window covering the front of the eye. This contains 78% of water[10]. An adult cornea has a front surface of radius about 8mm. The cornea helps in the image formation by refracting light entering into the eye. The cornea is a non-vascular structure

as the capillaries that supply nutrients to the cornea terminate in loops at its circumference.

1.3.2 Vascular Layer

The vascular layer is the middle layer of the eye tissue which contains much of the eye's pigment. The vascular layer or uvea consists of three regions.

(a) Choroid

(b) Iris

(c) Ciliary body

(a) Choroid: The choroid is also known as the choroidea or choroid coat and it lies in the region between the retina and sclera. This section of vascular layer is dark brown in colour containing blood vessels and gives nourishment to our eyes.

(b) Iris: The iris is that part of the vascular layer of the eye tissue which determines a person's eye colour (blue, green, brown). This is a pigmented tissue which lies behind the cornea and infront of the natural lens. The iris acts as a camera shutter which controls the amount of light entering the eye. There is a small opening in the center of iris called pupil. The pupil is small in bright light and large in dim light. The size of pupil usually varies with age.

(c) Ciliary body: The ciliary body is located behind the iris and acts as an instrument for controlling the focusing of the eye and the production of aqueous fluid. The ciliary body is a well vascularized tissue with high rate of blood flow.

1.3.3 Retina

The retina is a multi-layered sensory tissue that lies at the back of eye and contains millions of photo-receptors that capture light rays and convert them into the electrical impulses which travel to brain through optic nerve where they are turned into images. Retina is separated into two layers - the outer layer or pigmented layer which absorbs light as well as removes damaged and dead photoreceptor cells. This layer also helps to recycle the vitamin A product that is very essential for eye's nourishment. The second layer is the inner layer or the neural layer which contains the photo-receptors and other cells that allow a person to see.

There are mainly two types of shapes for the photo-receptors in the retina namely rods and cones. There are nearly about 6 million cones which are contained in macula, that portion of retina which is responsible for vision. Cones are used for day vision and in order to function these needs a lot of light. The rod type of photo-receptors are about 125 million in number, they are responsible for night vision and lack of them causes night blindness.

1.3.4 Lens

Lens is located directly behind the iris which helps to focus the rays of light onto the retina. The softer material called cortex surrounds the innermost part of the lens (nucleus). The lens is encased in a capsule like bag and suspended within the eye by tiny wires called Zonules[9]. There is about 65 % of water in lens and it decreases with ageing[8]. The lens is separated from aqueous chamber by capsule posterior and the epithelium capsule anterior, so any damage to the capsule may lead to the occurrence of the cataracts.

1.3.5 Aqueous humour

The watery fluid that is continually secreted by the ciliary body fill the space between the cornea and iris. This fluid nourishes the cornea and the lens, and also gives the front of eye its shape and form.

1.3.6 Vitreous humour

The chamber lying behind lens and infront of the retina is filled with a gelatinous fluid called the vitreous humour. It is composed of water and comprises about 2/3 of the eye's volume. The main function of vitreous humour is to retain the eye to its actual shape when compressed.

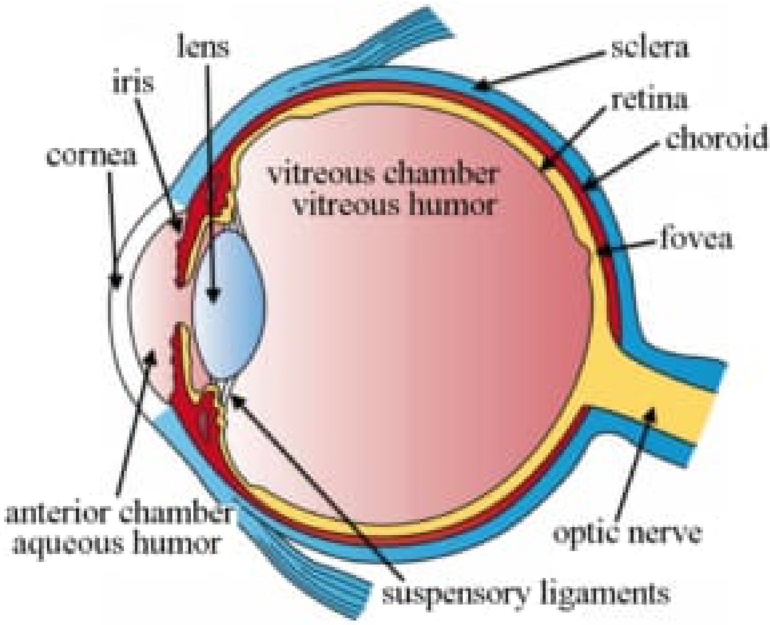

Figure 1.1: Schematic diagram of multilayered human eye
(https://en.m.wikipedia.org/wiki/Intravitreal administration)

Chapter 2

Heat Transfer Mechanism in Human Eye

Section 2.1

Introduction

Heat transfer is that science which seeks to predict the energy transfer that may take place between material bodies as a result of a temperature difference. The science of heat transfer seeks not only to explain how heat energy may be transformed, but also to predict the rate at which the exchange will take place under certain specified conditions.

Heat transfer describes the exchange of thermal energy between physical systems depending on the temperature and pressure by dissipating heat. Heat transfer is the exchange of kinetic energy of particles through the boundary between two systems which are at different temperatures from each other or from their surroundings[28]. Heat transfer always occurs from a region of higher temperature to another region of lower temperature. Heat transfer changes the internal energy of both bodies involved according to the first law of thermodynamics.

The principles of heat transfer in engineering systems can be applied to the human body in order to determine how the body transfers heat. Heat is produced in the body by continuous metabolism of nutrients which provides energy for the systems of the body.

The human body must maintain a consistent internal temperature in order to maintain healthy body functions. Therefore, excess heat must be dissipated from the body to avoid overheating. When a person engages in elevated levels of physical activity, the body requires additional fuel which increases the metabolic rate and the rate of heat production.

The body temperature of human beings remains relatively constant, despite considerable change in the external conditions[29]. In order to maintain a constant core temperature, the body must balance the amount of heat it produces and absorbs with the amount it loses; this is thermoregulation[30]. Thermoregulation maintains the core temperature at a constant set point, average 37°C, despite fluctuations in heat absorption, production and loss.

The body temperature is determined by the balance between heat produced and heat lost. Heat is lost or gained by radiation, evaporation, conduction and convection. Heat gain occurs due to internal metabolism and blood flow throughout the body. The metabolic heat is produced in the body due to breakdown, synthesis and utilization of food.

Besides conduction, convection, radiation and evaporation, there are many factors that directly affect heat transfer mechanism in human eye. One of the most important factor is blinking of eyelids. Although there is no skin layer that covers eye permanently as internal body organs, there are eyelids which covers eye frequently during blinking. In extreme environmental conditions like hot/cold temperatures, high wind flow, accidental exposure to UV/IR rays etc., eyelid blinking increases or covers eye surface completely for a while. With the eyelid closed, the temperature of the anterior eye is increased about 2°C by blood flow in the eyelid[24]. With the increased eyelid blinking, it picks/deposits heat energy from/to cornea to maintain constant eye temperature.

Another important factor is tearing. With each blink, a warm tear secreted at body core temperature is layered across the cornea. In extreme conditions this tear layer heats up or cool down the eye surface temperature. Mapstone indicated that a rapid increase in corneal temperature (i.e. in a matter of second) can result only from tearing since other factors need some time to act[26]. In cold outdoor conditions, eye may require instant heat (cold weather in combination with wind) to maintain constant temperature and this instant heating is possible via tearing. In hot outdoor conditions, eye surface evaporates water to cool the cornea. The cooling rate depends on how fast the water evaporates from the surface. Moist surface evaporates more water than dry surface. Tearing continuously moistens the cornea during blinking. This helps to evaporate more water from eye surface and hence cooling the cornea.

Blood perfusion also plays an important role in human eye heat transfer mechanism. As we mentioned in section 1.3, only retina, choroid, iris and ciliary body have blood flow inside the eye. The continuous chain of retina/choroid, ciliary body and iris covers vitreous humor, lens and posterior part of aqueous humor. Outside this chain there is sclera and cornea, the outer cover of eye. The temperature in internal parts (vitreous humor, lens, posterior aqueous) is maintained by high blood perfusion and metabolism in these four layers (choroid, iris, retina and ciliary body). The blood perfusion rate in choroid/iris is the highest among any other tissue in the human body[27]. The metabolic heat generation rate is the highest in the retina among any other tissue in human body[22]. These statements reveal that retina/choroid/iris/ciliary body are the power sources of eye and these play an important role in human eye heat transfer.

1.4.1 Conduction

Conduction is heat transfer by means of molecular agitation within the human body without any motion of the molecules as a whole[41]. Heat conduction occurs as hot and rapidly moving or vibrating molecules interact with neighboring molecules, transferring some of their energy (heat) to neighboring molecules. Conduction is the most significant means of heat transfer within a solid or between solid objects in thermal contact[23]. Solids are highly conductive and fluids specially gases are less conductive. Approximately 3% of body heat is lost through conduction. The mathematical model for heat transfer through conduction is described by Fourier's law of conduction. When a temperature gradient exists in a body, experiments have shown that there is an energy transfer from the high temperature region to low temperature region. We say that energy is transferred by conduction and that the heat transfer rate per unit area is proportional to the normal temperature gradient.

$$\frac{Q_{cond}}{A} \propto \frac{\partial T}{\partial x} \tag{1.1}$$

When the proportionality constant is inserted,

$$Q_{cond} = -KA\frac{\partial T}{\partial x} \tag{1.2}$$

where Q_{cond} is the heat transfer rate and $\frac{\partial T}{\partial x}$ is the temperature gradient in the direction of the heat flow. The positive constant K is called thermal conductivity of the material and the minus sign is inserted so that the second principle of thermodynamics will be satisfied i.e. heat must flow downhill (high temperature region to low). Equation (1.2) is called Fourier's law of heat conduction.

1.4.2 Convection

Convection is the exchange of body heat with the external environment. If the body temperature is higher than its external ambient temperature, then heat flows from the body to the surrounding air causing it to heat up, is replaced by the more dense and cool air at the peripheral regions of the body. Thus, cool air which moves continuously up to the body surface gets warmed by body heat and then flows away. This results heat loss from the body surface. Therefore, the transfer of heat to a moving fluid is termed as convection.

