CONSTANT MEAN AND CONSTANT GAUSSIAN CURVATURE SURFACES IN THREE DIMENSIONAL SPACES

Dissertation submitted to the Department of Mathematics
in partial fulfillment of the requirements for the award of
Master's Degree in Mathematics

by

Rayees Ahmad Shah	(21068120022)
Ishfaq Mohi Ud Din Dar	(21068120023)
Mansoor Fayaz Sheikh	(21068120024)
Mudasir Ahmad Raina	(21068120029)
Bilal Ahmad Wani	(21068120036)

(Batch 2021)

under the supervision of

DR. MOHAMD SALEEM LONE

Lecturer

Department of Mathematics

UNIVERSITY OF KASHMIR, HAZRATBAL, SRINAGAR

JAMMU & KASHMIR, 190006.

Department of Mathematics

University of Kashmir

(*NAAC Accredited "A+" University*)

Hazratbal, Srinagar, Jammu & Kashmir, 190006

CERTIFICATE

This is to certify that the dissertation entitled, **"CONSTANT MEAN AND CONSTANT GAUSSIAN CURVATURE SURFACES IN THREE DIMENSIONAL SPACES"** *being submitted by the students with the enrollments* ***21068120022, 21068120023, 21068120024, 21068120029, 21068120036*** *to the* ***Department of Mathematics, University of Kashmir, Srinagar,*** *for the award of Master's degree in Mathematics, is an original project work carried out by them under my guidance and supervision.*

The project dissertation meets the standard of fulfilling the requirements of regulations related to the award of the Master's degree in Mathematics. The material embodied in the project dissertation has not been submitted to any other institute, or to this university for the award of Master's degree in Mathematics or any other degree.

DR. M. A. KHANDAY

PROFESSOR AND HEAD

Department of Mathematics

University of Kashmir, Srinagar

DR. MOHAMD SALEEM LONE

LECTURER

Department of Mathematics

University of Kashmir, Srinagar

(Supervisor)

Acknowledgement

All praise be to Almighty Allah, the Beneficent, the Merciful and the Lord of all that exists. It is by His grace and blessings that the present work has been accomplished.

We are extremely grateful to all those who have supported us throughout our project journey in completing the Master's degree in Mathematics. We would like to express our sincere gratitude to our worthy supervisor, Dr. Mohamd Saleem Lone, for his unwavering support, encouragement, cooperation, and the freedom he granted us throughout the project. Dr. Mohamd Saleem Lone's demeanour, supervision and support during challenging times, the conducive environment he fostered, and the space he provided for our project work are truly commendable. Working with him has been a truly enriching experience. His guidance, patience and understanding have been instrumental in shaping our academic path and have played a crucial role in the development of the content presented in this project dissertation. We have been consistently inspired, enlightened and motivated to pursue a complex and rewarding project journey under his mentorship.

We are grateful to Prof. M. A. Khanday, Head Department of Mathematics, University of Kashmir and Prof. S. Pirzada (former Head of Department) for providing the required infrastructure in the department and facilitating us in the official procedures.

We are deeply grateful to the faculty members of the department, especially Prof. N.A. Rather, Prof. M.H. Gulzar, Prof. B.A. Zargar, and Dr. M.A. Mir, for their warm companionship, which greatly contributed in making our stay a pleasant and an enriching experience in the department. We are also thankful to Dr. Ishfaq Ahmad Malik, Dr. Aijaz A Malla and Dr. Shahnawaz A Rather, who have been always there for their constant support and insightful guidance. Their collective wisdom and invaluable advice have served as a constant source of enlightenment and motivation throughout the journey.

We are also grateful to the non-teaching staff for providing the required infrastructural facilities in the department and facilitating us in the official procedures.

Lastly, we express our deepest gratitude and utmost respect to our families whose tremendous moral support and unfaltering belief in us have been a constant source of strength throughout this entire endeavour. Their unwavering faith, love, care and sacrifices have been always invaluable for us.

OCTOBER 2023

RAYEES AHMAD SHAH (21068120022)
ISHFAQ MOHI UD DIN DAR (21068120023)
MANSOOR AHMAD SHEIKH (21068120024)
MUDASIR AHMAD RAINA (21068120029)
BILAL AHMAD WANI (21068120036)

Abstract

The project dissertation comprises four chapters that focuses on constant mean and constant Gaussian curvature surfaces in three dimensional spaces. The first chapter provides an introductory overview of the differential geometry of surfaces in three-dimensional spaces.

In the second chapter, we discuss the classification of minimal and constant mean curvature translation surface in Euclidean 3-space E^3. We also discuss the affine minimal and constant mean curvature translation surface.

In the third chapter, we discuss the translation surfaces with zero and non-zero constant mean curvature and non-zero constant Gaussian curvature in Minkowski 3-spaces. We also discuss the affine translation surface with zero mean curvature in the same space of E_1^3.

In the last chapter, we take into consideration the upper half plane model of hyperbolic space $\mathbb{H}^3$ and discuss the classification of minimal translation surfaces in this setting.

Contents

Chapter 1

Preliminaries

Introduction

In this chapter, we give a brief introduction to the differential geometry of surfaces in three-dimensional spaces. The main purpose of this chapter is to provide the basic notions of differential geometry and with the essential formulas that will be needed in the upcoming chapters. Most of the notions, formulas and definitions in this chapter are included in consultation with [1, 2, 3, 4, 5, 6, 7, 8, 9].

Let E^3 be the 3-dimensional Euclidean space with the metric

$$<,>= dx^2 + dy^2 + dz^2.$$

We denote a regular surface S in E^3 by $r(x,y) = (X(x,y), Y(x,y), Z(x,y))$, and is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure. The idea is to define a set that is, in a certain sense, two-dimensional and that also is smooth enough so that the usual notions of calculus can be extended to it.

***Definition* 1.0.1.** *A subset $S \subset \mathbb{R}^3$ is a **regular suface** if, for each $p \in S$, there exists a neighbourhood V in $\mathbb{R}^3$ and a map $\boldsymbol{r} : U \to V \cap S \subset \mathbb{R}^3$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

*1. **r** is differentiable. This means that if we write*

$$\boldsymbol{r}(x,y) = (X(x,y), Y(x,y), Z(x,y)), \quad (x,y) \in U,$$

the functions $X(x,y)$, $Y(x,y)$, $Z(x,y)$ have continuous partial derivatives of all orders in U.

*2. **r** is homeomorphism. Since **r** is continuous by condition 1, this means that **r** has an inverse $\boldsymbol{r}^{-1} : V \cap S \to U$, which is continuous.*

3. For each $q \in U$, the differential $dr_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one.

***Definition* 1.0.2.** *Let S be a regular surface, $p \in S$, consider all the curves defined on S passing through p. We define the tangent plane at p denoted by T_pS as the vector space of dimension 2 which contains all vectors tangent to the family of curves at point p.*

***Definition* 1.0.3.** *Let $p \in S$ and $w \in T_pS$, the quadratic form $I_p : T_pS \to \mathbb{R}$, defined by:*

$$I_p(w) = \langle\, w,\, w \rangle = \|w\|^2 \geq 0,$$

*is called the **first fundamental form** of the regular surface S at p.*

If r_x and r_y are the partial derivatives with respect to (w.r.t) x and y respectively, then the first fundamental form can be expressed in the base $\{r_x, r_y\}$ associated with a parametrization $r(x,y)$ at p as follows:

Let $w = \alpha'(0) = r_x x' + r_y y' \in T_pS$.

Then

$$\begin{aligned} I_p(w) &= \langle\, r_x x' + r_y y', r_x x' + r_y y' \,\rangle \\ &= E(x')^2 + 2Fx'y' + G(y')^2, \end{aligned}$$

where $E = \langle\, r_x, r_x \,\rangle$, $F = \langle\, r_x, r_y \,\rangle$ and $G = \langle\, r_y, r_y \,\rangle$ are the coefficients of the first fundamental form in the base $\{r_x, r_y\}$ of T_pS.

Definition* 1.0.4.** *A regular surface S is* ***orientable *if it is possible to cover S with a family of coordinate neighbourhoods so that if a point $p \in S$ is in two neighbourhoods of this family, then the change of coordinates has positive Jocobian at p. The choice of family that satisfies this condition is called an orientation of S and S is called oriented. If it is not possible to find such a family, then S is called non-orientable.*

Fix a parametrization $\mathbf{r} : U \subset \mathbb{R}^2 \to S$, we calculate the *normal vector* at each point $q(U)$ as

$$\mathbf{N}(q) = \frac{r_x \times r_y}{\|r_x \times r_y\|}(q).$$

Definition* 1.0.5.** *Let $S \subset \mathbb{R}^3$ be a surface with an orientation, we have the* ***Gauss map *$\boldsymbol{N} : S \to S^2 \subset \mathbb{R}^3$ defined to be $p \to \boldsymbol{N}(p)$.*

The Gauss map can be defined (globally) if and only if the surface is orientable. The Gauss map can always be defined locally (that is on a small piece of the surface).
The differential of the Gauss map $d\mathbf{N}_p : T_pS \to T_pS$ is a self-adjoint linear operator.
That is

$$\langle\, dN_p(w_1), w_2 \,\rangle = \langle\, w_1, d\mathbf{N}_p(w_2) \,\rangle, \quad w_1, w_2 \in T_pS.$$

Therefore, we can associate $d\mathbf{N}_p$ with a quadratic form Q in T_pS given by

$$Q(w) = \langle\, d\mathbf{N}_p(w), w \,\rangle, \quad w \in T_pS.$$

***Definition* 1.0.6.** *Let $p \in S$, the quadratic form $II_p : T_pS \to \mathbb{R}$ defined by*

$$II_p = -\langle\, d\boldsymbol{N}_p(w), w \,\rangle$$

is called the ***second fundamental form*** *of the regular surface S at p.*

The second fundamental form can be expressed in the base $\{r_x, r_y\}$ associated with a parametrization $r(x,y)$. In fact, let $\mathbf{N}$ be the normal vector to S at $p \in S$ and $\alpha(s) = r(x(s), y(s))$ a parameterized curve in S with $\alpha(0) = p$. Therefore, the tangent vector to $\alpha(s)$ at p is $\alpha' = r_x x' + r_y y'$. If we indicate by $\mathbf{N}$ the restriction of normal vector to the curve $\alpha(s)$, we have

$$\langle\, \mathbf{N}(s), \alpha'(s)\, \rangle = 0.$$

This implies that

$$\langle\, \mathbf{N}(s), \alpha''(s)\, \rangle = -\langle \mathbf{N}'(s), \alpha'(s)\, \rangle.$$

Let $w = \alpha'(0) = r_x x' + r_y y' \in T_pS$.
Then

$$\begin{aligned} II_p &= -\langle d\mathbf{N}(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle \mathbf{N}'(0), \alpha'(0)\rangle \\ &= -\langle \mathbf{N}(0), \alpha''(0)\rangle \\ &= -\langle \mathbf{N}(0), r_{xx}(x')^2 + r_x x'' + 2r_{xy}x'y' + r_{yy}(y')^2 - r_y y''\rangle. \end{aligned}$$

Since $\langle \mathbf{N}, r_x\rangle = \langle \mathbf{N}, r_y\rangle = 0$, it follows that

$$II_p(w) = L(x')^2 + 2Mx'y' + N(y')^2, \tag{1.0.1}$$

where $L = \langle \mathbf{N}, r_{xx}\rangle$, $M = \langle \mathbf{N}, r_{xy}\rangle$ and $N = \langle \mathbf{N}, r_{yy}\rangle$ are the coefficients of the second fundamental form in the base $\{r_x, r_y\}$ of T_pS. Also r_{xx}, r_{xy} and r_{yy} are the second order partial derivatives of $r(x,y)$ w.r.t. x and y.

