SOME FIXED POINT THEOREMS AND THEIR APPLICATIONS

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CERTIFICATE

This is to certify that the dissertation entitled, **"SOME FIXED POINT THEOREMS AND THEIR APPLICATIONS"** *being submitted by the students with the enrollments* **21068120001, 21068120008, 21068120009, 21068120012, 21068120033, 21068120049** *to the* **Department of Mathematics, University of Kashmir, Hazratbal, Srinagar,** *for the award of Master's degree in Mathematics,is an original project work carried out by them under my guidance and supervision.*

The project dissertation meets the standard of fulfilling the requirements of regulations related to the award of the Master's degree in Mathematics. The material embodied in the project dissertation has not been submitted before to any other institute, or to this university for the award of Master's degree in Mathematics or any other degree.

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Abstract

A fixed point is a point that remains unchanged under the action of a mapping or function. Fixed point theory aims to understand the existence, uniqueness and properties of fixed points and their applications in various areas of mathematics and other disciplines. There are many fixed point theorems, whose applications provide powerful mathematical tools for proving the existence of solutions, equilibrium points and optimal points in various disciplines including mathematics, economics, game theory and optimization which can be found in [1, 6, 5].

In the proposed work we studied two important fixed point theorems:

1. Brouwer's Fixed Point Theorem states that any continuous function from a closed ball in n dimensional Euclidean space to itself has at least one fixed point. This theorem has significant applications in topology, game theory and economics. For example, it is used to prove the existence of equilibrium points in economic models and to study the behavior of dynamical systems.

2. Banach's Fixed Point Theorem, also known as the contraction mapping theorem states that in a complete metric space, any contraction mapping has a unique fixed point. A contraction mapping is a function that contracts the distance between points in the space. This theorem has applications in various fields such as economics and physics. It is used to prove the existence and uniqueness of solutions to equations and systems of equations.

Contents

Chapter 1

Introduction

A fixed point theory is a beautiful mixture of Mathematical analysis to explain some conditions in which maps give excellent solutions. A fixed point of a function is a point that remains unchanged under that function. In geometric terms, it's a point where the graph of the function intersects the identity line, where the input and output are the same more about it can be found in [1, 2, 5, 7].

1.1 Definitions used in Dissertation

Definition 1.1.1 (Fixed point) *Suppose we have a non empty set X and we have a function* $f : X \to X$ *(need not be continuous) then we say* $x \in X$ *is a fixed point of* f *if* $f(x) = x$.

Problem-1: Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^2$. Determine the fixed point of T.
Solution:
Given that $T(x) = x^2$. From the definition of fixed point we have

$$\begin{aligned} T(x) &= x, \\ \Rightarrow \quad x^2 &= x \\ or \quad x(x-1) &= 0. \end{aligned}$$

therefore $x = 0, 1$.

Thus, the fixed points of T are 0 and 1.

Problem-2: Does a translation mapping $T(x) = x + a$ where a is non-zero fixed number have fixed points.

Solution:

Given that $T(x) = x + a$. From the definition of fixed point we have,

$$T(x) = x$$
$$\Rightarrow x + a = x$$
$$or\ a = 0. \qquad [By\ Left\ Cancellation\ Law]$$

Since $T(x) = x + a$ is a translation mapping, so $a \neq 0$.

Thus, the translation mapping $T(x) = x + a$ has no fixed point.

Problem-3: Show that $f(x) = -x$ for $x \in [-2, -1] \cup [1, 2]$ has no fixed point.

Solution:

Given that $f(x) = -x$. From the definition of fixed point we have,

$$f(x) = x$$
$$\Rightarrow -x = x.$$

It is clear that no point of $[-2, -1] \cup [1, 2]$ will satisfy the above equation.

Thus, $f(x) = -x$ has no fixed point for $x \in [-2, -1] \cup [1, 2]$.

Problem-4: Let T be a mapping on $\mathbb{R}$ into itself defined by $T(x) = \frac{1}{2}x$. Show that T has a unique fixed point.