The Newton's law of cooling provides a physical model of convection heat transfer. According to Newton's law of cooling, heat flux due to convection is directly proportional to the difference in temperature between the surface and the fluid.

That is

$$\frac{Q_{conv}}{A} \propto (T_s - T_f) \tag{1.3}$$

where Q_{conv} is heat transfer rate due to convection, T_s is surface temperature and T_f is the fluid temperature far away from the surface.
Introducing a proportionality constant, we obtain

$$Q_{conv} = hA(T_s - T_f) \tag{1.4}$$

The constant of proportionality h is called heat transfer coefficient. Equation (1.4) is the rate equation due to convection

1.4.3 Radiation

Radiation is another means of heat exchange between human body and surroundings through infrared rays. All objects including human body that are not at the absolute temperature radiate heat energy from such rays. Human body radiate heat rays in all directions. However, walls and the other objects radiate heat rays towards the body surface.

In human eye, the outer surface is transparent cornea. The thermal radiation from transparent body surface is negligible. However, inside cornea there is a highly pigmented tissue layer, iris. Iris is a colorful organ (blue or black or brown) and having high blood perfusion. Its surface temperature is supposed to be at 37°C. Because of highly pigmented tissue and of nearly black color, it exerts thermal radiation, which passes through the cornea to the environment. Thus, thermal radiation exerts from cornea and heat loss occurs via radiation.

Thermodynamic consideration shows that an ideal thermal radiation, or black body, will emit energy at the rate proportional to the fourth power of the absolute temperature of the body and its surface area. Thus

$$Q_{black\ body} = \sigma A T^4 \tag{1.5}$$

where σ is proportionality constant and is called the Stefan - Boltzmann constant with the value $5.67 \times 10^{-8} W/m^2K^4$. Equation (1.5) is called Stefan-Boltzmann law of thermal radiation and it applies only to black bodies.

1.4.4 Evaporation

Evaporation is the process whereby a liquid can be transformed into vapor. This implies a phase change, or change of state, and it is an example of a latent heat change. In latent heat change, the evaporative energy loss depends on the mass of the liquid and the energy required to vaporize the liquid. The energy which changes a gram of a liquid into the gaseous state at the boiling point is called the latent heat of vaporization (L). The evaporation of 1 gram of sweat removes 580 kcal or 2426 kJ of heat energy. If Qe denotes the rate of heat exchange from the body due to evaporation, A denotes the area and H denotes the rate of sweat evaporation,
then

$$\begin{aligned} Qe &= ALH \\ &= AE \end{aligned} \tag{1.6}$$

where

$$E = LH.$$

Section 2.2

Mathematical Techniques

A number of problems in science and technology can be addressed by formulating their suitable models with the help of differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. These methods produce good results in many boundary value problems where the analytical methods and exact solutions are not available. These methods are of even greater importance when we realise the availability of computing machines for the simulation of results by means of different softwares like MATLAB, MAPLE, MATHEMATICA etc., which considerably reduce the numerical work.

2.2.1 FINITE DIFFERENCE METHOD

The simple one dimensional heat equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}; \quad 0 \leq x \leq L \text{ and } t \geq 0 \tag{2.7}$$

where, $T = T(x,t)$ is a dependent variable and α is constant.

Let the boundary conditions and the initial conditions of equation (2.7) respectively be the following

$$T(0,t) = T_0 \quad \text{and} \quad T(L,t) = T_L \tag{2.8}$$

$$T(x,0) = f_0(x). \tag{2.9}$$

One way to numerically solve this equation is to approximate all the derivatives by finite differences. We form the mesh by partitioning the total length by points $x_0, x_1, x_2, \ldots, x_n$ and the time interval as $t_o, t_1, t_2, \ldots, t_m$. We assume a uniform partition both in space and in time, so that difference between two consecutive space points will be $\triangle x$ and between the two consecutive time points will be $\triangle t$. The points are denoted in the short form as

$$T(x_i, t_j) = T_{i,j}$$

2.2.1.1 Forward Time Centered Space

Replacing the time derivative in equation (2.7) with the forward difference

$$\left.\frac{\partial T}{\partial t}\right|_{T_{j+1,x_i}} = \frac{T_{i,j+1} - T_{i,j}}{\triangle t}$$

and $\dfrac{\partial^2 T}{\partial x^2}$ by its central difference approximations.

i.e.

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\triangle x^2}$$

Hence equation (2.7) becomes

$$\frac{T_{i,j+1} - T_{i,j}}{\triangle t} = \alpha \left(\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\triangle x^2} \right) \tag{2.10}$$

Solving for $T_{i,j+1}$ in terms of other values of T, equation (2.10) becomes

$$T_{i,j+1} = T_{i,j} + \alpha \frac{\triangle t}{\triangle x^2}(T_{i-1,j} - 2T_{i,j} + T_{i+1,j})$$
$$T_{i,j+1} = rT_{i+1,j} + (1-2r)T_{i,j} + rT_{i-1,j} \tag{2.11}$$

where, $r = \frac{\triangle t}{\triangle x^2}$
Equation (2.11) is called **Forward Time Centered Space or FTCS** approximation of heat equation. Equation (2.11) can be expressed in matrix form as

$$T_{j+1} = AT_j$$

Where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ r & 1-2r & r & 0 & \dots & 0 & 0 \\ 0 & r & 1-2r & r & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & 0 & r & 1-2r & r \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{2.12}$$

is a tridiagonal matrix, T_j and T_{j+1} are the vectors of T values respectively at the time steps j and $(j+1)$. The first and the last rows of A are adjusted so that the boundary values are not changed.

Equation (2.11) is used to estimate the values of the solution T(x,t) at the points on the $(j+1)^{th}$ time line using only values from the j^{th} time line. This kind of numerical procedure is called explicit finite difference method.

2.2.1.2 Backward Time Centered Space

The time derivative in equation (2.7) can be replaced with the back- ward difference

$$\left.\frac{\partial T}{\partial t}\right|_{T_{j+1,x_i}} = \frac{T_{i,j} - T_{i,j-1}}{\triangle t}$$

and the second order partial derivative by its central difference approximations.

Hence equation (2.7) becomes

$$\frac{T_{i,j} - T_{i,j-1}}{\Delta t} = \alpha \left(\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} \right) \tag{2.13}$$

The three terms in equation (2.13) refer to T at the jth level and only one term refers to T at $(j-1)th$ level. If T is known at the mesh points on $(j-1)th$ level, then the values at the jth level can be computed from equation (2.13) by solving the system of equations.

Rewriting (2.13) in the following form

$$T_{i,j} - T_{i,j-1} = r(T_{i+1,j} - 2T_{i,j} + T_{i-1,j})$$

$$-rT_{i-1,j} + (1+2r)T_{i,j} - rT_{i+1,j} = T_{i,j-1}, \quad (1 \le i \le n-1) \tag{2.14}$$

Equation (2.14) is called **Backward Time Centered Space or BTCS** approximation of heat equation. Equation (2.14) can be expressed in matrix form as

$$BT_j = T_{j-1}$$

Where,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -r & 1+2r & -r & 0 & \dots & 0 & 0 \\ 0 & -r & 1+2r & -r & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & . & \dots & 0 & 0 \\ . & . & . & 0 & -r & 1+2r & -r \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{2.15}$$

where T_j and T_{j-1} are the vectors at jth and $(j-1)th$ time level respectively. This kind of numerical procedure is called implicit finite difference method. BTCS scheme requires solving a system of equations at each time step. The computational effort per time step for BTCS is much greater than the computational effort per time step of FTCS.

2.2.2 CRANK-NICHOLSON METHOD

The algorithm introduced by John Crank and Phyllis Nicholson in 1947 is used mostly for solving the heat equation. In this algorithm, we replace the second order partial derivative in heat equation by an average of two central difference quotients. Hence equation (2.7) can be approximated as

$$\frac{T_{i,j+1} - T_{i,j}}{\triangle t} = \frac{\alpha}{2}\left[\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\triangle x^2} + \frac{T_{i+1,j+1} - 2T_{i,j+1} + T_{i-1,j+1}}{\triangle x^2}\right]$$

or, it can be written as

$$T_{i,j+1} - T_{i,j} = \frac{r}{2}\left[T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + T_{i+1,j+1} - 2T_{i,j+1} + T_{i-1,j+1}\right]$$

or

$$-T_{i-1,j+1} + \beta T_{i,j+1} - T_{i+1,j+1} = T_{i+1,j} - \gamma T_{i,j} + T_{i-1,j} \tag{2.16}$$

$for\ j = 0, 1, 2, \ldots, m-1$ and i = 1,2,...,n-1 where $\beta = 2(\frac{1}{r}+1)$ and $\gamma = 2(1-\frac{1}{r})$ For each choice of j, the difference equation (2.16) for $i = 1, 2, \ldots, n-1$ gives $(n-1)$ equations in $(n-1)$ unknowns $T_{i,j+1}$. Due to the boundary conditions the values $T_{i,j+1}$ are known for $i = 0$ and $i = n$. Equation (2.16) can be used to determine the values of T on the $(j+1)st$ time line. In matrix form equation (2.16) can be expressed as

$$AX = B.$$

Where A is a tridigonal matrix.

$$i.e,\ A = \begin{bmatrix} \beta & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & \beta & -1 & 0 & \ldots & 0 & 0 \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & \beta & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & \beta \end{bmatrix} \tag{2.17}$$

and $B = [b_1, b_2, \ldots, b_{n-1}]$ is a column matrix with entries

$b_1 = T_{2,j} - \gamma T_{1,j} + T_{0,j} + T_{0,j+1}$

$b_2 = T_{3,j} - \gamma T_{2,j} + T_{1,j}$

$b_3 = T_{4,j} - \gamma T_{3,j} + T2, j$

$\dots$

$\dots$

$\dots$

$b_{n-1} = T_{n,j} - \gamma T_{n-1,j} + T_{n-2,j} + T_{n,j+1}.$

2.2.3 FINITE ELEMENT METHOD

The FEM is one of the most popular and advanced mathematical cum numerical technique for obtaining an approximate solution to the complex boundary value problems in various engineering and science fields. In the variational finite element method, the domain of the problem is divided into a finite number of sub-domains called elements and variational functional is obtained at each of the sub-domain. The approximate solution for each element is expressed in terms of undetermined nodal values as appropriate shape functions or interpolating functions. The algebraic equations for the elements are assembled over the entire region and boundary conditions are suitably incorporated. The equations are solved for the nodal values and the approximate solution is obtained as piecewise interpolation function. The various steps involved in the FEM technique are summarised as :-

Step 1: Definition of the Problem and its Domain

In this step, the characteristics of the problem and its domain are defined. The domain may be defined physically and geometrically.