Equation (1.0.1) is also written as

$$
\begin{aligned}
II &= Ldx^2 + 2Mdxdy + Ndy^2, \\
L &= \frac{1}{\sqrt{|EG-F^2|}} \det(r_x, r_y, r_{xx}), \\
M &= \frac{1}{\sqrt{|EG-F^2|}} \det(r_x, r_y, r_{xy}), \\
N &= \frac{1}{\sqrt{|EG-F^2|}} \det(r_x, r_y, r_{yy}).
\end{aligned}
$$

Definition* 1.0.7.** *Let $p \in S$ and let $d\mathbf{N}_p : T_pS \to T_PS$ be the differential of the Gauss map. The determinant of $d\mathbf{N}_p$ is called* ***Gaussian curvature *K of S at p and the half of trace of dN_p is called as mean curvature H of S at p defined as*

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{LG - 2MF + NE}{2(EG - F^2)} \tag{1.0.2}$$

respectively. Also for $K = 0$, $H = 0$, the surface is called as developable and minimal surface, respectively.

***Definition* 1.0.8.** *A surface S in E^3 is called a translation surface, if it can be parameterized by*

$$r(x,y) = (x, y, f(x) + g(y)),$$

where f and g are smooth functions of x and y respectively.

***Definition* 1.0.9.** *A generalized notion of translation surface appear in the form of affine translation surface and is defined as:*

$$
\begin{aligned}
r(x,y) &= (X(x,y), Y(x,y), Z(x,y)) \\
&= (x, y, f(x) + g(y + ax))
\end{aligned}
$$

for some non zero constant a.

***Definition* 1.0.10.** *The Minkowski space is the space $E_1^3 = (\mathbb{R}^3, \langle , \rangle)$, where the metric $\langle , \rangle$ is*

$$\langle u, v \rangle = u_1v_1 + u_2v_2 - u_3v_3, \quad u = (u_1, u_2, u_3), v = (v_1, v_2, v_3),$$

which is called the Lorentzian metric.

***Definition* 1.0.11.** *A vector $v \in E_1^3$ is said to be*
(1) spacelike if $\langle v, v\rangle > 0$ or $v = 0$,
(2) timelike if $\langle v, v\rangle < 0$ and
(3) lightlike if $\langle v, v\rangle = 0$ or $v \neq 0$.

***Definition* 1.0.12.** *A Hyperbolic 3-space denoted by $\mathbb{H}^3$, is defined to be a 3 dimensional complete, simply connected space form with a sectional curvature of -1.*
A Hyperbolic 3 space has various models and in this work we deal with the half-space model of $\mathbb{H}^3$ defined below.

***Definition* 1.0.13.** *A half-space model of the hyperbolic space $\mathbb{H}^3$ is denoted by $\mathbb{R}^3_+$ and it is defined by*

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3; z > 0\}.$$

The metric of $\mathbb{R}^3_+$ is given by the following line element

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

***Definition* 1.0.14.** *A unit normal vector field $\boldsymbol{n}$ to S with respect to the hyperbolic metric determines a unit normal vector field $\boldsymbol{N}$ to S with respect to the Euclidean metric by the relation $\boldsymbol{N} = \frac{\boldsymbol{n}}{z}$.*
The hyperbolic principal curvatures $k_i's$ are related to the Euclidean principal curvatures k_i^e by

$$k_i = zk_i^e + \boldsymbol{N}_3,$$

where $\boldsymbol{N}_3$ is the third component of the unit normal vector $\boldsymbol{N}$. If we denote by H and H_e the hyperbolic and Euclidean mean curvature on a surface S respectively, we have the relation

$$H(x, y, z) = zH_e(x, y, z) + \boldsymbol{N}_3(x, y, z). \tag{1.0.3}$$

***Definition* 1.0.15.** *Consider the half-space model of $\mathbb{H}^3$. A surface S in hyperbolic space $\mathbb{H}^3$ is a translation surface if it is given by $r : U \subset \mathbb{R}^2 \to \mathbb{R}^3_+$ and is written as*

$$r(x,y) = (x,y,f(x)+g(y)),\ (x,y) \in U\ (typeI), \tag{1.0.4}$$

$$r(x,z) = (x,f(x)+g(z),z),\ (x,z) \in U\ (typeII), \tag{1.0.5}$$

where f and g are smooth functions on open subsets of $\mathbb{R}$.

Chapter 2

Translation surfaces in Euclidean 3-space E^3

In this chapter, we discuss the classification of minimal and constant mean curvature translation surface in Euclidean 3-space E^3. We also discuss the affine minimal and constant mean curvature translation surface. This chapter is a survey of the articles in [4, 5, 6, 9].

***Theorem* 2.0.1.** *Let S be a translation surface with zero mean curvature in 3-dimensional Euclidean space E^3. Then S is congruent to the following surface*

$$r(x,y) = \left(x, y, \frac{1}{a}\log\left|\frac{cos(ax)}{cos(ay)}\right|\right), \quad 0 \neq a \in \mathbb{R}.$$

Proof. Consider the translation surface parameterized as

$$r(x,y) = (x, y, f(x) + g(y)). \tag{2.0.1}$$

Thus

$$r_x = (1, 0, f'(x)), \quad r_y = (0, 1, g'(y)),$$

$$r_{xx} = (0, 0, f''(x), \quad r_{yy} = (0, 0, g''(y)), \quad r_{yx} = r_{xy} = (0, 0, 0).$$

Hence

$$E = r_x \cdot r_x = 1 + f'(x)^2, \quad F = r_x \cdot r_y = f'(x)g'(y), \quad G = r_y \cdot r_y = 1 + g'(y)^2.$$

Also

$$EG - F^2 = \left(1 + f'(x)^2\right)\left(1 + g'(y)^2\right) - f'(x)^2 g'(y)^2 = 1 + f'(x)^2 + g'(y)^2.$$

Now, we have

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f'(x) \\ 0 & 1 & g'(y) \end{vmatrix} = \left(-f'(x), -g'(y), 1\right).$$

Thus the unit normal **N** is obtained as

$$\mathbf{N} = \frac{(-f', -g', 1)}{\sqrt{1 + f'^2 + g'^2}}.$$

Also

$$\begin{aligned} L &= r_{xx} \cdot \mathbf{N} = \frac{f''}{\sqrt{1 + f'^2 + g'^2}}, \\ M &= r_{xy} \cdot \mathbf{N} = 0, \\ N &= r_{yy} \cdot \mathbf{N} = \frac{g''}{\sqrt{1 + f'^2 + g'^2}}. \end{aligned}$$

Therefore on substituting the values of first and second fundamental form coefficients in (1.0.2), we obtain

$$H = \frac{f''(1 + g'^2) + g''(1 + f'^2)}{2(1 + f'^2 + g'^2)}. \tag{2.0.2}$$

For $H = 0$, we have

$$f''(1 + g'^2) + g''(1 + f'^2) = 0,$$

or

$$\frac{g''}{(1 + g'^2)} + \frac{f''}{(1 + f'^2)} = 0.$$

Since f and g are independently the functions of x and y alone respectively, so we have two cases:

<u>Case I</u>:

$$\frac{1+f'^2}{f''}=0 \quad \text{and} \quad \frac{1+g'^2}{g''}=0,$$

which gives

$$f(x)=c_1 \pm \imath x \quad \text{and} \quad g(y)=c_2 \pm \imath y.$$

Thus $r(x,y)=(x,y,c_1x+c_2y+c_3)$, which is congruent to a plane.

<u>Case II</u>:

$$\frac{f''}{1+f'^2}=-a \quad \text{and} \quad \frac{g''}{1+g'^2}=a,$$

where a is a non-zero constant.

Solving these two, we get

$$f(x)=\frac{1}{a}\log|\cos(ax+c_1)|+c_2$$

and

$$g(y)=c_2-\frac{1}{a}\log|\cos(ay+c_1)|.$$

Hence

$$z=f(x)+g(y)=\frac{1}{a}\left[\log|\cos(ax+c_1)|-\log|\cos(ay-c_1)|\right]+2c_2. \tag{2.0.3}$$

Setting $c_2=0$, we can write

$$z=\frac{1}{a}\log|\cos(ax)|-\frac{1}{a}\log|\cos(ay)|,$$

or

$$z=\frac{1}{a}\log\left|\frac{\cos(ax)}{\cos(ay)}\right|. \tag{2.0.4}$$

This proves the theorem. □

Remark: The surface (2.0.4) is called as Scherk Surface which is the minimal surface of the form (2.0.1). Next, we discuss the classification of affine minimal translation surface.

***Theorem* 2.0.2.** *Let $r(x,y) = (x,y,z(x,y))$ be a minimal affine translation surface in Euclidean 3-space. Then either $z(x,y)$ is linear or can be written as*

$$z(x,y) = \frac{1}{c}\log\left|\frac{\cos(c\sqrt{1+a^2}x)}{\cos[c(y+ax)]}\right|, \tag{2.0.5}$$

where a and c are constants and $ac \neq 0$.

Proof. Let $r(x,y) = (x,y,f(x)+g(y+ax))$, $a \neq 0$ be an affine translation surface. Therefore

$$r_x = (1,0,f'(x)+ag'(y+ax)),\ r_y = (0,1,g'(y+ax)),$$

$$r_{xx} = (0,0,f''(x)+a^2g''(y+ax)),\ r_{yy} = (0,0,g''(y+ax)),$$

$$r_{xy} = (0,0,ag''(y+ax).$$

Hence

$$E = r_x \cdot r_x = 1+(f'+ag')^2, \quad F = r_x \cdot r_y = (f'+ag')g', \quad G = r_y \cdot r_y = 1+g'^2,$$

where $f' = f'(x)$ and $g' = g'(y+ax)$.

Now

$$\begin{aligned} EG-F^2 &= \left[1+(f'+ag')^2\right]\left[1+g'^2\right] - \left[(f'+ag')^2g'^2\right] \\ &= (1+g'^2)+(f'+ag')^2(1+g'^2-g'^2), \end{aligned}$$

or

$$EG-F^2 = 1+(f'+ag')^2+g'^2. \tag{2.0.6}$$

Hence, we have

$$\mathbf{N} = \frac{r_x \times r_y}{\|r_x \times r_y\|} = \frac{r_x \times r_y}{\sqrt{EG-F^2}} = \frac{(-f'-ag',-g',1)}{\sqrt{1+(f'+ag')^2+g'^2}}.$$

Therefore

$$L = r_{xx}.\mathbf{N} = \frac{f''+a^2g''}{\sqrt{1+(f'+ag')^2+g'^2}},$$

$$M = r_{xy}.\mathbf{N} = \frac{ag''}{\sqrt{1+(f'+ag')^2+g'^2}},$$

$$N = r_{yy}\cdot\mathbf{N} = \frac{g''}{\sqrt{1+(f'+ag')^2+g'^2}}.$$

Now

$$LG-2MF+NE = (f''+a^2g'')(1+g'^2)-2ag''g'(f'+ag')+g''(1+f'^2+a^2g'^2+2af'g'). \tag{2.0.7}$$

Thus from (1.0.2), $H=0$ takes the following form

$$(f''+a^2g'')(1+g'^2)-2ag''g'(f'+ag')+g''(1+f'^2+a^2g'^2+2af'g') = 0,$$

or

$$f''(1+g'^2)+a^2g''+g''+f'^2g'' = 0,$$

or

$$f''(1+g'^2)+g''(1+a^2+f'^2) = 0.$$

The above can be rewritten as

$$\frac{f''}{1+a^2+f'^2}+\frac{g''}{1+g'^2} = 0. \tag{2.0.8}$$

Differentiating (2.0.8) w.r.t. y, we get

$$\frac{d}{d(y+ax)}\left(\frac{g''}{1+g'^2}\right)\frac{d}{dy}(y+ax) = 0,$$

that is

$$\frac{d}{d(y+ax)}\left(\frac{g''}{1+g'^2}\right) = 0. \tag{2.0.9}$$

Again differentiating (2.0.8) w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{f''}{1+a^2+f'^2}\right)+\frac{d}{d(y+ax)}\left(\frac{g''}{1+g'^2}\right)\frac{d}{dx}(y+ax) = 0,$$

or

$$\frac{d}{dx}\left(\frac{f''}{1+a^2+f'^2}\right)+a\frac{d}{d(y+ax)}\left(\frac{g''}{1+g'^2}\right) = 0. \tag{2.0.10}$$

Using (2.0.9) in (2.0.10), we get

$$\frac{d}{dx}\left(\frac{f''}{1+a^2+f'^2}\right)=0.$$

Therefore, we have

$$\frac{f''}{1+a^2+f'^2}=-\frac{g''}{1+g'^2}=-c,$$

where c is constant.