Solution:

Given $T(x) = \frac{1}{2}x$

$$T(y) = y$$
$$\frac{1}{2}y = y,$$

which holds good at $y = 0$.

Note that

$$\|T(x) - T(y)\| = \left\| \frac{1}{2}x - \frac{1}{2}y \right\| = \frac{1}{2} \|x - y\|.$$

Thus T is a contraction mapping. Hence, by Banach fixed point theorem, T has a unique fixed point.

Problem-5: Give an example to show that T satisfies $\|T(x)-T(y)\| = \|x-y\|$ may not have any fixed point?

Solution:

Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) = \begin{cases} x - \frac{1}{2}e^x & \text{for } x \leq 0 \\ -\frac{1}{2} + \frac{1}{2}x & \text{for } x \geq 0 \end{cases} \tag{1.1.1}$$

Now for $x, y \leq 0$

$$\begin{aligned} \|T(x)-T(y)\| &= \left\| x - \frac{1}{2}e^x - y + \frac{1}{2}e^y \right\| \\ &= \left\| (x-y) - \frac{1}{2}(e^x - e^y) \right\| \\ &\leq \|x-y\|. \end{aligned}$$

For $x, y \geq 0$

$$\begin{aligned} \|T(x)-T(y)\| &= \left\| -\frac{1}{2} + \frac{1}{2}x + \frac{1}{2} - \frac{1}{2}y \right\| \\ &= \left\| \frac{1}{2}(x-y) \right\| \\ &\leq \|x-y\|. \end{aligned}$$

Thus T satisfies, $\|T(x)-T(y)\| \leq \|x-y\|$.

From the definition of fixed point we have $T(x) = x$.

Now for $x \leq 0$

$$\begin{aligned} T(x) &= x \\ \Rightarrow x - \frac{1}{2}e^x &= x \\ or \ -\frac{1}{2}e^x &= 0 \\ \Rightarrow e^x &= 0, \end{aligned}$$

which is not possible for finite x.

For $x \geq 0$

$$\begin{aligned} T(x) &= x \\ \Rightarrow -\frac{1}{2} + \frac{1}{2}x &= x \\ or\ \frac{1}{2}x &= -\frac{1}{2} \\ \Rightarrow x &= -1, \end{aligned}$$

is not acceptable as $x \geq 0$.
Thus, T defined in (1.1.1) is an example which satisfies the given condition (Banach contraction theorem) but have no fixed point.

Definition 1.1.2 (Lipschitz continuous function) *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be Lipschitz continuous if there exists a constant $\alpha \geq 0$ such that*

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$.

(a) If $\alpha = 1$, then T is said to be nonexpansive.

(b) If $\alpha \in (0, 1)$, then T is said to be contraction.

(c) If $d(T(x), T(y)) < \alpha d(x, y)$ for all $x \neq y$, then T is said to be contractive.

The number α is called Lipschitz constant of T.

Definition 1.1.3 Let (X, d) be a metric space and let $\{x_n\}$ be a sequence of points in X,

1. We say that $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $i, j \geq n$ gives
$$d(x_i, x_j) < \varepsilon.$$
2. We say that $\{x_n\}$ converges to a point $x \in X$ if
$$\lim_{n \to \infty} d(x_n, x) = 0.$$

Definition 1.1.4 A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point x in X.

Definition 1.1.5 Let X be a nonempty set. A function $f : X \to \mathbb{R}$ is said to be

(a) bounded above if there exists a real number k such that $f(x) \leq k$ for all $x \in X$;

(b) bounded below if there exists a real number k such that $k \leq f(x)$ for all $x \in X$;

(c) bounded if it is both bounded above as well as bounded below.

Definition 1.1.6 A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be upper semicontinuous from the right if $r_n \downarrow r \geq 0$ implies $\limsup_{n\to\infty} \psi(r_n) \leq \psi(r)$.