Step 2: Discretization of the Domain

The domain is divided into a number of sub-domains called elements. The system is usually divided into uniform or non-uniform line segments, triangles, rectangles or quadrilaterals depending upon the geometry of the domain. The elements are joined to each other at the limited number of points called nodes. The collection of elements is called the finite element mesh of the domain. The domain is divided into elements because of two main reasons-firstly to represent the geometry of the domain and secondly to approximate the solution over each element of the mesh in order to better represent the solution over the entire domain.

Step 3: Derivation of Element Equations

For the given differential equation, a variational formulation is constructed over a typical element. The derivation of finite element equations involves three steps:

i) Construct the weighted - residual or weak form of the differential equation.

ii) Assume the form of the approximate solution over a typical finite element.

iii) Derive the finite element equations by substituting the approximate solution into the weighted residual or weak form.

. **Step 4: Connectivity of elements**

While deriving the element equations, we isolate a particular element from the mesh and formulate the variational problem and develop its finite element model. To solve the total problem, we put the elements back into the original positions and the assembly of elements is carried out by imposing intermediate element continuity conditions. Then the entire system takes the matrix form.

Step 5: Imposition of the boundary conditions

The system of equations as obtained in Step 4 are modified by using the boundary conditions of the problem.

Step 6: Solution of Equations

After incorporating the boundary conditions, the simultaneous system of equations are solved by using standard techniques or by using MATLAB software.

2.2.4 VARIATIONAL FINITE ELEMENT METHOD

Variational finite element method is an extension of the finite element method. Like in FEM, the steps involved in the variational finite element method are summarised below:

Step 1: Transform differential equation into variational form.

The given differential equation is transformed into variational integral form. From the Calculus of variations, it follows that the variational integral and the given differential equation are equivalent as the function that satisfy the differential equation and the boundary conditions also extremize the functional.

Variational FEM is used for finding the solution of the functional

$$I = \int_a^b F(x, T(x), T'(x))dx \tag{2.18}$$

subject to the conditions

$$T(a) \ = \ T_1 \ \ and \ \ T(b) \ = \ T_2 \tag{2.19}$$

We wish to find the function $T(x)$ that satisfies equation (2.19) and minimizes the functional (2.18). From the Calculus of variations, the necessary condition for $T(x)$ to minimize $I(T)$ is to satisfy Euler-Lagrange's equation which is given as:

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0$$

where $T' = \frac{dT}{dx}$.

Step 2: Discretization

The domain is divided into a finite number of elements. The number, shape, size and configuration of these elements is selected in such a manner that the original structure of the domain is represented by the mesh of elements as closely as possible. The variational integral is defined for each element. So, we may write

$$I(T) = \sum_{k=1}^{n} I_k \tag{2.20}$$

where n represents the number of elements of domain under consideration.

Step 3: Selection of the interpolation function

An approximate function is chosen over each element such that it satisfies certain conditions. The polynomial represents the simplest form of the approximating function. If we take linear interpolation, then the function $T(x)$ may be approximated over the element (e) by

$$T^{(i)} = A_i + B_i x, \quad (i = 0, 1, \ldots, n)$$
$$where\ A_i = \frac{x_{i+1}T_i - x_i T_{i+1}}{x_{i+1} - x_i} \quad and \quad B_i = \frac{T_{i+1} - T_i}{x_{i+1} - x_i}$$

This is also called the shape function.

Step 4: Finding element matrices and equations

The element matrices and equations are obtained by using shape function and performing the necessary integration or differentiation wherever needed. Integration is performed for each element and the results are obtained in terms of the nodal values.

Step 5: Assembly of element equations

The element equations are assembled such that the total solution is continuous. Assemble all the element equations so as to get

$$I(T) = \sum_{k=1}^{n} I_k$$

where I_k represents the element functional.

Step 6: Differentiation

$I(T)$is differentiated w.r.t. the nodal values and equated to zero to obtain the system of equations i.e.,

$$\frac{\partial I}{\partial T_k} = 0, \quad (k = 0, 1, 2, \ldots, n)$$

Step 7: Imposition of boundary conditions

Boundary conditions are imposed on the simultaneous system of equations to reduce them into the condensed form.

Step 8: Solution of equations

The system of equations are solved to determine the unknown nodal values which can be substituted in the shape function to find the nature of the variable within each element.

Section 2.3

Bio-heat equation

Heat transfer in living tissues is a complex process as it includes conduction, convection, radiation, metabolism, evaporation and inherent temperature regulation. Blood perfusion has a remarkable effect on the temperature distribution in the living tissues. In 1948, Pennes was the first to propose and validate experimentally an analytical bio-heat transfer model with a heat loss term due to blood perfusion[39]. He suggested that the rate of heat transfer between blood and tissue is proportional to the product of the volumetric perfusion rate and the difference between the arterial blood temperature and the local tissue temperature.The following mathematical relationship in this direction is given below

$$h_p = \omega \rho_b c_b (1 - \nu)(T_A - T) \tag{3.21}$$

where h_p is the rate of heat transfer per unit volume of tissue, ω is the perfusion rate per unit volume of tissue, ρ_b is the density of blood, c_b is the specific heat of blood, $\nu(0 \leq \nu \leq 1)$ is a factor that accounts for incomplete thermal equilibrium between blood in tissue, T_A is the arterial blood temperature and T is the tissue temperature. He assumed $\nu = 0$ when he computed his theoretical curves and also incorporated the effect of metabolism.

The bio-heat equation is extensively used in investigation of many heat transfer problems with bio-medical applications. The general bio-heat equation is

$$\rho c \frac{\partial T}{\partial t} = \nabla(\lambda \nabla T) + \omega \rho_b c_b (T_a - T) + S \tag{3.22}$$

where,

c is tissue specific heat

ρ is tissue density

∇ is divergence

λ is tissue thermal conductivity

S is rate of metabolic heat generation

t is time.

The first term on the right side of the equation accounts for the heat being conducted through the various layers of tissues with differing thermal properties using Fourier's law. The second term of the bio-heat equation accounts for the heat transfer due to the blood flow (also referred to as perfusion) within the body's circulatory system. Finally, the third term is used to represent the heat that is generated due to natural metabolic processes in the body and external heat generation sources. The metabolic heat generation refers to the heat produced in the body as a result of metabolic activities.

Section 2.4

Survey Work

Thermal modelling of the eye is important as it can provide us with the tool to investigate the effect of external heat sources as well as the abnormalities within the eye. Heat transfer models of the eye have been developed during the past thirty years. Taflove and Brodwin in 1975 studied microwave radiation effects on the human eye and obtained the transient solution using the finite difference methods[21]. In this model, eye was considered as a homogeneous tissue with thermal parameters similar to that of the water. The major drawback of this model was that the heat loss from the eye was considered due to constant convective heat transfer coefficient over the entire eye ball surface and as such it did not distinguish the heat transfer between the cornea and the environment, sclera and the body. Amara studied laser-ocular media interaction through a numerical heat transfer model[14]. Lagendijk in 1982 used the simple explicit forward difference heat balance technique to study the temperature distribution in human and rabbit eyes[15].

A finite element model of heat transport in the human eye was presented by Scott in 1988[20].The model was based on the bio-heat transfer equations. In that model it could be seen some temperature distribution of the human eye with ambient temperature of 20°C and blood temperature 37°C. This model took in consideration steady state temperature variation in the human eye when exposed to microwave radiation, but the analytic method of solution did not take in consideration transient temperature variations in the human eye. Scott showed that the temperature variation in the anterior segment of the eye can occur if an increasing evaporation from the anterior corneal surface and rapid blink factors appear simultaneously. This model did not include blood flow in the iris and ciliary body; that is a deficiency in the model.

In the literature, one of the models that were presented earlier in the simple heat transfer model to analyse thermal effect of microwave radiation of the human eye (Al-

Badwahy and Youssef, 1976)[37]. The model used an analytical approach for the solution of steady state temperature variation.

E.H.Ooi and Kin Wei Ng studied two and three dimensional models of the human eye in which Finite Element Method simulation and the investigation of thermal effects of laser ocular media interaction were discussed[19]. Cvetkovic et al., developed the model describing the thermal stress of the human eye exposed to laser radiation[16]. Khanday M.A. and Saxena (2009) confirmed that when the outer surface of eye is exposed to the atmosphere, the temperature changes take place at various ambient temperatures[18]. Rafiq A. and Khanday M.A.(2013) analyzed the heat transfer in human eye. Khanday et al., (2014) estimated the heat distribution in the multi-layered human eye[42]. In 2016, Rafiq A. studied thermal behaviour of human eye in relation with change in blood perfusion, porosity, evaporation and ambient temperature[32].

Gursu E. and Berberoglu K. (2020) focussed on the exposition of heat transfer processes and the mass transfer processes that govern drug delivery methods to the retinal especially in eye drops[43] . O.S. Zadorozhnyy et al.,(2022) also reviewed the heat exchange in human eye[44].

Chapter 3

Mathematical study of one dimensional heat distribution in human eye

The present chapter is devoted to find the solution of steady state heat distribution in the five layers of human eye viz., cornea, vitreous humor, lens, aqueous humor and retina by using the variational finite element method. The physiology and parameters responsible for heat transfer in the human eye have been taken into account with their significant importance. It is understood that the outer surface of eye is exposed to the atmosphere and thereby the temperature changes takes place at various ambient temperatures as confirmed by Khanday and Saxena in human head regions. The heat loss from the surface takes place due to convection, radiation and evaporation.The main purpose of the study is to estimate the temperature profiles at various nodal points of the multi-layered human eye by using variational finite element technique. The advantage of this technique over other numerical approximations is due to the applicability of this method over irregular geometries. Since the nature of the eye is irregular, therefore, the method guarantees the reasonable outcome.