If $c=0$ then $f''=g''\equiv 0$ that is $f=ax+c_1$ and $g=by+c_2$

Hence $r(x,y)$ is a plane.

If $c\neq 0$ then

$$\frac{f''}{1+a^2+f'^2}=-c,$$

which gives

$$f(x)=\frac{\log\left|\cos\left(\sqrt{a^2+1}(cx+k_1)\right)\right|}{c}+k_2,$$

where k_1 and k_2 are constants.

Setting constants feasibly, we get

$$f(x)=\frac{\log\left|\cos(c\sqrt{1+a^2}x)\right|}{c}.$$

Now for

$$\frac{g''}{1+g'^2}=c,$$

which gives

$$g(y+ax)=\frac{k_2-\log|\cos(c(y+ax)+k_1)|}{c}.$$

Again setting constants feasibly, we get

$$g(y+ax)=-\frac{\log|\cos(c(y+ax))|}{c}.$$

Therefore

$$f(x)+g(y+ax)=\frac{1}{c}\log\left|\frac{\cos(c\sqrt{1+a^2}x)}{\cos(c(y+ax))}\right|.$$

That proves the result. □

***Theorem* 2.0.3.** *Let S be a translation surface with constant Gaussian curvature K in 3-dimensional Euclidean space E^3. Then S is congruent to a cylinder.*

Proof. In E^3, by a transformation the translation surface S can be written as

$$z = g(x) - h(y).$$

That is

$$r(x,y) = (x, y, g(x) - h(y)). \tag{2.0.11}$$

So

$$r_x = (1, 0, g'(x)),\ r_y = (0, 1, -h'(y))$$

and

$$r_{xx} = (0, 0, g''(x)),\ r_{yy} = (0, 0, -h''(y)),\ r_{xy} = (0, 0, 0).$$

Now we calculate the coefficients of first and second fundamental form as

$$E = r_x \cdot r_x = 1 + g'(x)^2, \quad F = r_x \cdot r_y = g'(x)h'(y), \quad G = r_y \cdot r_y = 1 + h'(y)^2. \tag{2.0.12}$$

Therefore

$$EG - F^2 = 1 + g'(x)^2 + h'(y)^2. \tag{2.0.13}$$

Now

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & g'(x) \\ 0 & 1 & -h'(y) \end{vmatrix} = (-g'(x), h'(y), 1).$$

So, the unit normal vector is given as

$$\mathbf{N} = \frac{r_x \times r_y}{||r_x \times r_y||} = \frac{(-g'(x), h'(y), 1)}{\sqrt{1 + g'(x)^2 + h'(y)^2}}.$$

Therefore,

$$L = r_{xx} \cdot \mathbf{N} = \frac{g''(x)}{\sqrt{1 + g'(x)^2 + h'(y)^2}}, \quad M = r_{xy} \cdot \mathbf{N} = 0, \quad N = r_{yy} \cdot \mathbf{N} = \frac{-h''(y)}{\sqrt{1 + g'(x)^2 + h'(y)^2}}. \tag{2.0.14}$$

Now

$$LN - M^2 = \frac{-g''(x)h''(y)}{1+g'(x)^2+h'(y)^2}.$$

Substituting (2.0.12) and (2.0.14) in (1.0.2), we get

$$K = \frac{-g''(x)h''(y)}{(1+g'(x)^2+h'(y)^2)^2}. \tag{2.0.15}$$

Let $K = c(constant)$ and $g''(x) \neq 0$, then from (2.0.15), we have

$$\frac{h''(y)}{(1+g'(x)^2+h'(y)^2)^2} = \frac{c}{-g''(x)}.$$

Differentiating w.r.t. y on both sides, we get

$$\left(1+g'(x)^2+h'(y)^2\right)^2 h'''(y) - 4h'(y)h''(y)^2 = 0. \tag{2.0.16}$$

Similarly let $h''(y) \neq 0$, then from (2.0.15), we have

$$\frac{g''(x)}{(1+g'(x)^2+h'(y)^2)^2} = \frac{c}{-h''(y)}.$$

Differentiating w.r.t. x on both sides, we get

$$(1+g'(x)^2+h'(y)^2)^2 g'''(x) - 4g'(x)g''(x)^2 = 0. \tag{2.0.17}$$

Comparing (2.0.16) and (2.0.17), we get

$$\frac{h'''(y)}{h'(y)h''(y)^2} = \frac{g'''(x)}{g'(x)g''(x)^2},$$

which is not possible as $f(x)$ and $g(y)$ are independent.

Hence, $g''(x) = 0$ or $h''(y) = 0$.

Let $g''(x) = 0$, we get

$$g(x) = ax + b, \quad a, b \in \mathbb{R}.$$

Therefore from (2.0.11), we have

$$r(x,y) = (x, y, ax + b - h(y)) = (0, y, b - h(y)) + x(1, 0, a),$$

which is a cylinder. □

***Theorem* 2.0.4.** *Let S be a translation surface with constant mean curvature $H \neq 0$ in 3-dimensional Euclidean space E^3. Then S is congruent to the following surface*

$$r(x,y) = \left(x, y, -\frac{\sqrt{1+a^2}}{2H} \times \sqrt{1-4H^2x^2} - ay \right), \quad a \in \mathbb{R}.$$

Proof. Let S be the surface with constant mean curvature $H \neq 0$.
Substituting the values of fundamental coefficients from equation (2.0.12) and (2.0.14) in (1.0.2), we have

$$H = \frac{g''(x)(1+h'(y)^2) - h''(y)(1+g'(x)^2)}{2\sqrt{1+g'(x)^2+h'(y)^2}(1+g'(x)^2+h'(y)^2)},$$

or

$$H = \frac{g''(x)(1+h'(y)^2 - h''(y))(1+g'(x)^2)}{2(1+g'(x)^2+h'(y)^2)^{\frac{3}{2}}}. \tag{2.0.18}$$

Since H is constant, therefore $H' = 0$.
Therefore, (2.0.18) can be written as

$$2H = \left[g''(x)(1+h'(y)^2) - h''(y)(1+g'(x)^2)\right]\left[1+g'(x)^2+h'(y)^2\right]^{-\frac{3}{2}}.$$

Differentiating w.r.t. x and taking $g = g(x)$ and $h = h(y)$, we have

$$\begin{aligned}
0 &= \left[g'''(1+h'^2) - 2g'g''h''\right]\left[1+g'^2+h'^2\right]^{-\frac{3}{2}} + \left[g''(1+h'^2) - h''(1+g'^2)\right] \\
&\quad \left[-3(1+g'^2+h'^2)^{-\frac{5}{2}}g'g''\right] \\
&= \left[g'''(1+h'^2) - 2g'g''h''\right]\left[1+g'^2+h'^2\right]^{-\frac{3}{2}} - 3g'g''\left[g''(1+h'^2) - h''(1+g'^2)\right] \\
&\quad \left[1+g'^2+h'^2\right]^{-\frac{5}{2}} \\
&= \left[g'''(1+h'^2) - 2g'g''h''\right]\left[1+g'^2+h'^2\right]^{-\frac{3}{2}} \\
&\quad - \left[\frac{6g'g''\left[g''(1+h'^2) - h''(1+g'^2)\right]\left[1+g'^2+h'^2\right]^{-1}}{2\left[1+g'^2+h'^2\right]^{\frac{3}{2}}}\right].
\end{aligned}$$

Using (2.0.18), we get

$$= \left[g'''(1+h'^2) - 2g'g''h''\right]\left[1+g'^2+h'^2\right]^{-\frac{3}{2}} - 6Hg'g''\left[1+g'^2+h'^2\right]^{-1},$$

that is

$$\left[g'''(1+h'^2)-2g'g''h''\right]\left[1+g'^2+h'^2\right]^{-\frac{1}{2}}=6Hg'g''. \tag{2.0.19}$$

Differentiating (2.0.19) w.r.t. y, we get

$$0=\left[2h'h''g'''-2g'g''h'''\right]\left[1+g'^2+h'^2\right]^{-\frac{1}{2}}-\frac{1}{2}\left[1+g'^2+h'^2\right]^{-\frac{3}{2}}$$
$$\times 2h'h''\left[g'''(1+h'^2)-2g'g''h''\right].$$

Using (2.0.19), we get

$$=\left[2h'h''g'''-2g'g''h'''\right]\left[1+g'^2+h'^2\right]^{-\frac{1}{2}}-6Hg'g''h'h''\left[1+g'^2+h'^2\right]^{-1},$$

or

$$\left[2h'h''g'''-2g'g''h'''\right]\left[1+g'^2+h'^2\right]^{\frac{1}{2}}-6Hg'g''h'h''=0. \tag{2.0.20}$$

Assume $g''(x)\neq 0$ and $h''(y)\neq 0$.

Dividing (2.0.20) both sides by $g'g''h'h''$, we get

$$\left[\frac{g'''}{g'g''}-\frac{h'''}{h'h''}\right]\left[1+g'^2+h'^2\right]^{\frac{1}{2}}=3H. \tag{2.0.21}$$

Differentiating (2.0.21) w.r.t. x, we get

$$\begin{aligned}0&=\left[\frac{g'''}{g'g''}\right]'[1+g'^2+h'^2]^{\frac{1}{2}}+\left[\frac{g'''}{g'g''}\right]\frac{1}{2}\left[1+g'^2+h'^2\right]^{-\frac{1}{2}}2g'g''\\&\quad-\frac{1}{2}\frac{h'''}{h'h''}\left[1+g'^2+h'^2\right]^{-\frac{1}{2}}2g'g''\\&=\left[\frac{g'''}{g'g''}\right]'\left[1+g'^2+h'^2\right]^{\frac{1}{2}}+\left[1+g'^2+h'^2\right]^{-\frac{1}{2}}g'g''\left[\frac{g'''}{g'g''}-\frac{h'''}{h'h''}\right]\\&=\left[\frac{g'''}{g'g''}\right]'\left[1+g'^2+h'^2\right]^{\frac{3}{2}}+\frac{[1+g'^2+h'^2]^{\frac{1}{2}}}{[1+g'^2+h'^2]}g'g''\left[\frac{g'''}{g'g''}-\frac{h'''}{h'h''}\right].\end{aligned}$$

Using (2.0.21), we get

$$\left[\frac{g'''(x)}{g'(x)g''(x)}\right]'[1+g'(x)^2+h'(y)^2]^{\frac{3}{2}}+3Hg'(x)g''(x)=0. \tag{2.0.22}$$

So $g''(x)\neq 0$ and $h''(y)\neq 0$,

implies that $H=0$, which is contradiction as $H\neq 0$.

Hence $g''(x) = 0$ or $h''(y) = 0$.

Assume $h''(y) = 0$. Let $h(y) = ay$, where a is constant.

Then from (2.0.18), we have

$$H = \frac{g''(x)(1+a^2)}{2\left[1+g'(x)^2+a^2\right]^{\frac{3}{2}}},$$

or

$$g''(x)(1+a^2) = 2H\left[1+g'(x)^2+a^2\right]^{\frac{3}{2}}.$$

Solving this equation, we get

$$g(x) = -\frac{\sqrt{1+a^2}}{2H}\sqrt{1-4H^2(x+c_1)^2}+c_2, \qquad c_1, c_2 \in \mathbb{R}.$$

Therefore, the surface is

$$z = -\frac{\sqrt{1+a^2}}{2H}\sqrt{1-4H^2(x+c_1)^2}+c_2-ay.$$

This proves the theorem. □

Theorem 2.0.5. *Let $r(x,y) = (x, y, f(x)+g(y+ax))$ be an affine transation surface with non-zero constant mean curvature H, then $r(x,y)$ is of the form*
$r(x,y) = \left(x, y, \frac{\pm\sqrt{1+b^2}}{2H}\sqrt{1-4H^2x^2}-abx+g(y+ax)+c_2\right)$,
where a,b,c_1 and c_2 are constants.