Definition 1.1.7 [Lipschitzian map] Let (X, d) be a metric space. A map $F\colon X \longrightarrow X$ is said to be $\mathbb{L}$*ipschitzian* if there exists a constant $\alpha \geq 0$ with

$$d(F(x), F(y)) \leq \alpha d(x, y) \ for\ all\ x, y \in X. \tag{1.1.2}$$

Notice that a Lipschitzian map is necessarily continuous. The smallest α for which (1.1.2) holds is said to be the Lipschitz constant for F and is denoted by L. If $L < 1$ we say that F is contraction, whereas if L=1, we say that F is nonexpansive.

Definition 1.1.8 (Contraction map) *A mapping $T : X \longrightarrow X$, where X is a subset of a normed linear space N, is called a contraction mapping or simply a contraction, if there is a positive number $a < 1$ such that*

$$\|T(x) - T(y)\| \leq a\|x - y\|,$$

for all $x, y \in X$.

Definition 1.1.9 (Metric space) *Let E be a non-empty set. A function d (which is a real valued function defined on $E \times E$) is said to be a metric X, if the following properties are satisfied:*
1. $d(x, y) \geq 0$ *(non-negativity)*
2. $d(x, y) = d(y, x)$ *(symmetry)*

3. $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*
4. $d(x, y) = 0 \;\; iff \;\; x = y$ *(Positiveness) for arbitrary elements* x, y, z *of* E *and* d *is called a metric on* E *and* E *together with* d *i.e;* (E, d) *is called a metric space.*
Note that (E, d) *is called semi metric space if property 4 is not satisfied.*

Definition 1.1.10 (Norm) *A norm defined on a vector space* X *is a non-negative real valued function* $\| \bullet \| : X \to \mathbb{R}$ *whose value at* x *is denoted by* $\|x\|$ *such that for any* $x, y \in X$ *and for any scalar* α *we have the following properties:*

(a). $\|x\| \geq 0 \quad \forall\, x \in X$

(b). $\|x\| = 0$ iff $x = 0$

(c). $\|\alpha x\| = |\alpha| \|x\|$

(d). $\|x + y\| \leq \|x\| + \|y\|$

then $\|.\|$ is said to be a norm on X and the ordered pair $(X, \|\|)$ is called a normed linear space.

Definition 1.1.11 The normed linear space X is said to be complete, if every Cauchy sequence in X converges to some $x \in X$, that is, every Cauchy sequence is convergent in X. In otherwords, a complete normed linear space is called a Banach space or simply a B-space.

Chapter 2

Banach's Fixed Point Theorem

The Banach contraction theorem is one of the most important and useful results in the fixed point theory. It is perhaps one of the most widely used fixed point theorems in all analysis. This is because the contraction condition on the mapping is simple and easy to verify, because it requires only completeness assumption on the underlying metric space. It finds almost canonical applications in the theory of differential and integral equations. Although the basic idea was known earlier, the theorem first appeared in explicit form in the setting of $C[0,1]$ in Banach's 1922 Ph.D. thesis where it was used to establish the existence of a solution of an integral equation [8].

This chapter deals with some important and useful results in metric fixed point theory. We present the Banach contraction theorem and some of its applications. An important generalization of Banach contraction theorem, obtained by Boyd and Wong [4] in 1969, is also given.

2.1 Results

Theorem 2.1.1 [Banach contraction principle for metric space] Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point.

Proof: We construct a sequence $\{x_n\}$ by the following iterative method.
Choose any arbitrary point $x_0 \in X$.

Then $x_0 \neq T(x_0)$, otherwise x_0 is a fixed point of T and there is nothing to prove.
Now, we define

$$x_1 = T(x_0), x_2 = T(x_1), x_3 = T(x_2), \ldots, x_n = T(x_{n-1}) \;\; \forall\, n \in \mathbb{N}.$$

We claim that this sequence $\{x_n\}$ of points of X is a Cauchy sequence.
Since T is a contraction mapping with Lipschitz constant $0 < \alpha < 1$, for all $p = 1, 2, \ldots,$ we have

$$\begin{aligned} d(x_{p+1}, x_p) &= d\left(T(x_p), T(x_{p-1})\right) \\ &\leq \alpha d\left(x_p, x_{p-1}\right) \\ &= \alpha d\left(T(x_{p-1}), T(x_{p-2})\right) \\ &\leq \alpha^2 d(x_{p-1}, x_{p-2}) \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &= \alpha^{p-1} d\left(T(x_1), T(x_0)\right) \\ &\leq \alpha^p d(x_1, x_0). \end{aligned}$$