Section 3.1

Formulation of Mathemetical Model

The Pennes-Bioheat equaiton in one dimensional steady state case is given as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)-mc_b\left(T-T_A\right)+S=0 \tag{1.1}$$

Where $k,m,c_b,S$and T_A respectively denote the thermal conductivity, mass flow rate, specific heat of blood,metabolic heat generation rate and arterial blood temperature. T_i represents the nodal temperatures respectively at distances $l_i(i=1,2,3,4,5)$ from the inner layer of eye.

The outer surface of the eye i.e., cornea is exposed to the environment and heat loss at the cornea takes place due to conduction,convection, radiation and evoperation. Thus, the boundary condition at the cornea based on Newton's law of cooling is given by

$$-k\frac{\partial T}{\partial x}=h\left(T-T_a\right)+LE. \tag{1.2}$$

The temperature at the inner layer of the eye (retina) is assumed to be similar to that of the body core temperature. The presence of blood in capillaries and regulatory mechanism at retina maintains the thermoregulatory mechanaism intact at the region. Hence, boundary condition at that layer is given as

$$T_1=T_b=37^0C. \tag{1.3}$$

Solution of the model

The variational integral

$$I = \int F\left(T, T', x\right) dx \tag{1.4}$$

in optimum form is equivalent to the Euler-Lagrange differential equation

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0 \tag{1.5}$$

where

$$T' = \frac{\partial T}{\partial x}$$

0n comparing equation (1.1) with Euler-Lagrange equation (1.5), we have the following variational integral by using equation (1.2)

$$I = \frac{1}{2}\int_{l_i}^{l_{i+1}} k[\left(\frac{\partial T}{\partial x}\right)^2 + mc_b(T - T_A)^2 - 2ST]dx + \frac{1}{2}h(T - T_a)^2 + LET_M \tag{1.6}$$

$$= I_k + I_m - I_S + J$$

where

$$I_k = \frac{1}{2}\int_{l_i}^{l_{i+1}} k\left(\frac{\partial T}{\partial x}\right)^2 dx$$

$$I_m = \frac{1}{2}\int_{l_i}^{l_{i+1}} mc_b(T - T_A)^2 dx$$

$$I_s = \int_{l_i}^{l_{i+1}} ST\, dx$$

and

$$J = \frac{1}{2}h(T - T_a)^2 + LET_M$$

The integral **I** to be minimised will be a function of the nodal temperatures $I = I(T_1\ T_2 \ldots T_i\ T_{i+1} \ldots T_M)$.

We write

$$I = I(\theta) \tag{1.7}$$

where θ represents a column vector of M nodal temperatures.

$$\theta = [T_1\ T_2 \ldots T_i\ T_{i+1} \ldots T_M]^T$$

The solution leading to temperature distribution at different layers of human eye can be estimated by optimizing equation (1.7) using first order derivative of **I** with respect to θ and equating to zero. In this process we can establish temperature profile at nodal points $T_1\ T_2 \ldots T_i\ T_{i+1} \ldots T_M$. Later on, we reassemble the variational integrals that is $I_k,\ I_m,\ I_s\ and\ J$ to obtain temperature distribution throughout the layers of human eye using physiological parameters given in the table (3.2) we can get required solution of the model.

Section 3.2

Numerical Computation

The equation (1.4) is the general model equation in which the domain can be discretized into any number of nodal points. In the present case as already described the subdomains and their corresponding nodal temperatures, thus for the solution of the equation (1.4), the numerical values of the physiological parameters given in Table (3.2) have been used.

These values were taken from Ooi E.H. and Ng E.Y.[19]

S.NO	Quantity	Value
01	l_1	0 m
02	l_2	0.035 m
03	l_3	0.0385 m
04	l_4	0.042 m
05	l_5	0.045 m
06	$k^{(1)}$	$0.594J/m$°C
07	$k^{(2)}$	$0.4J/m$°C
08	$k^{(3)}$	$0.578J/m$°C
09	$k^{(4)}$	$0.58J/m$°C
10	$(mc_b)^{(1)}$	$\approx 0.30 \times 10^{-3} J/m^3$ °C
11	$(mc_b)^{(2)}$	$\approx 0.28 \times 10^{-3} J/m^3$ °C
12	$(mc_b)^{(3)}$	$\approx 0.33 \times 10^{-3} J/m^3$ °C
13	$(mc_b)^{(4)}$	$\approx 0.27 \times 10^{-3} J/m^3$ °C
14	L	$579 \times 10^{-3} cal/kg$
15	S	$0J/m^3$
16	T_A	37°C

Table (3.2): Numerical values of the physiological parameters[19].

Section 3.3

Discussion and Conclusion

For the process of thermoregulation in human body, it is important to highlight the important factors responsible in the system. The thermal conductivity of the tissues, blood

perfusion, metabolic heat generation, perfusion and evaporation, etc are mainly associated with the heat distribution in biological tissues. The eye is the only organ without the immediate protective layer in terms of skin. Therefore, it is important to address the issues related to the thermal stability in the human eye at various environmental disturbances. The present model is based on Pennes bio-heat equation together with appropriate boundary conditions. Variational finite element method is realistically applicable for the estimation of temperature profiles on irregular geometrical objects for the reasonable results. The general heat distribution model has been constructed and later confined to the four layers of the human eye to estimate temperature profiles at various ambient temperatures. The results were compared by researchers with the results obtained by other methods[42]. Their results have shown considerable amount of effect due to perfusion and evaporation on eye thermostat. Also, the temperature profiles at various environmental factors are discussed as:
(i) Temperature distribution of different layers of human eye at $T_a = 30, 25, 20 \textit{ and } 15(^\circ C)$ is represented in Fig. (3.1) by the curves 1, 2, 3 and 4 respectively with $h = 8W/m^{2\circ}C$ and $E = 40W/m^2$. (ii) Temperature distribution of different layers of human eye at $T_a = 30, 25, 20 \textit{ and } 15(^\circ C)$ is represented in Fig. (3.2) by the curves 1, 2, 3 and 4 respectively with $h = 8W/m^{2\circ}C$ and $E = 20W/m^2$.

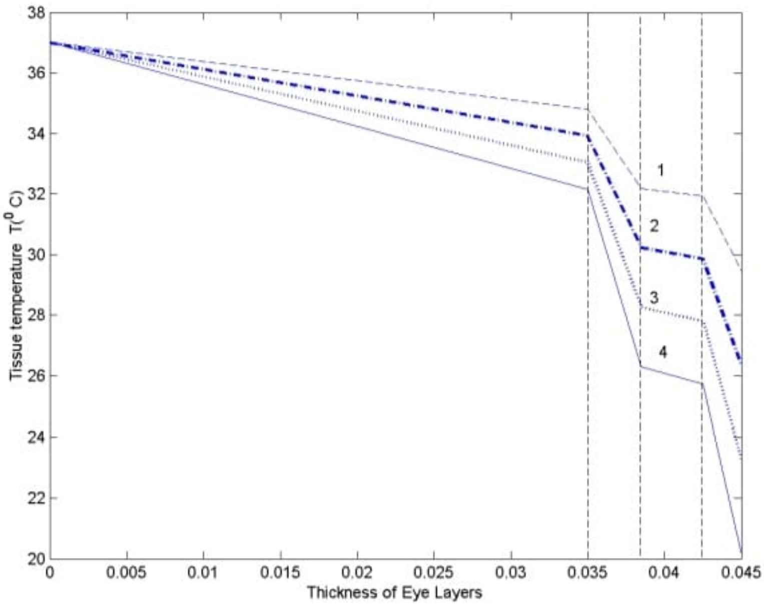

Figure 3.1: Temperature vaiation at $E = 40W/m^2$

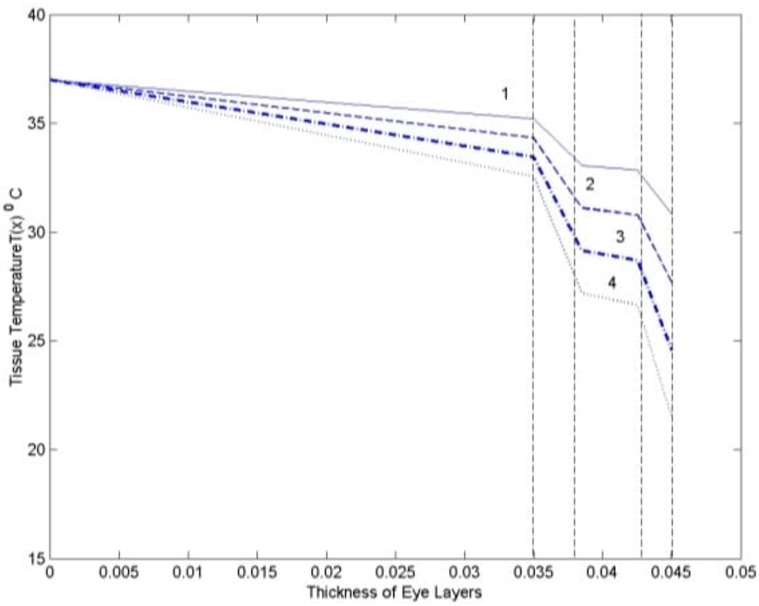

Figure 3.2: Temperature vaiation at $E = 20W/m^2$

Chapter 4

Review of the Paper "Thermal Behaviour of Human Eye"

Introduction

The heat distribution in human eye is basically due to conduction, convection, evaporation, etc. The thermal stability of human eye is a subject of great concern due to its insufficient blood flow and lack of skin as a protecting layer. The physiology of the human eye operates the thermoregulatory mechanism up to large extent in various physiological and moderate ambient conditions; however, the severity of heat and cold causes adverse effects on its thermal equilibrium. Such disturbances lead to damage the sensitive tissues of the human eye and thereby eye vision. Thus, it is imperative to study the role of physiological and environmental conditions on the thermal stability and other homeostasis of human eye. The information in this direction can be useful not only in clinical situations but can help to maintain the heat distribution while performing laser surgeries and other medical diagnosis.

In this chapter, a mathematical model has been established to estimate the thermal stress in human eye in terms of changes in porosity, evaporation, perfusion rate and other environmental disturbances.