Proof. Let S be the affine translation surface with non-zero constant mean curvature in E^3.

Then on substituting the values from (2.0.6) and (2.0.7) in (1.0.2), we get

$$H = \frac{f''\left(1+g'^2\right)+g''\left(1+a^2+f'^2\right)}{2\left[1+(f'+ag')^2+g'^2\right]^{\frac{3}{2}}}. \tag{2.0.23}$$

It is clear that $f''^2+g''^2 \neq 0$. If $f'' = 0$ that is $f' = b$ is constant, (2.0.23) becomes

$$g''\left(1+a^2+b^2\right) = 2H\left[1+(b+ag')^2+g'^2\right]^{\frac{3}{2}},$$

or

$$\begin{aligned}
g'' &= \frac{2H}{1+a^2+b^2}\left[1+b^2+2abg'+(1+a^2)g'^2\right]^{\frac{3}{2}} \\
&= \frac{2H\left(1+a^2\right)^{\frac{3}{2}}}{1+a^2+b^2}\left[g'^2+\frac{2ab}{1+a^2}g'+\frac{1+b^2}{1+a^2}-\frac{a^2b^2}{(1+a^2)^2}+\frac{a^2b^2}{(1+a^2)^2}\right]^{\frac{3}{2}} \\
&= \frac{2H(1+a^2)^{\frac{3}{2}}}{1+a^2+b^2}\left[\left(g'+\frac{ab}{1+a^2}\right)^2+\frac{1}{1+a^2}\left(1+b^2-\frac{a^2b^2}{1+a^2}\right)\right]^{\frac{3}{2}} \\
&= \frac{2H\left(1+a^2\right)^{\frac{3}{2}}}{1+a^2+b^2}\left[\left(g'+\frac{ab}{1+a^2}\right)^2+\frac{1}{1+a^2}\left(\frac{1+a^2+b^2}{1+a^2}\right)\right]^{\frac{3}{2}}
\end{aligned}$$

$$= \frac{2H(1+a^2)^{\frac{3}{2}}}{1+a^2+b^2}\left[\left(g'+\frac{ab}{1+a^2}\right)^2+\frac{1+a^2+b^2}{\left(1+a^2\right)^2}\right]^{\frac{3}{2}}. \tag{2.0.24}$$

Putting

$$A^2=\frac{1+a^2+b^2}{(1+a^2)^2} \quad \text{and} \quad B=\frac{2H\left(1+a^2\right)^{\frac{3}{2}}}{1+a^2+b^2},$$

that is

$$g''=B\left[\left(g'+\frac{ab}{1+a^2}\right)^2+A^2\right]^{\frac{3}{2}},$$

or

$$\frac{dg'}{d(y+ax)}=B\left[\left(g'+\frac{ab}{1+a^2}\right)^2+A^2\right]^{\frac{3}{2}},$$

or

$$\frac{dg'}{\left[\left(g'+\frac{ab}{1+a^2}\right)^2+A^2\right]^{\frac{3}{2}}}=Bd(y+ax). \tag{2.0.25}$$

For solving (2.0.25), let $x=g'+\frac{ab}{1+a^2}$, then $dx=dg'$.

Hence on integrating, we get

$$\int\frac{dx}{(x^2+A^2)^{\frac{3}{2}}}=B(y+ax).$$

We have to find

$$\frac{d}{dx}\left(\frac{x}{(x^2+A^2)^{\frac{1}{2}}}\right) = \frac{(x^2+A^2)^{\frac{1}{2}}\cdot 1 - \frac{x}{2}(x^2+A^2)^{-\frac{1}{2}}\cdot 2x}{(x^2+A^2)}$$
$$= \frac{(x^2+A^2)-x^2}{(x^2+A^2)^{\frac{3}{2}}}$$
$$= \frac{A^2}{(x^2+A^2)^{\frac{3}{2}}}.$$

That is

$$\frac{A^2}{(x^2+A^2)^{\frac{3}{2}}} = \frac{d}{dx}\left[\frac{x}{(x^2+A^2)^{\frac{1}{2}}}\right],$$

or

$$\int \frac{dx}{(x^2+A^2)^{\frac{3}{2}}} = \frac{x}{A^2\sqrt{x^2+A^2}},$$

or

$$\frac{g'+\frac{ab}{1+a^2}}{A^2\sqrt{\left(g'+\frac{ab}{1+a^2}\right)^2+A^2}} = B(y+ax)+c_1.$$

This gives

$$g'+\frac{ab}{1+a^2} = \pm\frac{A^3B(y+ax)}{\sqrt{1-A^4B^2(y+ax)^2}}. \tag{2.0.26}$$

Where c_1 is an integral constant, for simplicity we choose $c_1 = 0$.

Therefore solving (2.0.26), we obtain

$$g(y+ax) = \pm\frac{1}{AB}\sqrt{1-A^4B^2(y+ax)^2} - \frac{ab}{1+a^2}(y+ax)+c_2$$
$$= \pm\frac{\sqrt{1+a^2+b^2}}{2H\sqrt{1+a^2}}\sqrt{1-\frac{4H^2}{1+a^2}(y+ax)^2} - \frac{ab}{1+a^2}(y+ax)+c_2.$$

That is

$$g(y+ax) = \pm\frac{\sqrt{1+a^2+b^2}}{2H(1+a^2)}\sqrt{1+a^2-4H^2(y+ax)^2} - \frac{ab}{1+a^2}(y+ax)+c_2,$$

where c_2 is an integral contant.

Now if $f'' \neq 0$, then from (2.0.23), we have

$$f''(1+g'^2)+g''(1+a^2+f'^2) = 2H\left[1+(f'+ag')^2+g'^2\right]^{\frac{3}{2}}. \tag{2.0.27}$$

Differentiating (2.0.27) w.r.t. y, we have

$$2f''g'g''+(1+a^2+f'^2)g'''=3H\left[1+(f'+ag')^2+g'^2\right]^{\frac{1}{2}}\left[2(f'+ag')ag''+2g'g''\right],$$

that is

$$2f''g'g''+(1+a^2+f'^2)g'''=6H\left[1+(f'+ag')^2+g'^2\right]^{\frac{1}{2}}(af'g''+a^2g'g''+g'g''). \tag{2.0.28}$$

Differentiating (2.0.27) w.r.t. x, we have

$$\begin{aligned}&f'''(1+g'^2)+2af''g'g''+ag'''+a^3g'''+2f'f''g''+af'^2g'''\\&=3H[1+(f'+ag')^2+g'^2]^{\frac{1}{2}}[2(f'+ag')(f''+a^2g'')+2ag'g'']\\&=6H\left[1+(f'+ag')^2+g'^2\right]^{\frac{1}{2}}\left[f'f''+a^2f'g''+ag'f''+a^3g'g''+ag'g''\right],\end{aligned}$$

that is

$$\begin{aligned}&f'''(1+g'^2)+2f'f''g''+2af''g'g''+ag'''+a^3g'''+af'^2g'''\\&\qquad=6H\left[1+(f'+ag')^2+g'^2\right]^{\frac{1}{2}}\left[f'f''+a^2f'g''+ag'f''+a^3g'g''+ag'g''\right],\end{aligned}$$

or

$$\begin{aligned}&f'''(1+g'^2)+2f'f''g''+2af''g'g''+a(1+a^2+f'^2)g'''\\&=6H\left[1+(f'+ag')^2+g'^2\right]^{\frac{1}{2}}\left[a(af'g''+a^2g'g''+g'g'')+f'f''+ag'f''\right].\end{aligned} \tag{2.0.29}$$

Multiplying (2.0.28) both sides by a and comparing with (2.0.29), we get

$$f'''(1+g'^2)+2f'f''g''=6HD(f'f''+ag'f''),\quad \textit{where}\quad D=\left[1+(f'+ag')^2+g'^2\right]^{\frac{1}{2}}. \tag{2.0.30}$$

Using (2.0.24), we get

$$g''=\left[\frac{2HD^2-f''(1+g'^2)}{1+a^2+f'^2}\right].$$

Equation (2.0.30) becomes

$$f'''(1+g'^2)+2f'f''\left[\frac{2HD^3-f''(1+g'^2)}{1+a^2+f'^2}\right]=6HD(f'f''+ag'f''),$$

or

$$f'''(1+g'^2)(1+a^2+f'^2)+2f'f''\left[2HD^3-f''(1+g'^2)\right] \\ =6HD(f'f''+ag'f'')(1+a^2+f'^2),$$

or

$$\begin{aligned}
&f'''(1+g'^2)(1+a^2+f'^2)-2f'f''^2(1+g'^2)\\
&=6HD(f'f''+ag'f'')(1+a^2+f'^2)-4Hf'f''D\cdot D^2\\
&=6HD(f'f''+ag'f'')(1+a^2+f'^2)-4HDf'f''\left[1+(f'+ag')^2+g'^2\right]\\
&=2HD\left[3f'f''+3f'f''a^2+3f'^3f''+3ag'f''+3a^3g'f''+3ag'f'^2f''\right]\\
&\qquad\left[-2f'f''(1+f'^2+a^2g'^2+2af'g'+g'^2)\right]\\
&=2HD\left[3f'f''+3f'f''a^2+3f'^3f''+3ag'f''+3a^3g'f''+3ag'f'^2f''\right]\\
&\qquad\left[-2f'f''-2f'^3f''-2a^2f'f''g'^2-4af'^2f''g'-2f'f''g'^2\right]\\
&=2HD\left[f'f''+3a^2f'f''+f'^3f''+(3af''+3a^3f''-af'^2f'')g'-2f'f''(1+a^2)g'^2\right].
\end{aligned}$$

Squaring both sides and put the value of D, we get

$$\left[f'''(1+g'^2)(1+a^2+f'^2)-2f'f''^2(1+g'^2)\right]^2=4H^2\left[1+(f'+ag')^2+g'^2\right] \\ \left[f'f''+3a^2f'f''+f'^3f''+(3af''+3a^3f''-af'^2f'')g'-2f'f''(1+a^2)g'^2\right]^2. \tag{2.0.31}$$

The coefficient of highest order 6 of g' in above equation (2.0.31) is $16H^2(1+a^2)^3f'^2f''^2$. Therefore $f''\neq 0$ means that g' is a constant.

Put $g'=b$, (2.0.23) becomes

$$(1+b^2)f''=2H\left[1+b^2+(ab+f')^2\right]^{\frac{3}{2}}. \tag{2.0.32}$$

On solving equation (2.0.32), we get

$$f(x)=\pm\frac{\sqrt{1+b^2}}{2H}\sqrt{1-4H^2x^2}-abx+c_3,$$

which proves the theorem. □

Chapter 3

Translation surfaces in Minkowski 3-space E_1^3

In this chapter, we discuss the translation surfaces with zero and non-zero constant mean curvature and non-zero constant Gaussian curvature in E_1^3 and affine translation surface with zero mean curvature. This chapter is a survey of the articles in [4, 8].