Section 4.1

Model Development

The governing equation used for modeling heat flow inside the biological tissues is based on Pennes' bio-heat equation which involves the role of conduction, metabolic heat generation and blood perfusion term[39] . Since human eye is mainly comprised of water and therefore in few regions, the metabolic heat generation is playing negligible role for its temperature regulation. Blood perfusion term is dropped in the outer regions of the human eye while blood flow in sclera /iris plays a vital role in maintaining the eye temperature close to the other body organs. This region is modelled as a porous medium and incorporating blood circulation through the tissue. Therefore, the modified Pennes' bio-heat equation has been used incorporated earlier by Nakayama and Kuwahara [36] and Khanafer and Vafia [33] is given as

$$(1-\phi)\rho c\frac{\partial T}{\partial t} = \nabla((1-\phi)k\nabla T) + \rho_b c_b \omega(T_b - T) + S \tag{1.1}$$

where $k,\ T,\ t,\ S,\ \phi,\ \omega, \rho\ and\ c$ represents the thermal conductivity, temperature, time, metabolic heat generation, porosity, perfusion rate, density and specific heat of the tissues.

In order to make use of variational finite element method, the domain of the study is assumed to be consisting of sub-domains − cornea, aqueous humor, lens, viterous humor and sclera with the size of regions as $l_1, l_2 - l_1, l_3 - l_2, l_4 - l_3$ and $l_5 - l_4$ respectively. Also $T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)}$ and $T^{(4)}$ represents the temperatures of the respective regions. Conduction is dominant heat transfer mechanism in cornea, aqueous humor, lens and vitreous humor parts of the eye and as such the Pennes' bio-heat equation for heat flow in these regions reduces to classical heat equation:

$$\rho_i c_i \frac{\partial T^{(i)}}{\partial t} = \nabla(k_i \nabla T^{(i)});\ (i = 0, 1, 2, 3) \tag{1.2}$$

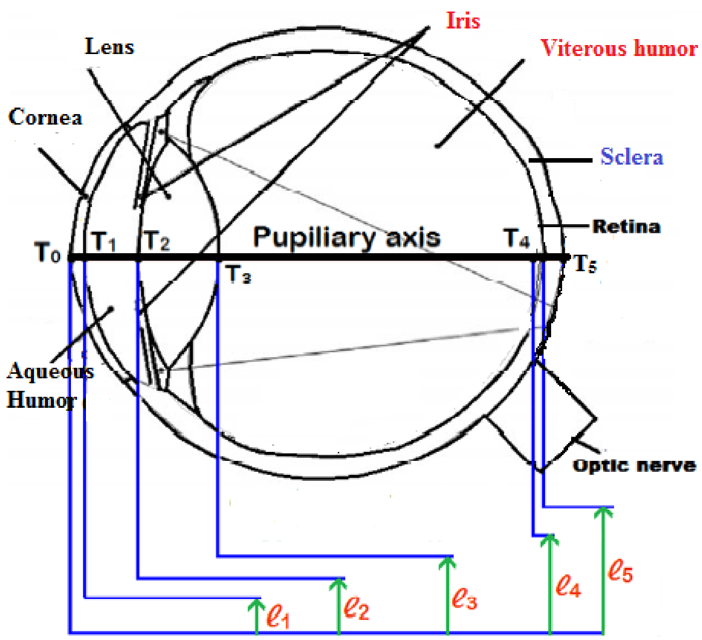

Figure 4.1: Schematic diagram of human eye [46]

Blood flow occurs in the sclera /iris region which we considered to be the porous media as such blood perfusion and porosity accounts for the heat transfer. The governing equation used for the heat transfer in the sclera region is given as

$$(1-\phi_4)\rho_4 c_4 \frac{\partial T^{(4)}}{\partial t} = \nabla((1-\phi_4)k_i \nabla T^{(4)}) + \rho_b c_b \omega (T_b - T^{(4)}) \qquad (1.3)$$

Boundary Conditions

The thermal exchange between the eye and blood flow at the sclera occurs through convection and therefore the boundary condition at this interface is given by Newtons law of cooling

$$k\frac{\partial T}{\partial n} = h_b(T - T_b); \qquad \text{at } x = l_5 \qquad (1.4)$$

where n is the normal direction to the surface of the boundary, h_b is the blood convection coefficient and T_b is the blood temperature.

At the exterior region (cornea), heat loss from the eye occurs through convection and evaporation and hence the boundary condition is given as

$$k\frac{\partial T}{\partial n} = h_a(T - T_a) + E; \qquad \text{at } x = l_0 \tag{1.5}$$

where h_a is the ambient convection coefficient, T_a is the ambient temperature and E is the evaporative heat loss due to blinking and tears.

Section 4.2

Solution of the Model

The solution of the model was given by the researchers (Aasma Rafiq and M.A.Khanday) based on variational finite element method . On comparing equation (1.1), (1.4) and (1.5) with Euler-Lagrange differential equation, the layer-wise variational integrals are given below [18]

$$I_i = \int_{l_i}^{l_{i+1}} \frac{1}{2}\left[(1-\phi_i)k_i\left(\frac{\partial T^{(i)}}{\partial x}\right)^2 + \rho_i c_i(1-\phi_i)\left(\frac{\partial (T^{(i)})^2}{\partial t}\right) + \gamma_i\rho_b c_b \omega(T_b - T^{(i)})^2\right] dx$$
$$+\delta_i\left(\frac{1}{2}h_a(T_0 - T_a)^2 + ET_0\right) + \gamma_i\frac{h_b}{2}(T_5 - T_b)^2 \tag{2.6}$$

$$\text{where} \quad \delta_i = \begin{cases} 1, i = 0 \\ 0, \text{otherwise} \end{cases} \qquad \text{and} \quad \gamma_i = \begin{cases} 0, i = 0,1,2,3 \\ 1, i = 4 \end{cases}$$

The solution of the problem was established over each of the sub-domains using Lagrange linear shape functions given as

$$T^{(i)} = \frac{l_{i+1}T_i - l_iT_{i+1}}{l_{i+1} - l_i} + \frac{T_{i+1} - T_i}{l_{i+1} - l_i}x; \quad i = 0,1,\ldots,4. \tag{2.7}$$

Using equations (2.7) to equations (2.6), and by virtue of finite element method, the heat regulation in the complete domain can be computed by assembling these variational integrals as

$$I = \sum_{i=0}^{4} I_i \tag{2.8}$$

On equating the partial derivatives of I w.r.t. $T_i\,(i = 0,1,2,3,4,5)$ to zero leads to the optimization of I through the following system of differential equations

$$\begin{aligned}
L_1\dot{T}_0 + L_2\dot{T}_1 + A_1T_0 + A_2T_1 &= Q_1\\
M_1\dot{T}_0 + M_2\dot{T}_1 + M_3\dot{T}_2 + B_1T_0 + B_2T_1 + B_3T_2 &= 0\\
N_1\dot{T}_1 + N_2\dot{T}_2 + N_3\dot{T}_3 + C_1T_1 + C_2T_2 + C_3T_3 &= 0\\
O_1\dot{T}_2 + O_2\dot{T}_3 + O_3\dot{T}_4 + D_1T_2 + D_2T_3 + D_3T_4 &= 0\\
P_1\dot{T}_3 + P_2\dot{T}_4 + P_3\dot{T}_5 + E_1T_3 + E_2T_4 + E_3T_5 &= Q_2\\
R_1\dot{T}_4 + R_2\dot{T}_5 + F_1T_4 + F_2T_5 &= Q_3
\end{aligned} \tag{2.9}$$

where the coefficients of the equations (2.9) are given in Appendix.

The above system of equations (2.9) in matrix form can be written as

$$M\dot{T} + NT = Q \tag{2.10}$$

where

$$M = \begin{pmatrix} L_1 & L_2 & & & & \\ M_1 & M_2 & M_3 & & & \\ & N_1 & N_2 & N_3 & & \\ & & O_1 & O_2 & O_3 & \\ & & & P_1 & P_2 & P_3 \\ & & & & R_1 & R_2 \end{pmatrix} \qquad N = \begin{pmatrix} A_1 & A_2 & & & & \\ B_1 & B_2 & B_3 & & & \\ & C_1 & C_2 & C_3 & & \\ & & D_1 & D_2 & D_3 & \\ & & & E_1 & E_2 & E_3 \\ & & & & F_1 & F_2 \end{pmatrix}$$

$$Q = \begin{pmatrix} Q_1 & Q_2 & 0 & 0 & 0 & Q_3 \end{pmatrix}^T \qquad T = \begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 \end{pmatrix}^T$$

Section 4.3

Numerical values

To solve equations (2.9), the following values of parameters given by Scott and Ng et al., [40] and [19] were considered.

Table 4.1: **Numerical values of the control physiological parameters**

Parameters	Values
Blood temperature, T_b	$37\,^0C$
Blood convection coefficient, h_b	$65\,Wm^{-2}\,^0C^{-1}$
Ambient convection coefficient, h_a	$10\,Wm^{-2}\,^0C^{-1}$
Thermal conductivity of cornea, k_0	$0.58\,Wm^{-1}\,^0C^{-1}$
Thermal conductivity of aqueous humor, k_1	$0.58\,Wm^{-1}\,^0C^{-1}$
Thermal conductivity of lens, k_2	$0.40\,Wm^{-1}\,^0C^{-1}$
Thermal conductivity of viterous humor, k_3	$0.603\,Wm^{-1}\,^0C^{-1}$
Thermal conductivity of sclera, k_4	$1.0042\,Wm^{-1}\,^0C^{-1}$

The numerical estimates for steady state temperature distribution in different regions of human eye with different values of porosity, blood perfusion, ambient temperatures and evaporation rates have been carried out.

Therefore, the solution of the system given in equations (2.10) has been carried out using Crank-Nicolson method. The successive temperature profiles in terms of time are given by the relation

$$\left(M+\frac{\Delta t}{2}N\right)T^{(i+1)}=\left(M-\frac{\Delta t}{2}N\right)T^{(i)}+\Delta t\,Q \tag{3.11}$$

where Δt is the time interval and $T^{(0)}$ represents the 6×1 matrix for the initial temperature.

Section 4.4

Discussion and Conclusion

In this study, the modified Pennes' bio-heat equation has been used to study the thermal stability of human eye. The variational form of finite element method has been employed to establish variational integrals at subregions of multi-layered eye. The formulation of the model incorporates various parameters including metabolic heat generation, blood density, specific heat, porosity of the tissue medium, evaporation and other physiological processes regulating heat transport in human eye. The eye as a domain is composed of five regions viz. cornea, aqueous humor, lens, viterous humor and sclera. The overall temperature distribution in human eye has been approximated by solving variational integrals corresponding to each sub-domain. Variational finite element method is realistically valid for the determination of temperature distribution in irregular geometrical objects for the reasonable output. It has been observed from the results that the main factors affecting the temperature distribution in the human eye are porosity, blood perfusion, ambient temperature and tear evaporation. The numerical value of the temperatures at the different regions were obtained and are found to fall within the range of the values given by other numerical and experimental data reported in the literature [25], [26], [40].

Figures (4.2) and (4.3) shows the steady state temperature distribution of the one dimensional multi-layered eye model. The warmer region is considered at the sclera which may be due to the presence of blood vessels whereas the cooler region is the cornea surface where the heat gets lost due to conduction, convection and tear evaporation. Since sclera has porosity and blood perfusion, it has been observed that increase in blood perfusion helps the eye to maintain its temperature to that of internal body temperature.

Effect of porosity and blood perfusion:- Figure (4.2) shows the impact of the blood perfusion rate on the temperature distribution at different layers. Keeping porosity at the constant value of 0.4 and changing perfusion rates as

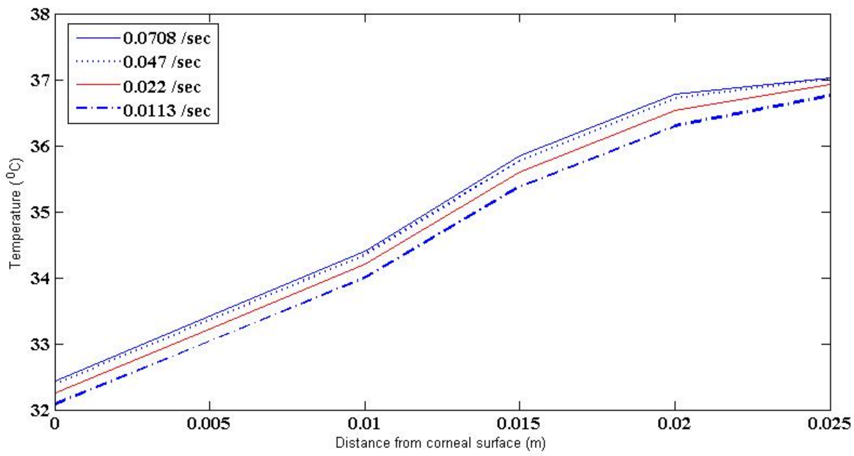

Figure 4.2: Effect of blood perfusion rate on temperature distribution at $\phi = 0.4$, $E = 40Wm^{-2}$ and $T_a = 25^0C$.

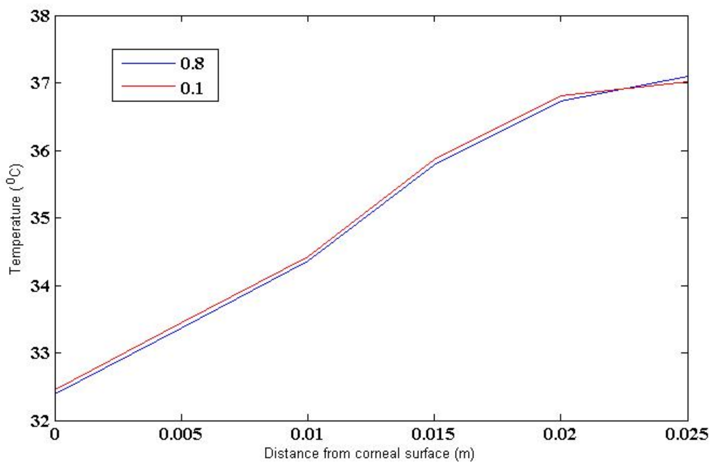

Figure 4.3: Effect of porosity on temperature distribution at $\omega = 0.0708/sec$, $E = 40Wm^{-2}$ and $T_a = 25^0C$.

0.0113, 0.022, 0.047 and 0.0708 $/sec$, it is evident that the corneal temperature varies between 32.08^0C to 32.44^0C and that of the sclera region ranges from 36.76^0C to 37.03^0C. It reveals from the study that higher blood flow rate towards the sclera helps the eye to maintain its temperature same to that of the body temperature. Figure (4.3) shows the

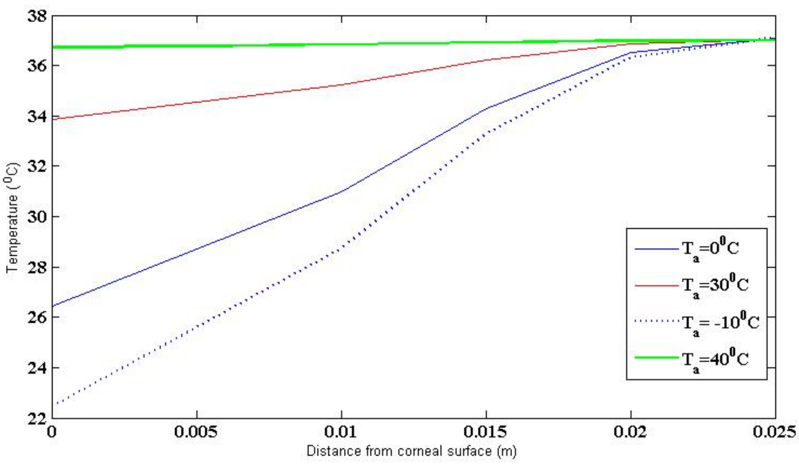

Figure 4.4: Effect of ambient temperature on eye tissue temperature distribution at $\phi = 0.4$, $\omega = 0.0708/sec$ and $E = 40Wm^{-2}$.

effect of porosity on temperature distribution in human eye where two values of porosity 0.1 and 0.8 were considered. It is observed from the curves that increase in the value of porosity keeps the temperature within the range of 32.38^0C to 37.1^0C while as at 0.1 porosity, it takes the values 32.5^0C to 37^0C.

Effect of atmospheric temperature:- Figure (4.4) shows the effect of the different ambient conditions- four different sets of data values of T_a equals to $0^0C, 30^0C, 40^0C$ and -10^0C are considered. At $T_a = 30^0C$, the temperature distribution along the corneal surface varies from 33.86^0C to 37.02^0C; At $T_a = 0^0C$, it varies between 26.44^0C to 37.08^0C; At $T_a = 40^0C$, temperature ranges from 36.7^0C to 37^0C and at $T_a = -10^0C$, margin lies between 22.45^0C to 37.1^0C which shows that in cold conditions, corneal temperature is very low which may have adverse effects on the thermal stability of human eye; while as at the higher environmental temperatures, the inbuilt mechanism of human eye tries to bring it close to 37^0C. Thus increase or decrease in the ambient temperature may be very helpful while performing some laser surgeries.

Effect of evaporation:- To analyse the effect of evaporation rate on temperature distribu-

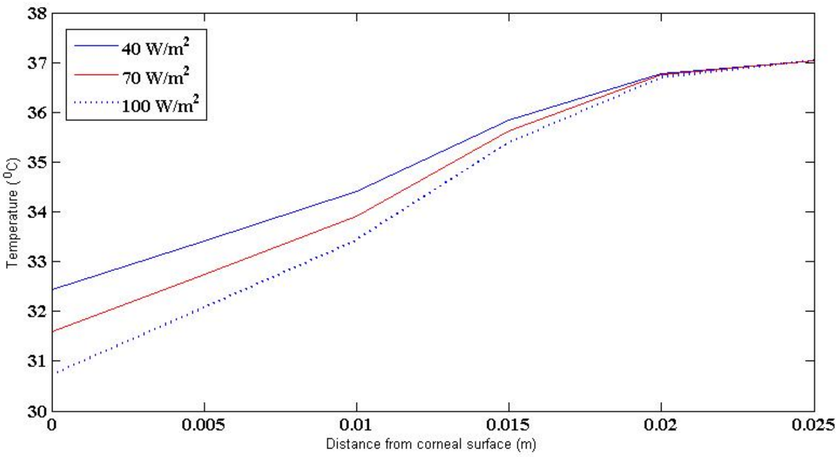

Figure 4.5: Effect of evaporation rate on temperature distribution at $\phi = 0.4$, $\omega = 0.0708/sec$ and $T_a = 25^0C$.

tion in human eye, three cases were considered - E $= 40Wm^{-2}$, $70Wm^{-2}$ and $100Wm^{-2}$. From Figure (4.5), it has been observed that with increase in evaporation rate, the corneal temperature decreases having a slight variations on temperatures of lens, viterous humor and sclera. Thus slight variation in temperature may be due to the blood supply at the sclera region which tries to keep the temperature of surrounding tissue similar to that of the body temperature.

The thermal models become more accurate while incorporating more physiological properties related to the tissue. Since any change in temperature can lead to the change in the values of the parameters, so the present model can be used for both the normal tissue as well as to the unhealthy eye. The present work done by Aasma Rafiq and M.A.Khanday can be applied to all those tissues which behave similar to that of human eye with appropriate physiological changes in parameters. These thermal models on human eye can be useful in predicting certain diseases related to temperature changes.

Section 4.5

Modelling Blood Flow in Human Eye

We know that blood is carried from heart to various parts of the eye and eventually returned to heart. In fact, blood is carried through system of elastic tubes-the arteries, capillaries and veins. The blood returns to the heart without actually leaving the system. This process is known as circulation of blood or flow of blood.

We also know that proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the eye in human beings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and cause damages ranging from minor discomfort to death, in worst case. Therefore a better understanding of the physiology of the system is essential. Mathematical modeling of the system is aimed at this[31]:

As a first step in modeling, we shall first identify the essential characteristics of blood flow. We list them below:

i) Blood is a non-homogenous fluid.

ii) Blood vessels are elastic, they branch repeatedly.

iii) Blood flow is unsteady or pulsatile.

iv) Blood flow is generally laminar except for flow near heart.

Viscosity

Suppose a force is applied to a portion of a mass of a fluid, it will begin to flow but if the force is removed the movement will be brought to rest. On the other hand, if a similar portion of a fluid is kept in moving, the movement will be transferred to the rest of the fluid. This property is analogous to that of friction between solid bodies.