***Theorem* 3.0.1.** *Let S be a translation surface with constant Gaussian curvature K in 3-dimensional Minkowski space E_1^3. Then S is congruent to a cylinder, so $K = 0$.*

Proof. In Minkowski space E_1^3, by a transformation in E_1^3 the translation surface S can be written as

$$z = g(x) - h(y),$$

or

$$x = g(y) - h(z).$$

Accordingly, we have

$$r(x,y) = (x, y, g(x) - h(y)), \tag{3.0.1}$$

or

$$r(y,z) = (g(y) - h(z), y, z). \tag{3.0.2}$$

Then for (3.0.1), we have

$$E = r_x \cdot r_x = 1 - g'(x)^2, \quad F = r_x \cdot r_y = g'(x)h'(y), \quad G = r_y \cdot r_y = 1 - h'(y)^2.$$

Hence

$$EG - F^2 = 1 - g'(x)^2 - h'(y)^2. \tag{3.0.3}$$

Therefore

$$\mathbf{N} = \frac{(-g'(x), h'(y), 1)}{\sqrt{1 - g'(x)^2 - h'(y)^2}}.$$

Substituting the above found E, F, G and L, M, N in (1.0.2), we get

$$K = \frac{-g''(x)h''(y)}{(1 - g'(x)^2 - h'(y)^2)^2}. \tag{3.0.4}$$

Similarly for surface (3.0.2), we have

$$r_y = (g'(y), 1, 0), \quad r_z = (-h'(z), 0, 1), \quad r_{yy} = (g''(y), 0, 0), \quad r_{zz} = (-h''(z), 0, 0), \quad r_{yz} = (0, 0, 0).$$

Therefore

$$E = r_y.r_y = g'(y)^2 + 1, \; F = r_y.r_z = -h'(z)g'(y), \; G = r_z.r_z = h'(z)^2 - 1.$$

So

$$EG - F^2 = (1 + g'(y)^2)(h'(z)^2 - 1) - h'(z)^2 g'(y)^2 = h'(z)^2 - g'(y)^2 - 1.$$

Now

$$r_y \times r_z = (1, -g'(y), h'(z)).$$

So

$$\mathbf{N} = \frac{(1, -g'(y), h'(z))}{\sqrt{h'(y)^2 - g'(z)^2 - 1}}.$$

Therefore

$$L = r_{yy} \cdot \mathbf{N} = \frac{g''(y)}{(h'(z)^2 - g'(y)^2 - 1)^{\frac{1}{2}}}, \quad M = r_{yz} \cdot \mathbf{N} = 0, \quad N = r_{zz} \cdot \mathbf{N} = \frac{-h''(z)}{(h'(z)^2 - g'(y)^2 - 1)^{\frac{1}{2}}}.$$

On using these values of L, M, N in (1.0.2), we get

$$K = \frac{-g''(y)h''(z)}{(h'(z)^2 - g'(y)^2 - 1)^2}. \tag{3.0.5}$$

Now if K is constant, then from (3.0.4) and (3.0.5), we have $g'' = 0$ or $h'' = 0$, which is again not possible like as in theorem (2.0.3). So we again get a contradiction. Hence the surface is a cylinder. □

***Theorem* 3.0.2.** *Let S be a translation surface with constant mean curvature $H \neq 0$ in 3-dimensional Minkowski space E_1^3. Then*

(i) if S is spacelike, it is congruent to the following surfaces or a part in E_1^3:

(a)

$$z = \frac{\sqrt{1-a^2}}{2H}\sqrt{1+4H^2x^2} - ay, \quad |a| < 1,$$

or

(b)

$$x = ay - \frac{\sqrt{a^2+1}}{2H}\sqrt{4H^2z^2 - 1},$$

or

(c)

$$x = \frac{\sqrt{a^2-1}}{2H}\sqrt{1+4H^2y^2} - az, \quad |a| > 1;$$

(ii) if S is timelike, it is congruent to the following surfaces or a part in E_1^3:

(d)

$$z = -\frac{\sqrt{1-a^2}}{2H}\sqrt{4H^2x^2 - 1} - ay, \quad |a| < 1,$$

or

(e)

$$z = \frac{\sqrt{a^2-1}}{2H}\sqrt{1-4H^2x^2} - ay, \quad |a| > 1,$$

or

(f)

$$x = ay + \frac{\sqrt{1+a^2}}{2H}\sqrt{1+4H^2z^2},$$

or

(g)

$$x = -\frac{\sqrt{a^2-1}}{2H}\sqrt{4H^2y^2-1} - az, \quad |a| > 1,$$

or

(h)

$$x = \frac{\sqrt{1-a^2}}{2H}\sqrt{1-4H^2y^2} - az, \quad |a| < 1.$$

Proof. Let S be a surface with constant mean curvature $H \neq 0$ in Minkowski space E_1^3. Let the translation surface in E_1^3 be

$$r(x,y) = (x, y, g(x) - h(y)), \tag{3.0.6}$$

or

$$r(y,z) = (g(y) - h(z), y, z). \tag{3.0.7}$$

For surface (3.0.6), we have calculated the the value of E, F, G, L, M, N and $EG - F^2$ in (2.0.12), (2.0.14) and (2.0.13).

On substituting these values in (1.0.2), we get

$$H = \frac{-g''(1-h'^2) + h''(1-g'^2)}{2\sqrt{1-g'^2-h'^2}(1-g'^2-h'^2)},$$

where $g = g(x)$ and $h = h(y)$;

or

$$H = \frac{g''(1-h'^2) - h''(1-g'^2)}{2\left[1-g'^2-h'^2\right]^{\frac{3}{2}}}. \tag{3.0.8}$$

Assume in (3.0.8), $g''(x) \neq 0$ and $h''(y) \neq 0$, we have

$$2H = \left[g''(1-h'^2) - h''(1-g'^2)\right]\left[1-g'^2-h'^2\right]^{-\frac{3}{2}}.$$

Differentiating w.r.t. x, we get

$$\begin{aligned} 0 &= \left[g'''(1-h'^2)+2g'g''h''\right]\left[1-g'^2-h'^2\right]^{-\frac{3}{2}}+\left[g''(1-h'^2)-h''(1-g'^2)\right] \\ &\qquad \left[-\tfrac{3}{2}(1-g'^2-h'^2)^{-\frac{5}{2}}(-2g'g'')\right] \\ &= \left[g'''(1-h'^2)+2g'g''h''\right]\left[1-g'^2-h'^2\right]^{-\frac{3}{2}} \\ &\quad +3g'g''\left[g''(1-h'^2)-h''(1-g'^2)(1-g'^2-h'^2)\right]^{-\frac{5}{2}}. \end{aligned}$$

Using (3.0.8), we get

$$\left[g'''(1-h'^2)+2g'g''h''\right]\left[1-g'^2-h'^2\right]^{-\frac{3}{2}}+6g'g''H\left[1-g'^2-h'^2\right]^{-1}=0,$$

or

$$\left[g'''(1-h'^2)+2g'g''h''\right]\left[1-g'^2-h'^2\right]^{-\frac{1}{2}}=-6Hg'g''. \tag{3.0.9}$$

Differentiating (3.0.9) w.r.t. y, we get

$$\begin{aligned} 0 = \left[-2g'''h'h''+2g'g''h'''\right]\left[1-g'^2-h'^2\right]^{-\frac{1}{2}}-\frac{1}{2}\left[1-g'^2-h'^2\right]^{-\frac{3}{2}} \\ \left[-2h'h''\right]\left[g'''(1-h'^2)+2g'g''h''\right], \end{aligned}$$

that is

$$\left[-g'''h'h''+g'g''h'''\right]\left[1-g'^2-h'^2\right]^{\frac{1}{2}}=3Hg'g''h'h''.$$

Dividing both sides by $g'g''h'h''$, we get

$$3H=\left[\frac{-g'''}{g'g''}+\frac{h'''}{h'h''}\right][1-g'^2-h'^2]^{\frac{1}{2}}. \tag{3.0.10}$$

Differentiating (3.0.10) w.r.t x, we have

$$\begin{aligned} 0 &= \left[\frac{-g'''}{g'g''}\right]'[1-g'^2-h'^2]^{\frac{1}{2}}-\left[\frac{-g'''}{g'g''}\right]\frac{1}{2}[1-g'^2-h'^2]^{-\frac{1}{2}}2g'g'' \\ &\quad -\frac{1}{2}\frac{h'''}{h'h''}[1-g'^2-h'^2]^{-\frac{1}{2}}2g'g'' \\ &= \left[\frac{-g'''}{g'g''}\right]'[1-g'^2-h'^2]^{\frac{1}{2}}-[1-g'^2-h'^2]^{-\frac{1}{2}}g'g''\left[\frac{-g'''}{g'g''}+\frac{h'''}{h'h''}\right] \\ &= \left[\frac{-g'''}{g'g''}\right]'[1-g'^2-h'^2]^{\frac{3}{2}}-\frac{[1-g'^2-h'^2]^{\frac{1}{2}}}{[1-g'^2-h'^2]}g'g''\left[\frac{-g'''}{g'g''}+\frac{h'''}{h'h''}\right]. \end{aligned}$$

Using (3.0.10), we get

$$\left[\frac{-g'''(x)}{g'(x)g''(x)}\right]'[1-g'(x)^2-h'(y)^2]^{\frac{3}{2}}+3Hg'(x)g''(x)=0. \tag{3.0.11}$$

Equation (3.0.11) yields $H=0$, which is a contradiction to the assumption that $H\neq 0$. Hence either $g''(x)=0$ or $h''(y)=0$.

So, for the surface (3.0.6), by a transformation in E_1^3 we can assume $h''(y)=0$ so $h(y)=ay$. Hence from (3.0.8), we have

$$g''(x)(1-a^2)=2H(1-a^2-g'(x)^2)^{\frac{3}{2}}, \tag{3.0.12}$$

or

$$g''(x)(1-a^2)=2H(g'(x)^2-1-a^2). \tag{3.0.13}$$

Solving (3.0.12) and (3.0.13), we obtain

$$g(x)=\frac{\sqrt{1-a^2}}{2H}\sqrt{1+4H^2(x+c_1)^2}+c_2, \qquad c_1,c_2\in\mathbb{R}, \qquad |a|<1,$$

the surface is spacelike and congruent to the surface (a) given by the theorem.

$$g(x)=-\frac{\sqrt{1-a^2}}{2H}\sqrt{4H^2(x+c_1)^2-1}+c_2, \qquad c_1,c_2\in\mathbb{R}, \qquad |a|<1,$$

the surface is timelike and is congruent to the surface (d) given by the theorem;

and

$$g(x)=\frac{a^2-1}{2H}\sqrt{1-4H^2(x+c_1)^2}+c_2, \qquad c_1,c_2\in\mathbb{R}, \qquad |a|>1,$$

the surface is timelike and congruent to the surface (e) given by the theorem.

Now for surface (3.0.7), let $g=g(y)$ and $h=h(z)$.

The values of E,F,G,L,M,N and $EG-F^2$ have been calculated in (2.0.12),(2.0.14) and (2.0.13).

Hence

$$H=\frac{g''(h'^2-1)-0+(1+g'^2)(-h'')}{2(h'^2-g'^2-1)(h'^2-g'^2-1)^{\frac{1}{2}}},$$

or

$$H=\frac{g''(h'^2-1)-h''(g'^2+1)}{2|h'^2-g'^2-1|^{\frac{3}{2}}}, \tag{3.0.14}$$

or

$$2H=[g''(h'^2-1)-h''(g'^2+1)][h'^2-g'^2-1]^{-\frac{3}{2}}.$$

Differentiating w.r.t. y and assume $g''(y)\neq 0$ and $h''(z)\neq 0$, we get

$$\begin{aligned}0&=\left[g'''(h'^2-1)-2g'g''h''\right]\left[h'^2-g'^2-1\right]^{-\frac{3}{2}}+\left[g''(h'^2-1)-h''(g'^2+1)\right]\\&\quad\left[-\frac{3}{2}(h'^2-g'^2-1)^{-\frac{5}{2}}(-2g'g'')\right]\\&=\left[g'''(h'^2-1)-2g'g''h''\right][h'^2-g'^2-1]^{-\frac{3}{2}}+6g'g''\left[\frac{g''(h'^2-1)-h''(g'^2+1)}{2(h'^2-g'^2-1)^{\frac{3}{2}}}\right]\\&\quad\left[h'^2-g'^2-1\right]^{-1}.\end{aligned}$$

Using (3.0.14), we get

$$0=\left[g'''(h'^2-1)-2g'g''h''\right]\left[h'^2-g'^2-1\right]^{-\frac{3}{2}}+6Hg'g''\left[h'^2-g'^2-1\right]^{-1},$$

or

$$\left[g'''(h'^2-1)-2g'g''h''\right]\left[h'^2-g'^2-1\right]^{-\frac{1}{2}}=-6Hg'g''. \tag{3.0.15}$$