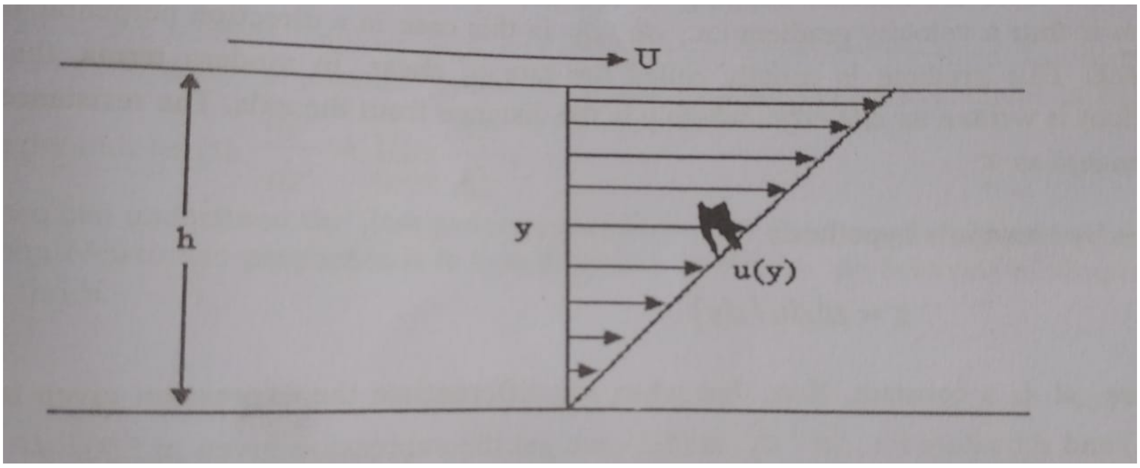

Figure 4.6: Motion of fluid between parallel plates

Now, we shall explain the concept of Viscosity of a fluid based on the following simple experiment([31]). Consider the motion of a fluid between two long parallel plates one of which is at rest and the other one is moving with a constant velocity U parallel to itself as shown in the figure(4.6). Let the distance between the plates be h and the fluid velocity be u. Assume that the fluid pressure is constant throughout the fluid. Due to cohesive nature of fluid it adheres to the plates. The fluid velocity at the lower plate is zero and that at upper plate is U. This is because the upper plate is moving and the lower plate is at rest. So, we get

$u = 0$, when $y = 0$

$u = U$, when $y = h$.

Expermentally, it is observed that the fluid velocity distribution is linear and as such it is given by

$$u(y) = \frac{U}{h}y \tag{5.12}$$

where y is the direction at right angles to the flow. In order to support the motion it is necessary to apply a tangential force to the upper plate. Experimentally it is observed that this force, taken per unit area, is proportional to the velocity U of the upper plate and inversely to the distance h. If τ denote the force, then τ is directly proportional to $\frac{U}{h}$. This is denoted by

$$\tau \propto \frac{U}{h} \tag{5.13}$$

Many researchers have studied this property; the first theoretical consideration was made by Newton in which he considered the motion imparted to a large volume of fluid by the rotation of a long cylinder suspended in it. The hypothesis on which he based his derivation was that the resistance which arises from the defect of slipperiness of the parts of the liquid, other things being equal is proportional to the velocity with which the parts of the liquid are separated form one another. "Defect of slipperiness" was the term used to describe what we now call viscosity. This hypothesis emphasizes immediately that in a fluid moving relative to a surface there are laminae slipping on one another and so moving at different velocities. There is thus a velocity gradient i.e., du/dy in this case in a direction perpendicular to the surface. This gradient is usually called the rate of shear. In modern terms, the velocity gradient is written as du/dy, where y is the distance from the axis. The resistance or force is denoted by τ.

Then by Newton's hypothesis

$$\tau = \mu(du/dy) \tag{5.14}$$

where μ is a constant. Note that when we differentiate the expression given in equation (5.12) and substitute for du/dy in (5.14), we get the expression given in (5.13) is called the proportionality constant which gives the measure of the viscosity of the fluid, μ is also called the coefficient of viscosity.

The unit of viscosity is called Poise. The viscosity of water at 20.2°C is 0.01 Poise and at 37°C it is 0.0069 Poise. At 20°C water it is approximately 60 times more viscous than air.

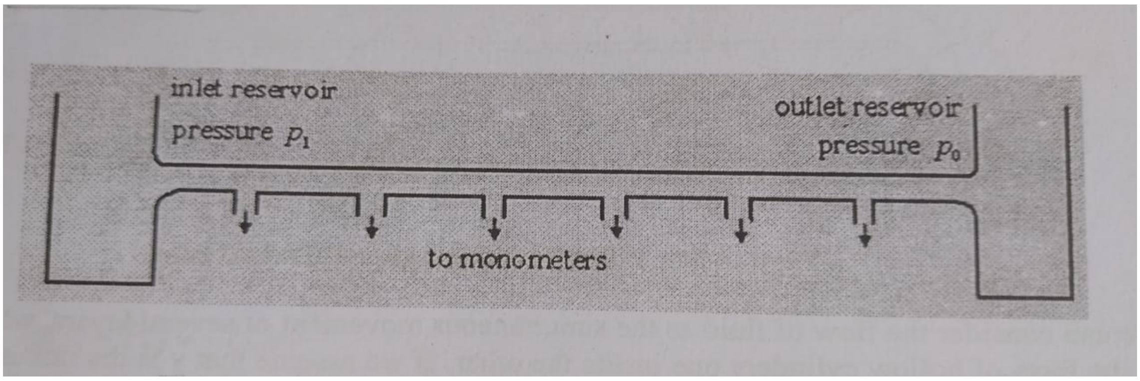

Figure 4.7: Flow properties of simple fluid

Over the range 0°C - 30°C the viscosity of air increases by about 9% while that of water decreases by 45%. Using this we can say that the viscous components of resistances to motion (the frictional drag) is about 9% higher for birds flying in tropics than for the same bird flying in arctic. Also fish (and other marine creatures) have considerably easier way for moving about in tropics than in arctic water.

For convenience, the viscosity of any fluid is expressed relative to the viscosity of water. This viscosity of fluid is known as relative viscosity.

Equation (5.13) is known as Newton's law of friction. Fluids obeying this law are called Newtonian otherwise non-Newtonian. Most of the common fluids obey this law.

Poiseuille's Law

Poiseuille's law is the relation between flow rate and pressure gradient for fluid flow in a rigid cylindrical tube under a pressure gradient. (Note that the pressure gradient is the pressure drop per unit lenght $\frac{dp}{dz} = \lim_{\triangle(x)\to 0} \frac{\triangle p}{\triangle z}$.)

In order that we can understand the flow properties of biological fluids such as blood which may exhibit non-Newtonian properties, it is first necessary to discuss the behavior of simple or Newtonian fluids. Let us look at the flow properties of a simple liquid like water in a very long horizontal pipe. Imagine that the pipe is circular in cross-section and d units in diameter as shown in above figure(4.7).

Its entrance and exit are connected to large reservoirs so that the pressure drops between the ends of the tube may be maintained constant and a steady flow of water through the pipe is achieved. Small side hole, or lateral, pressure tappings are made in the pipe at frequent intervals along its length and these tappings are connected to a series of manometers. It is thus possible to measure the pressure drop per unit length or pressure gradient along the pipe.

If the pressure at the inlet to the pipe is p_1 and that at the outlet p_0, then we shall observe that is $p_1 - p_0$ (or $\triangle p$) is increased by raising the level in the upstream reservoir, so is the flow rate V through the pipe.

It was Poiseuille in 1840, who as a first step towards understanding the mechanism of the circulation, published a quantitative study of the flow properties in a pipe very remote from the entrance, and flow conditions in this region are now named after him. In addition to varying the flow rate and tube size, Poiseuille also studied the effect of viscosity on the flow conditions. Here we found that as viscosity was increased so was the pressure gradient necessary to maintain a given flow-rate.

Now to derive Poiseuille's formula, we make use of Newton's second law of motion which say that

$$Mass \times acceleration = body\ force + pressure\ gradient\ force + Viscous\ force. \quad (5.15)$$

Let us consider the fluid flow through a circular tube with length L and diameter $D = 2R$, which is small compared with the length. We assume that the rate of flow is constant i.e. flow is steady. We also assume that the fluid velocity everywhere inside the tube is laminar stream lined. As we know for a laminar flow, the velocity is purely in the direction parallel to the axis of the cylinder. The fluid velocity at the inner surface is zero and it reaches maximum value on the axis (here axis means axis of the cylinder.)

We can consider the flow of fluid as the simultaneous movement of several layers, which are in the form of hollow cylinders one inside the other. If we assume that y is the radius of any one of these cylinders, then y varies from 0 to R, i.e., $0 < y < R$ as shown in the

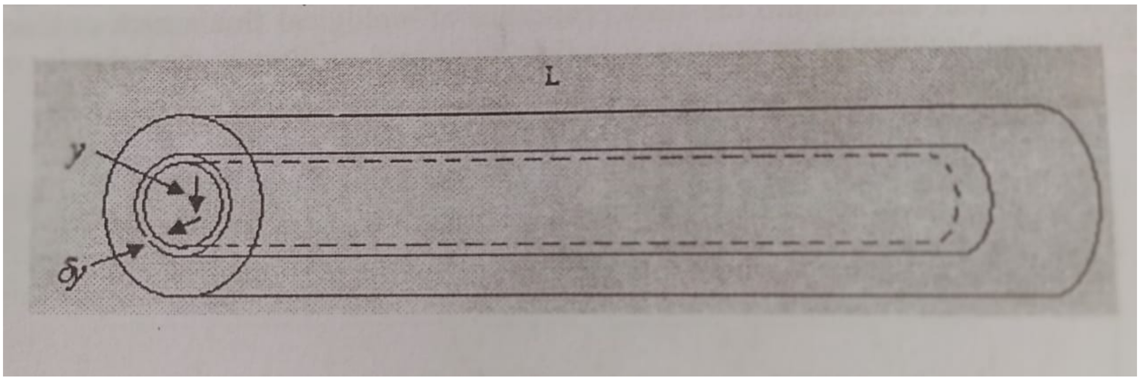

Figure 4.8: Fluid flow through cylindrical tube

figure(4.8).

If we consider the fluid flow to be due to pressure differences at the ends of the tube from the higher to the lower one then the only force opposing this flow is viscous resistance. We know that this force is $\mu(du/dy)$, and we find that the fluid particles are accelerated by the pressure difference and retarded by viscous resistance. If we look at equation (5.14), then we will find that the only forces present are pressure gradient force and viscous force. This is because, since the flow is a steady flow in a straight tube, the fluid is not subjected to any acceleration (i.e. when the flow is steady, things do not change with respect to time). Therefore, LHS of equation (5.15) is zero also, since we are considering the flow in horizontal pipes, gravitational forces are not relevant and therefore the body force term also vanishes. Thus, equation (5.14) reduces to pressure gradient force =-Viscouos force. Now if F_{visc} denote the viscous force and F(P) denote the pressure difference, then we have

$$F(P) = -F_{visc} \tag{5.16}$$

(The negative sign indicates one force accelerates the motion, the other retards.)

Now we will calculate the LHS and RHS of equation (5.16), we first consider the RHS of equation (5.16). Here note that each flow is in the form of cylindrical layer of length L and radius y, y varying from 0 to R. The viscous force acts on the surface and it is given by the following formula:

$$F_{visc} = surface\ area\ of\ the\ cylinder \times viscosity \times the\ velocity\ gradient. \quad (5.17)$$

We have denoted the viscosity as μ and we know that the velocity gradient is given by du/dy. Therefore we can write equation (5.17) as

$$F_{visc} = 2\pi y L \left(\mu \frac{du}{dy} \right) \quad (5.18)$$

Next we shall find the difference pressure.

Note that the force exerted by the pressure at an end of the cylinder is pressure at that end multiplied by the cross sectional area. Now if P_1 and P_2 respectively denote the pressures at either end of length L of the cylinder considered, then the required pressure difference is

$$F(P) = \pi y^2 (P_1 - P_2) \quad (5.19)$$

Substituting for F(P) and F_{visc} in equation (5.16), we get

$$\pi y^2 (P_1 - P_2) = -2\pi y L \left(\mu \frac{du}{dy} \right)$$

or

$$y(P_1 - P_2) = -2L\mu \frac{du}{dy}.$$

This gives the velocity gradient du/dy as

$$\frac{du}{dy} = \frac{-y(P_1 - P_2)}{2L\mu} \quad (5.20)$$

(the negative sign here implies u decreases when y increases. Also, note $P_1 > P_2$).

Now, substituting this value of the velocity gradient in equation (5.14), we get the shear stress as

$$\begin{aligned} \tau &= \mu(du/dy) \\ &= \frac{\mu \times (-y) \times (P_1 - P_2)}{2L\mu} \\ &= \frac{-y(P_1 - P_2)}{2L} \end{aligned} \quad (5.21)$$

Now if we consider the wall of the tube, then the radius y of the wall is R, therefore from equation (5.21) we get

$$Shear\ stress\ at\ the\ wall\ of\ the\ tube = \frac{-R(P_1 - P_2)}{2L} \tag{5.22}$$

Thus, we have derived the equation describing the flow of fluid in a thin tube of length L with pressures P_1 and P_2 at the ends.

Now, we have to solve equation (5.20) to get the velocity u.

Let us consider the equation (5.20), this equation is a first order linear ordinary differential equation. To find the solution, we integrate on both sides of equation (5.20) and we get the velocity as

$$u(y) = -\frac{(P_1 - P_2)}{4\mu L} y^2 + C \tag{5.23}$$

where C is the constant of integration which is to be evaluated. To calculate C, it is necessary to prescribe the boundary conditions. Here, we make use of the assumption made by Newton that the fluid in contact with the wall is at rest,

i.e., u=0, when y=R

Substituting the condition in equation (5.23) we get

$$C = -\frac{(P_1 - P_2)}{4\mu L} R^2.$$

So that the equation(5.23) reduces to

$$\begin{aligned} u(y) &= -y^2 \frac{(P_1 - P_2)}{4\mu L} + R^2 \frac{(P_1 - P_2)}{4\mu L} \\ &= (R^2 - y^2) \frac{(P_1 - P_2)}{4\mu L} \end{aligned} \tag{5.24}$$

where u is the velocity component parallel to the axis, R is the radius of the cylinder, L is the length of the tube,μ is the viscosity of the fluid and $P_1 - P_2$, the pressure drop.

Therefore, equation (5.24) describes the velocity of the fluid in a steady laminar flow.

Now, let us see what equation (5.24) represents geometrically. Since equation (5.24) is an equation of a parabola where u=0 when y=R and u is maximum when y=0 i.e, at the axis

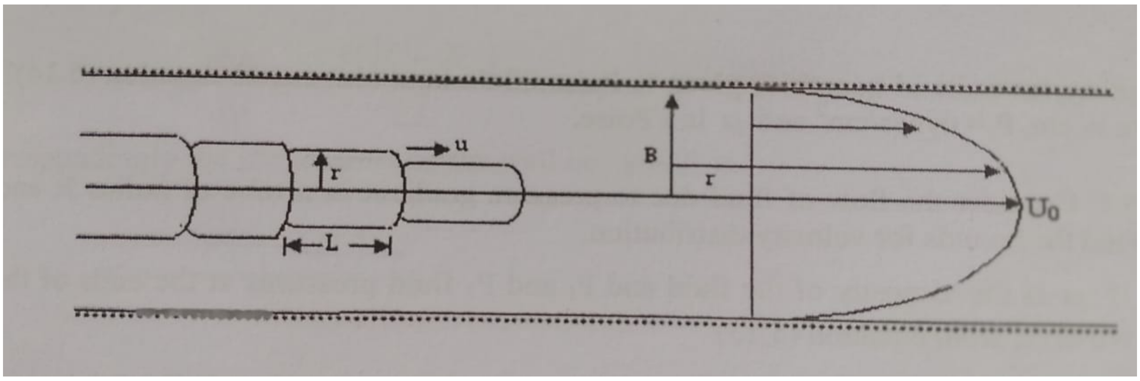

Figure 4.9: The parabola shows the velocity profile in a steady laminar flow

of the tube as shown in the figure (4.9).

Our boundary conditions say that the velocity is zero at the wall. If the principle of conservation of mass is to hold well, the same amount of fluid should come out of every cross-section. The loss of velocity at the wall has to be compensated by maximum velocity at the centre.

Thus, we find that the velocity distribution in a tube, with given pressure gradient is parabolic.

So, we have got an equation, which gives velocity distribution in a tube.
Let us now find the volume of fluid, flowing through a section of the tube per unit time. Here we shall see how we will use equation (5.24) along its axis. That is, we have to determine the volume of the solid of revolution of parabola.

The required volume V = Volume of parabola of revolution.

Then

$$V = \int_0^{2\pi} \int_0^R u(y) y dy d\theta$$

Now substituting for u in the above integral, we get

$$
\begin{aligned}
V &= \frac{2\pi(P_1 - P_2)}{4L\mu}\int_0^R (R^2 - y^2)ydy \\
&= \frac{(P_1 - P_2)\pi R^4}{8L\mu}
\end{aligned}
$$

i.e.,

$$V = \frac{(P_1 - P_2)\pi R^4}{8L\mu} \tag{5.25}$$

Equation (5.25) is called Poiseuille's law and it says that the volume is proportional to the first power of the pressure drop per unit length, $(P_1 - P_2)/L$, and to the fourth power of the radius of the pipe R^4 and it is inversely proportional to the length of the tube as well as the viscosity of the fluid. This equation is a general solution for any problem of fluid flow through cylindrical pipes, provided that the fluid flow satisfies all the assumptions, which are assumed in obtaining equation (5.25).The assumptions made are:

1) The fluid is homogeneous and its viscosity is the same at all rates of shear.

2) The fluid does not slip at the wall of the tube. This was the assumption that u=0 when y=R which was made in evaluating the constant of integration in equation (5.20).

3) The flow is laminar, i.e. the fluid is flowing parallel to the axis of inner surface wall of the tube.

4) The rate of flow is steady.

5) The tube is along with length much greater than the diameter of the tube.

The quantities given in Poiseuille's equation, i.e, in equation (5.25) R and L are in cm, $P = \ Dynes/cm^2$ and μ is Poise.

Case Study:

For any given flow of fluid due to pressure gradient in a tube of radius R and length L,we have evaluated the bounds for velocity distribution by the formula

$$u(y) = \frac{(P_1 - P_2)}{4\mu L}(R^2 - y^2) : 0 \leq y \leq R$$

Where μ is the viscosity of the fluid and $P = P_1 - P_2$ is fluid pressure at the ends of the tube.

i.e.,

$$u(y) = \frac{P}{4\mu L}(R^2 - y^2), \tag{5.26}$$

At $y = 0, u(0) = \frac{PR^2}{4\mu L}$ and at $y = R, u(R) = o$

Therefore,

$$0 \leq u(y) \leq \frac{PR^2}{4\mu L}$$

Now, we have $P = 4 \times 10^3$ $Dyne/cm^2$, $R = 8 \times 10^{-3} cm$, $\mu = 0.027$ *Poise* and $L = 2cm$

Then the bounds of velocity distribution are given as;

At y=0,

$$\begin{aligned}
u(0) &= \frac{PR^2}{4\mu L} \\
&= \frac{4 \times 10^3 \times (8 \times 10^{-3})^2}{4 \times 0.027 \times 2} \\
&= \frac{10^{-3} \times 64}{0.027 \times 2} \\
&= \frac{32 \times 10^{-3}}{0.027} \\
&= \frac{32}{27} \\
&= 1.185 cm/s
\end{aligned}$$

At $y = R$,

$$u(R) = 0$$

$$\therefore 0 \leq u(y) \leq 1.185$$

Now we have evaluated shear stress(τ) on the wall i.e. y=R given as

$$\begin{aligned}
\tau &= \frac{-R(P_1 - P_2)}{2L} \\
&= -\frac{8 \times 10^{-3} \times 4 \times 10^{3}}{2 \times 2} \\
&= -8 \; Dynes/cm^2
\end{aligned}$$

In magnitude, shear stress (τ) is 8 $Dynes/cm^2$

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