Differentiating (3.0.15) w.r.t z, we get

$$\begin{aligned}0&=\left[2g'''h'h''-2g'g''h'''\right]\left[h'^2-g'^2-1\right]^{-\frac{1}{2}}-\frac{1}{2}\left[h'^2-g'^2-1\right]^{-\frac{3}{2}}\left[2h'h''\right]\\&\quad\left[g'''(h'^2-1)-2g'g''h''\right].\end{aligned}$$

Using (3.0.15), we get

$$0=\left[2g'''h'h''-2g'g''h'''\right]\left[h'^2-g'^2-1\right]^{-\frac{1}{2}}+6Hg'g''h'h''\left[h'^2-g'^2-1\right]^{-1},$$

or

$$\left[g'''h'h''-g'g''h'''\right]\left[h'^2-g'^2-1\right]^{\frac{1}{2}}=-3Hg'g''h'h''.$$

Dividing both sides by $g'g''h'h''$, we get

$$-3H = \left[\frac{g'''}{g'g''} - \frac{h'''}{h'h''}\right][h'^2 - g'^2 - 1]^{\frac{1}{2}}. \tag{3.0.16}$$

Differentiating (3.0.16) w.r.t. y, we get

$$\begin{aligned}
0 &= \left[\frac{g'''}{g'g''}\right]'[h'^2 - g'^2 - 1]^{\frac{1}{2}} + \left[\frac{g'''}{g'g''}\right]\frac{1}{2}[h'^2 - g'^2 - 1]^{-\frac{1}{2}}[-2g'g''] \\
&\quad - \frac{1}{2}\frac{h'''}{h'h''}[h'^2 - g'^2 - 1]^{-\frac{1}{2}}[-2g'g''] \\
&= \left[\frac{g'''}{g'g''}\right]'[h'^2 - g'^2 - 1]^{\frac{1}{2}} - [h'^2 - g'^2 - 1]^{-\frac{1}{2}}g'g''\left[\frac{g'''}{g'g''} - \frac{h'''}{h'h''}\right] \\
&= \left[\frac{g'''}{g'g''}\right]'[h'^2 - g'^2 - 1]^{\frac{1}{2}} - \frac{[h'^2 - g'^2 - 1]^{\frac{1}{2}}}{[h'^2 - g'^2 - 1]}g'g''\left[\frac{g'''}{g'g''} - \frac{h'''}{h'h''}\right].
\end{aligned}$$

Using (3.0.16), we get

$$\left[\frac{g'''(y)}{g'(y)g''(y)}\right][h'(z)^2 - g'(y)^2 - 1]^{\frac{1}{2}} + 3Hg'(y)g''(y) = 0. \tag{3.0.17}$$

Equation (3.0.17) yields $H = 0$, which is a contradiction to the assumption that $H \neq 0$. So $g''(y) = 0$ or $h''(z) = 0$ and hence for surface (3.0.7), we let $g''(y) = 0$.

Therefore, by a transformation in E_1^3, we assume $g(y) = ay$. Then from (3.0.14), we have

$$-h''(z)(1 + a^2) = 2H(h'(z)^2 - a^2 - 1)^{\frac{3}{2}}, \tag{3.0.18}$$

or

$$-h''(z)(1 + a^2) = 2H(a^2 + 1 - h'(z)^2)^{\frac{3}{2}}. \tag{3.0.19}$$

Solving (3.0.18) and (3.0.19), we obtain

$$h(z) = \frac{\sqrt{1 + a^2}}{2H}\sqrt{4H^2(z + c_1)^2 - 1} + c_2, \quad c_1, c_2 \in \mathbb{R},$$

the surface is spacelike and congruent to the surface (b) of the theorem;

$$h(z) = -\frac{\sqrt{1 + a^2}}{2H}\sqrt{4H^2(z + c_1)^2 + 1} + c_2, \quad c_1, c_2 \in \mathbb{R},$$

the surface is timelike and congruent to the surface (f) given by the theorem.

Now assume for surface (3.0.7), $h''(z)=0$. So let $h(z)=az$.

Then by (3.0.14), we have

$$g''(y)(a^2-1)=2H(a^2-1-g'(y)^2)^{\frac{3}{2}}, \tag{3.0.20}$$

or

$$g''(y)(a^2-1)=2H(g'(y)^2+1-a^2)^{\frac{3}{2}}. \tag{3.0.21}$$

Solving (3.0.20) and (3.0.21), we obtain

$$g(y)=\frac{\sqrt{a^2-1}}{2H}\sqrt{1+4H^2(y+c_1)^2}+c_2, \quad c_1,c_2\in\mathbb{R}, \quad |a|>1,$$

the surface is spacelike and congruent to the surface (c) given by the theorem;

$$g(y)=-\frac{\sqrt{a^2-1}}{2H}\sqrt{4H^2(y+c_1)^2-1}+c_2, \quad c_1,c_2\in\mathbb{R}, \quad |a|>1,$$

the surface is timelike and congruent to the surface (g) given by the theorem;

$$g(y)=\frac{\sqrt{1-a^2}}{2H}\sqrt{1-4H^2(y+c_1)^2}+c_2, \quad c_1,c_2\in\mathbb{R}, \quad |a|<1,$$

the surface is timelike and congruent to the surface (h) given by the theorem. □

***Theorem* 3.0.3.** *Let $r(x,y)=(x,y,z(x,y))$ be a minimal affine translation surface in Mikowski space E_1^3. Then either $z(x,y)$ is linear or can be written as*

$$z(x,y)=\frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{1+a^2}x\right]}{\cosh[c(y+ax)]}\right| \tag{3.0.22}$$

Proof. Let $r(x,y)=(x,y,f(x)+g(y+ax)), a\neq 0$ be the minimal affine translation surface in E_1^3.

Therefore

$$r_x = (1, 0, f'(x) + ag'(y+ax)),\ r_y = (0, 1, g'(y+ax)),$$

$$r_{xx} = (0, 0, f''(x) + a^2 g''(y+ax)),\ r_{yy} = (0, 0, g''(y+ax)),$$

$$r_{xy} = (0, 0, ag''(y+ax).$$

Now

$$E = 1 - (f' + ag')^2,\ F = -(f' + ag')g',\ G = 1 - g'^2,$$

where $f = f(x)$ and $g = g(y+ax)$.

Hence

$$\begin{aligned} EG - F^2 &= \left[1 - (f' + ag')^2\right]\left[1 - g'^2\right] + \left[(f' + ag')^2 g'^2\right] \\ &= (1 - g'^2) - (f' + ag')^2(1 + g'^2 - g'^2) \\ &= 1 - (f' + ag')^2 - g'^2. \end{aligned}$$

Now

$$r_x \times r_y = (-(f' + ag'), -g', 1).$$

So

$$\mathbf{N} = \frac{(-(f' + ag'), -g', 1)}{\sqrt{1 - (f' + ag')^2 - g'^2}}.$$

Hence

$$L = r_{xx} \cdot \mathbf{N} = \frac{-(f'' + a^2 g'')}{\sqrt{1 - (f' + ag')^2 - g'^2}},$$

$$M = r_{xy} \cdot \mathbf{N} = \frac{-ag''}{\sqrt{1 - (f' + ag')^2 - g'^2}},$$

$$N = r_{yy} \cdot \mathbf{N} = \frac{-g''}{\sqrt{1 - (f' + ag')^2 - g'^2}}.$$

Since the surface is minimal, so $H = 0$.

On substituting the values of fundamental coefficients in (1.0.2), we get

$$\begin{aligned}0 &= -(f'' + a^2 g'')(1 - g'^2) - 2ag''g'(f' + ag') - g''(1 - f'^2 - a^2 g'^2 - 2af'g')\\ &= -f''(1 - g'^2) - a^2 g'' + a^2 g'^2 g'' - 2af'g'g'' - 2a^2 g'^2 g'' - g''\\ &\quad + f'^2 g'' + a^2 g'^2 g'' + 2af'g'g''\\ &= -f''(1 - g'^2) - g''(a^2 + 1 - f'^2).\end{aligned}$$

That is

$$\frac{f''}{1 + a^2 - f'^2} + \frac{g''}{1 - g'^2} = 0. \tag{3.0.23}$$

Differentiating (3.0.23) w.r.t. y, we get

$$\frac{d}{d(y + ax)}\left(\frac{g''}{1 - g'^2}\right)\frac{d}{dy}(y + ax) = 0,$$

or

$$\frac{d}{d(y + ax)}\left(\frac{g''}{1 - g'^2}\right) = 0. \tag{3.0.24}$$

Now differentiating (3.0.23) w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{f''}{1 + a^2 - f'^2}\right) + \frac{d}{d(y + ax)}\left(\frac{g''}{1 - g'^2}\right)\frac{d}{dx}(y + ax) = 0,$$

or

$$\frac{d}{dx}\left(\frac{f''}{1 + a^2 - f'^2}\right) + a\frac{d}{d(y + ax)}\left(\frac{g''}{1 - g'^2}\right) = 0. \tag{3.0.25}$$

Using (3.0.24) in (3.0.25), we get

$$\frac{d}{dx}\left(\frac{f''}{1 + a^2 - f'^2}\right) = 0.$$

Therefore, we have

$$\frac{f''}{1 + a^2 - f'^2} = -\frac{g''}{1 - g'^2} = -c,$$

where c is constant.

If $c = 0$, then $f'' = g'' \equiv 0$ i.e. $r(x, y)$ is a plane.

Now if $c \neq 0$, then we have the following system of differential equations

$$\frac{f''}{1+a^2-f'^2} = -c \tag{3.0.26}$$

and

$$\frac{g''}{1-g'^2} = c. \tag{3.0.27}$$

On solving (3.0.26) (3.0.27) respectively, we get

$$f(x) = \frac{1}{c}\log\left|2\cosh\left[c\sqrt{1+a^2}x\right]\right| \tag{3.0.28}$$

$$g(y+ax) = -\frac{1}{c}\log|2\cosh[c(y+ax)]|. \tag{3.0.29}$$

Adding (3.0.28) and (3.0.29), we have

$$f(x)+g(y+ax) = \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{1+a^2}x\right]}{\cosh[c(y+ax)]}\right|.$$

Hence the theorem is proved. □

Chapter 4

Translation surfaces in Hyperbolic 3-space $\mathbb{H}^3$

Introduction

Till now we have discussed the translation surfaces with zero and non-zero constant mean curvature in Euclidean 3-space $\mathbb{E}^3$ and in Minkowski's 3-space $\mathbb{E}^3_1$. Now in this chapter we consider minimal translation surfaces in three-dimensional hyperbolic space $\mathbb{H}^3$. This chapter is a survey of the article in [7].

***Theorem* 4.0.1.** *There are no minimal surfaces in $\mathbb{H}^3$ that are translation surfaces of type I as defined in Eq. 1.0.4.*

Proof. To prove this theorem, we assume that S is a translation surface of type I given by the parametrization (1.0.4). From (2.0.2), we know that

$$H_e = \frac{1}{2}\frac{(1+g'^2)f''+(1+f'^2)g''}{(1+f'^2+g'^2)^{\frac{3}{2}}}.$$

Let us now calculate $\mathbf{N}_3$,

let the surface be

$$r(x,y) = (x, y, f(x) + g(y)),$$

we have

$$r_x = (1,0,f'), \;\; r_{xx} = (0,0,f''), \;\; r_y = (0,1,g'), \;\; r_{yy} = (0,0,g''), \quad r_{xy} = (0,0,0).$$

Therefore

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f' \\ 0 & 1 & g' \end{vmatrix} = i(-f') - j(g') + k(1),$$

that is

$$r_x \times r_y = (-f', -g', 1).$$

Implies

$$\|r_x \times r_y\| = \sqrt{1 + f'^2 + g'^2}.$$

Now

$$\mathbf{N} = \frac{r_x \times r_y}{\|r_x \times r_y\|} = \frac{(-f', -g', 1)}{\sqrt{1 + f'^2 + g'^2}}.$$

Hence

$$\mathbf{N}_3 = \frac{1}{\sqrt{1 + f'^2 + g'^2}}.$$

Now from equation (1.0.3), we have

$$H = (f+g)\left[\frac{(1+g'^2)f'' + (1+f'^2)g''}{2(1+f'^2+g'^2)^{\frac{3}{2}}}\right] + \frac{1}{\sqrt{1+f'^2+g'^2}}.$$

If the surface is minimal that is $H = 0$ on S, we have

$$0 = (f+g)\left[\frac{(1+g'^2)f'' + (1+f'^2)g''}{(1+f'^2+g'^2)^{\frac{3}{2}}}\right] + \frac{2}{\sqrt{1+f'^2+g'^2}},$$

that is

$$\frac{1}{\sqrt{1+f'^2+g'^2}}\left[(f+g)\frac{(1+g'^2)f'' + (1+f'^2)g''}{1+f'^2+g'^2} + 2\right] = 0,$$

or

$$(f+g)\left[(1+g'^2)f''+(1+f'^2)g''\right]+2\left(1+f'^2+g'^2\right)=0.$$

Dividing on both sides by $(1+g'^2)(1+f'^2)$, we get

$$(f+g)\left[\frac{f''}{1+f'^2}+\frac{g''}{1+g'^2}\right]+\frac{2(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)}=0,$$

that is

$$(f+g)\left[\frac{f''}{1+f'^2}+\frac{g''}{1+g'^2}\right]=\frac{-2(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)}. \tag{4.0.1}$$

Differentiating equation (4.0.1) w.r.t. x, we get

$$\begin{aligned}
(f+g)\left(\frac{f''}{1+f'^2}\right)'+f'\left[\frac{f''}{1+f'^2}+\frac{g''}{1+g'^2}\right]
&=-2\left[\frac{(1+f'^2)(1+g'^2)2f'f''-(1+f'^2+g'^2)(1+g'^2)2f'f''}{(1+f'^2)^2(1+g'^2)^2}\right]\\
&=-2\left[\frac{2f'f''(1+f'^2)(1+g'^2)-2f'f''(1+f'^2+g'^2)(1+g'^2)}{(1+f'^2)^2(1+g'^2)^2}\right]\\
&=-4f'f''(1+g'^2)\left[\frac{1+f'^2-1-f'^2-g'^2}{(1+f'^2)^2(1+g'^2)^2}\right]\\
&=\frac{4f'f''g'^2}{(1+f'^2)^2(1+g'^2)},
\end{aligned}$$

that is

$$(f+g)\left(\frac{f''}{1+f'^2}\right)'+f'\left(\frac{f''}{1+f'^2}+\frac{g''}{1+g'^2}\right)=\frac{4f'f''}{(1+f'^2)^2}\cdot\frac{g'^2}{1+g'^2}. \tag{4.0.2}$$

Differentiating (4.0.2) w.r.t. y, we get

$$\begin{aligned}
g'\left(\frac{f''}{1+f'^2}\right)'+f'\left(\frac{g''}{1+g'^2}\right)'&=\frac{4f'f''}{(1+f'^2)^2}\left[\frac{(1+g'^2)2g'g''-2g'^2g'g''}{(1+g'^2)^2}\right]\\
&=\frac{8f'f''g'g''}{(1+f'^2)^2(1+g'^2)^2},
\end{aligned}$$

or

$$f'\left(\frac{g''}{1+g'^2}\right)'+g'\left(\frac{f''}{1+f'^2}\right)'=\frac{8f'f''g'g''}{(1+f'^2)^2(1+g'^2)^2}.$$

Dividing on both sides by $f'g'$, we get

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{1}{f'}\left(\frac{f''}{1+f'^2}\right)' = \frac{8f''g''}{(1+f'^2)^2(1+g'^2)^2}. \tag{4.0.3}$$

Now differentiating equation (4.0.3) w.r.t. x, we have

$$\begin{aligned}\frac{1}{f'}\left(\frac{f''}{1+f'^2}\right)'' + \left(\frac{1}{f'}\right)'\left(\frac{f''}{1+f'^2}\right)' &= \frac{8g''}{(1+g'^2)^2}\left[\frac{(1+f'^2)^2f''' - 2f''(1+f'^2)2f'f''}{(1+f'^2)^4}\right]\\ &= \frac{8g''}{(1+g'^2)^2}\left[\frac{(1+f'^2)^2f''' - 4f'f''^2(1+f'^2)}{(1+f'^2)^4}\right]\\ &= \frac{8g''(1+f'^2)}{(1+g'^2)^2}\left[\frac{(1+f'^2)f''' - 4f'f''^2}{(1+f'^2)^4}\right],\end{aligned}$$

that is

$$\frac{1}{f'}\left(\frac{f''}{1+f'^2}\right)'' + \left(\frac{1}{f'}\right)'\left(\frac{f''}{1+f'^2}\right)' = \frac{8g''}{(1+g'^2)^2}\left[\frac{(1+f'^2)f''' - 4f'f''^2}{(1+f'^2)^3}\right]. \tag{4.0.4}$$

Differentiating equation (4.0.4) w.r.t. y, we get

$$\begin{aligned}0 &= 8\left[\frac{(1+g'^2)^2g''' - 2g''(1+g'^2)2g'g''}{(1+g'^2)^4}\right]\left[\frac{(1+f'^2)f''' - 4f'f''^2}{(1+f'^2)^3}\right]\\ &= 8(1+g'^2)\left[\frac{(1+g'^2)g''' - 4g'g''^2}{(1+g'^2)^4}\right]\left[\frac{(1+f'^2)f''' - 4f'f''^2}{(1+f'^2)^3}\right]\\ &= 8\left[\frac{(1+g'^2)g''' - 4g'g''^2}{(1+g'^2)^3}\right]\left[\frac{(1+f'^2)f''' - 4f'f''^2}{(1+f'^2)^3}\right]\\ &= \left[(1+g'^2)g''' - 4g'g''^2\right]\left[(1+f'^2)f''' - 4f'f''^2\right].\end{aligned}$$

This implies either

$$(1+g'^2)g''' - 4g'g''^2 = 0, \tag{4.0.5}$$

or

$$(1+f'^2)f''' - 4f'f''^2 = 0. \tag{4.0.6}$$

We assume that

$$(1+f'^2)f''' - 4f'f''^2 = 0.$$

On integrating, we get

$$f'' = a(1+f'^2)^2, \tag{4.0.7}$$

for some constant a.

Now substituting equation (4.0.7) in equation (4.0.3), we get

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{1}{f'}\left(\frac{a(1+f'^2)^2}{(1+f'^2)}\right)' = \frac{8a(1+f'^2)^2 g''}{(1+f'^2)(1+g'^2)^2},$$

that implies

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{1}{f'}\left(a(1+f'^2)\right)' = \frac{8ag''}{(1+g'^2)^2},$$

or

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{a}{f'}(2f'f'') = \frac{8ag''}{(1+g'^2)^2},$$

that is

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + 2af'' = \frac{8ag''}{(1+g'^2)^2}. \tag{4.0.8}$$

Let us discuss several cases:

Case I: Let $a = 0$, then from equation (4.0.7), we have

$$f''(x) = 0,$$

this implies

$$f(x) = mx + n,$$

where $m, n \in \mathbb{R}$;

and from equation (4.0.5), we get

$$g'' = b(1+g'^2)$$

for some constant b.

Therefore, from equation (4.0.1)

$$(f+g)\left[\frac{f''}{1+f'^2} + \frac{g''}{1+g'^2}\right] = -2\frac{(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)},$$

or

$$(mx+n+g)\left[0+\frac{b(1+g'^2)}{(1+g'^2)}\right]=\frac{-2(1+m^2+g'^2)}{(1+m^2)(1+g'^2)},$$

or

$$(mx+n+g)\,b=\frac{-2(1+m^2+g'^2)}{(1+m^2)(1+g'^2)}. \tag{4.0.9}$$

<u>Subcase 1</u>: If $b\neq 0$, then $m=0$ and from equation (4.0.9), we have

$$(n+g)b=\frac{-2(1+g'^2)}{(1+g'^2)},$$

that is

$$(n+g)b=-2.$$

This implies that g is a contant function and so $g''=0$ and $b=0$, a contradiction.

<u>Subcase 2</u>: If $b=0$, then $g(y)=py+q; p,q\in\mathbb{R}$.

Now equation (4.0.1) can be written as

$$(mx+n+py+q)\left[\frac{0\cdot(1+g'^2)}{(1+g'^2)}\right]=\frac{-2(1+m^2+p^2)}{(1+m^2)(1+p^2)},$$

or

$$0=\frac{-2(1+m^2+p^2)}{(1+m^2)(1+p^2)},$$

which is again a contradiction.

<u>Case II</u>: Now suppose $a\neq 0$, from equation (4.0.8) and since x and y are independent variables, there exists a constant b such that

$$2af''=-b. \tag{4.0.10}$$

Combining (4.0.8) and (4.0.10), we have

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)'-\frac{8ag''}{(1+g'^2)^2}=-2af''=b,$$

that is

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)'-\frac{8ag''}{(1+g'^2)^2}=b.$$

In particular from equation (4.0.10), we get

$$f'' = \frac{-b}{2a}.$$

On integrating, we get

$$f(x) = \frac{-b}{4a}x^2 + mx + n, \qquad m, n \in \mathbb{R}.$$

From this expression of the function f together with the differential equation $f'' = a(1 + f'^2)^2$, we obtain a 4-degree polynomial on x whose coefficients on x must vanish. This yields $b = m = 0$.

Equation (4.0.3) implies that

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{1}{f'}\left(\frac{f''}{1+f'^2}\right)' = \frac{8f''g''}{(1+f'^2)(1+g'^2)^2},$$

that implies

$$\begin{aligned} 0 &= \frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{1}{f'}\left(\frac{a(1+f'^2)^2}{1+f'^2}\right)' \\ &= \frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{a(1+f'^2)'}{f'} \\ &= \frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' + \frac{a}{f'}\left(1 + \frac{b^2x^2}{4a^2}\right)', \end{aligned}$$

that is

$$\frac{1}{g'}\left(\frac{g''}{1+g'^2}\right)' = 0,$$

or

$$\left(\frac{g''}{1+g'^2}\right)' = 0.$$

Then $g'' = p(1+g'^2)$ for some constant $p \in \mathbb{R}$.

From equation (4.0.1), we have

$$(f+g)\left(\frac{f''}{1+f'^2} + \frac{g''}{1+g'^2}\right) = -2\frac{(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)},$$

or

$$\left(\frac{-b}{4a}x^2+mx+n+g(y)\right)\left(\frac{-b}{2a(1+f'^2)}+p\right)=-2\frac{\left(1+\frac{b^2}{4a^2}x^2+g'^2\right)}{\left(1+\frac{b^2}{4a^2}x^2\right)(1+g'^2)}.$$

Here $b=m=0$, we get

$$(n+g(y))p=-2,$$

which concludes that g is a constant function and $p\neq 0$, a contradiction with the fact that $g''=p(1+g'^2)$. □

***Theorem* 4.0.2.** *The only minimal surfaces in $\mathbb{H}^3$ that are surfaces of type II as defined in Eq. 1.0.5 are totally geodesic planes.*

Proof. Let S be a translation surface of type II, that is S is given by the parametrization $r(x,z)=(x,f(x)+g(z),z)$.
Now we compute H_e and $\mathbf{N}_3$ as

$$r_x=(1,f',0),\;\; r_z=(0,g',1).$$

Therefore

$$r_x\times r_z=\begin{vmatrix} i & j & k\\ 1 & f' & 0\\ 0 & g' & 1\end{vmatrix}=i(f')-j(1)+k(g')$$

$$r_x\times r_z=(f',-1,g').$$

So

$$\|r_x\times r_z\|=\sqrt{1+f'^2+g'^2}.$$

Therefore

$$\mathbf{N}=\frac{f',-1,g'}{\sqrt{1+f'^2+g'^2}}.$$

Hence

$$\mathbf{N}_3=\frac{g'}{\sqrt{1+f'^2+g'^2}}.$$

Also in this case, we have

$$H_e = \frac{(1+g'^2)f'' + (1+f'^2)g''}{2(1+f'^2+g'^2)^{\frac{3}{2}}}.$$

Therefore by (1.0.3), we have

$$H = -z\left[\frac{(1+g'^2)f'' + (1+f'^2)g''}{2(1+f'^2+g'^2)^{\frac{3}{2}}}\right] + \frac{g'}{\sqrt{1+f'^2+g'^2}}.$$

If S is minimal that is $H = 0$, then

$$0 = \sqrt{1+f'^2+g'^2}\left[-z\left((1+g'^2)f'' + (1+f'^2)g''\right) + 2g'(1+f'^2+g'^2)\right].$$

Dividing on both sides by $(1+f'^2)(1+g'^2)$, we get

$$0 = -z\left[\frac{f''}{1+f'^2} + \frac{g''}{1+g'^2}\right] + \frac{2g'(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)},$$

or

$$z\left[\frac{f''}{1+f'^2} + \frac{g''}{1+g'^2}\right] = \frac{2g'(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)}. \tag{4.0.11}$$

Differentiating equation (4.0.11) w.r.t. x, we get

$$\begin{aligned}
z\left(\frac{f''}{1+f'^2}\right)' &= \frac{2g'}{1+g'^2}\left[\frac{(1+f'^2)2f'f'' - (1+f'^2+g'^2)2f'f''}{(1+f'^2)^2}\right] \\
&= \frac{4g'f'f''}{(1+g'^2)(1+f'^2)^2}(1+f'^2-1-f'^2-g'^2) \\
&= \frac{4g'f'f''}{(1+g'^2)(1+f'^2)^2}(-g'^2) \\
&= \frac{-4f'f''g'^3}{(1+g'^2)(1+f'^2)^2}.
\end{aligned}$$

Hence we deduce the existance of a real number $a \in \mathbb{R}$, such that

$$\left(\frac{f''}{1+f'^2}\right)' = \frac{-4af'f''}{(1+f'^2)^2} \quad \text{and} \quad \frac{g'^3}{1+g'^2} = az. \tag{4.0.12}$$

If $a = 0$, then $g(y) = p$ is a constant function and from equation (4.0.11), we have

$$f(x) = mx + n, \qquad m, n \in \mathbb{R}.$$

Therefore, the surface can be reparametrized as

$$r(x,z) = (x, mx+n+p, z).$$

This surface is vertical Euclidean plane and the surface is a totally geodesic plane, which is the statement of given theorem.

Now we assume that $a \neq 0$ in equation (4.0.12) and we will arrive to a contradiction. In particular $g' \neq 0$ and from equation (4.0.12), we have

$$g'^3 = az(1+g'^2),$$

or

$$g'^3 - azg'^2 - az = 0. \tag{4.0.13}$$

Again from equation (4.0.12), we have

$$\left(\frac{f''}{1+f'^2}\right)' = \frac{-4af'f''}{(1+f'^2)^2}.$$

Integrating it, we get

$$\int \left(\frac{f''}{1+f'^2}\right)' dx = -2a \int \frac{2f'f''}{(1+f'^2)^2} dx.$$

Put $1+f'^2 = t$, we have

$$2f'f''dx = dt.$$

Therefore

$$\begin{aligned}
\frac{f''}{1+f'^2} &= -2a \int \frac{dt}{t^2} \\
&= -2a \int t^{-2} dt \\
&= 2a\frac{1}{t} + b, \quad b \in \mathbb{R} \\
&= \frac{2a}{1+f'^2} + b, \quad b \in \mathbb{R}.
\end{aligned}$$

Again from equation (4.0.12), we have

$$\frac{g'^3}{1+g'^2} = az. \tag{4.0.14}$$

Differentiate w.r.t. z, we get

$$\begin{aligned}
a &= \frac{(1+g'^2)3g'^2g'' - 2g'g'^3g''}{(1+g'^2)^2} \\
&= \frac{3g'^2g''(1+g'^2) - 2g'^4g''}{(1+g'^2)^2} \\
&= g''\left[\frac{3g'^2(1+g'^2) - 2g'^4}{(1+g'^2)^2}\right] \\
&= g''\left[\frac{3g'^2}{1+g'^2} - \frac{2g'}{1+g'^2}\cdot\frac{g'^3}{1+g'^2}\right].
\end{aligned}$$

Using (4.0.14), we get

$$\begin{aligned}
a &= g''\left[\frac{3g'^2}{1+g'^2} - \frac{2g'}{1+g'^2}\cdot az\right] \\
&= \frac{g''}{1+g'^2}[3g'^2 - 2azg'] \\
&= \frac{g'g''}{1+g'^2}[3g' - 2az],
\end{aligned}$$

or

$$g'g'' = \frac{a(1+g'^2)}{3g'-2az},$$

that is

$$g'' = \frac{a(1+g'^2)}{g'(3g'-2az)}. \tag{4.0.15}$$

Assume $(3g'-2az) \neq 0$, since $a \neq 0$ using equation (4.0.15) in (4.0.11), we get

$$z\left(\frac{f''}{1+f'^2} + \frac{g''}{1+g'^2}\right) = 2g'\frac{(1+f'^2+g'^2}{(1+f'^2)(1+g'^2)},$$

or

$$z\left(\frac{2a}{1+f'^2} + b + \frac{a(1+g'^2)}{(1+g'^2)g'(3g'-2az)}\right) = 2g'\frac{(1+f'^2+g'^2}{(1+f'^2)(1+g'^2)},$$

or

$$z\left(b+2a\frac{1}{1+f'^2}+a\frac{1}{g'(3g'-2az)}\right)=\frac{2g'(1+f'^2+g'^2)}{(1+f'^2)(1+g'^2)}.$$

That implies

$$\begin{aligned}\left[b+\frac{2a}{1+f'^2}+\frac{a}{g'(3g'-2az)}\right]&=\frac{2g'}{z(1+g'^2)}\frac{g'^2}{g'^2}\frac{(1+f'^2+g'^2)}{(1+f'^2)}\\&=\frac{g'^3}{z(1+g'^2)}\frac{2(1+f'^2+g'^2)}{g'(1+f'^2)}\\&=\frac{2a(1+f'^2+g'^2)}{g'^2(1+f'^2)},\end{aligned}$$

or

$$b+\frac{a}{g'(3g'-2az)}=\frac{2a(1+f'^2)}{g'^2(1+f'^2)}+\frac{2ag'^2}{g'^2(1+f'^2)}-\frac{2a}{(1+f'^2)},$$

or

$$b+\frac{a}{g'(3g'-2az)}=\frac{2a}{g'^2}+\frac{2a}{(1+f'^2)}-\frac{2a}{(1+f'^2)},$$

or

$$b+\frac{a}{g'(3g'-2az)}=\frac{2a}{g'^2},$$

or

$$\frac{bg'(3g'-2az)+a}{g'(3g'-2az)}=\frac{2a}{g'^2},$$

that is

$$bg'^2(3g'-2az)+ag'=2a(3g'-2az),$$

or

$$3bg'^3-2abzg'^2+ag'=6ag'-4a^2z.$$

That implies

$$3bg'^3-2abzg'^2-5ag'+4a^2z=0. \tag{4.0.16}$$

If $b=0$, use this in equation (4.0.16), we have

$$-5ag'=-4a^2z,$$

that is

$$g' = \frac{4}{5}az.$$

By using this value of g' in equation (4.0.13), we have

$$\begin{aligned} 0 &= \left(\frac{4}{5}az\right)^3 - az\left(\frac{4}{5}az\right)^2 - az \\ &= \frac{64}{125}a^3z^3 - a^3z^3\frac{16}{25} - az \\ &= -\frac{16}{25}a^3z^3 - az \end{aligned}$$

defined in some interval of $\mathbb{R}$, this leads to a contradiction.

Thus we assume $b \neq 0$ in equation (4.0.16).

Set $x = g'$ from equation (4.0.13) and (4.0.16), we have

$$3bx^3 - 2abzx^2 - 5ax + 4a^2z = 0 \tag{4.0.17}$$

and

$$x^3 - azx^2 - az = 0. \tag{4.0.18}$$

Multiplying equation (4.0.18) by $3b$ and then subtracting from equation (4.0.17), we get

$$3bx^3 - 2abzx^2 - 5ax + 4a^2z - 3bx^3 + 3abzx^2 + 3abz = 0,$$

or

$$abzx^2 - 5ax + 4a^2z + 3abz = 0,$$

that is

$$bzx^2 - 5x + 4az + 3bz = 0. \tag{4.0.19}$$

Similarly if we multiply equation (4.0.18) by $2b$ and then subtracting from equation (4.0.17), we get

$$3bx^3 - 2abzx^2 - 5ax + 4a^2z - 2bx^3 + 2abzx^2 + 2abz = 0,$$

or

$$bx^3 - 5ax + 4a^2z + 2abz = 0. \tag{4.0.20}$$

Now multiplying equation (4.0.19) by x and equation (4.0.20) by z then subtracting, we get

$$bzx^3 - 5x^2 + 4azx + 3bzx - bzx^3 + 5azx - 4a^2z^2 - 2abz^2 = 0.$$

Implies

$$-5x^2 + 9azx + 3bzx - 4a^2z^2 - 2abz^2 = 0,$$

or

$$-5x^2 + 3z(3a+b)x - 2az^2(2a+b) = 0. \tag{4.0.21}$$

Multiplying the above equation by bz, we have

$$-5bzx^2 + bz(9az+3bz)x - 2abz^3(2a+b) = 0. \tag{4.0.22}$$

On multiplying (4.0.19) by 5, we get

$$5bzx^2 - 25x + 20az + 15bz = 0. \tag{4.0.23}$$

Adding equation (4.0.22) and (4.0.23), we get

$$bz(9az+3bz)x - 25x - 2abz^3(2a+b) + 20az + 15bz = 0,$$

that is

$$(9abz^2 + 3b^2z^2 - 25)x - 4a^2bz^3 - 2ab^2z^3 + 20az + 15bz = 0,$$

or

$$(9abz^2 + 3b^2z^2 - 25)x = 4a^2bz^3 + 2ab^2z^3 - 20az - 15bz,$$

or

$$(9abz^2 + 3b^2z^2 - 25)x = z(-20a - 15b + 4a^2bz^2 + 2ab^2z^2).$$

Therefore, we have

$$x = \frac{z(-20a - 15b + 4a^2bz^2 + 2ab^2z^2)}{(9abz^2 + 3b^2z^2 - 25)}.$$

Replacing this expression of x in equation (4.0.19), we obtain a polynomial equation on z as

$$4a^2b^3(2a+b)^2z^7 - b^2(16a^3 - 109a^2b - 108ab^2 - 27b^3)z^5 - 125ab^2z^3 = 0$$

and z is defined in some interval of $\mathbb{R}$. This implies $a = b = 0$, a contradiction.

This completes the proof. □

Bibliography

[1] M. P. Do Carmo, *Differential geometry of curves and surfaces: revised and updated second edition*. Dover Publications, 2016.

[2] E. F. C. Huamani, *Affine Minimal surfaces with singularities*. PhD thesis, Masters dissertation, Rio de Janeiro, 2017.

[3] A. N. Pressley, *Elementary differential geometry*. Springer Science & Business Media, 2010.

[4] H. Liu, "Translation surfaces with constant mean curvature in 3-dimensional spaces," *Journal of Geometry*, vol. 64, pp. 141–149, 1999.

[5] H. Liu and Y. Yu, "Affine translation surfaces in Euclidean 3-space," 2013.

[6] H. Liu and S. D. Jung, "Affine translation surfaces with constant mean curvature in Euclidean 3-space," *Journal of Geometry*, vol. 108, no. 2, pp. 423–428, 2017.

[7] R. López, "Minimal translation surfaces in hyperbolic space," *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, vol. 52, no. 1, pp. 105–112, 2011.

[8] R. López, “Differential geometry of curves and surfaces in Lorentz-Minkowski space,” *International electronic journal of geometry*, vol. 7, no. 1, pp. 44–107, 2014.

[9] H. F. Scherk, “Bemerkungen über die kleinste fläche innerhalb gegebener grenzen.,” 1835.