Let m and n be any positive integers with $m > n$. Then, by the triangle inequality, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) d(x_1, x_0) \\ &\leq \alpha^n(\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + 1) d(x_1, x_0) \\ &\leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0). \end{aligned}$$

Since $\lim\limits_{n\to\infty} \alpha^n = 0$ and $d(x_1, x_0)$ is fixed, the right hand side of the above inequality approaches to 0 as $n \to \infty$.
It follows that $\{x_n\}$ is a Cauchy sequence in X.
Since X is complete, there exists $x \in X$ such that $x_n \to x$.
We show that this limit point x is a fixed point of T.

Since T is a contraction mapping, from the triangle inequality, we have

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_n) + d(x_n, T(x)) \\ &= d(x, x_n) + d\left(T(x_{n-1}), T(x)\right) \\ &\leq d(x, x_n) + \alpha d(x_{n-1}, x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $d(x, T(x)) = 0$ this gives $T(x) = x$.
Now we show that the fixed point of T is unique. Suppose to the contrary that x and y are two distinct fixed points of T.
$T(x) = x$ and $T(y) = y$.
Since T is a contraction mapping, we have

$$d(x, y) = d(T(x), T(y)) \leq \alpha d(x, y) < d(x, y)$$

a contradiction.
Hence $x = y$.

Remark 2.1.2 *If X is not complete in Theorem 2.1.1, then T may not have a fixed point. For example, consider $X = (0, 1)$ and the mapping $T : X \to X$ defined by $T(x) = \frac{x}{2}$. Then X is not a complete metric space with the usual metric and T does not have any fixed point.*
In fact, $T(0) = 0 \notin X$.

Remark 2.1.3 If T is not contraction in Theorem 2.1.1, then it may not have a fixed point. For example, consider the metric space $X = [1, \infty)$ with the usual metric and the mapping $T : X \to X$ given by $T(x) = x + \frac{1}{x}$. Then X is a complete metric space but T is not a contraction mapping. In fact,

$$\begin{aligned} |T(x)-T(y)| &= |(x+\frac{1}{x})-(y+\frac{1}{y})| \\ &= |x+\frac{1}{x}-y-\frac{1}{y}| \\ &= |x-y|\left(1-\frac{1}{xy}\right) \\ &< |x-y| \quad for\ all\ x,y\in X. \end{aligned}$$

So, T is contractive. Of course, T does not have any fixed point.

The following example shows that if X is a complete metric space and $T : X \to X$ is not a contraction mapping but $T^2 = T \circ T$ is contraction, even then T has a fixed point.

Example 2.1.4 Let $X = \mathbb{R}$ be a metric space with the usual metric and $T : X \to X$ be a mapping defined as

$$T(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then T is not continuous and hence not a contraction mapping. Now

$$T^2(x) = T(T(x)) = \begin{cases} T(1) = 1 & \text{if } x \in \mathbb{Q} \\ T(0) = 1 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then T^2 is a contraction mapping but both T^2 and T have the same fixed point that is 1.

The above example motivates us to present the following result.

Theorem 2.1.5 *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for some integer m, $T^m = \underbrace{T \circ T \circ \cdots \circ T}_{m\ times}$ is a contraction mapping. Then T has a unique fixed point.*

Proof: By theorem 2.1.1, T^m has a unique fixed point $x \in X$ that is, $T^m(x) = x$. Then

$$T(x) = T(T^m(x)) = T^m(T(x))$$

and so $T(x)$ is a fixed point of T^m. Since the fixed point of T^m is unique, so $T(x) = x$.
To prove the uniqueness, we assume that y is another fixed point of T.
Then $T(y) = y$ and so $T^m(y) = y$.
Again by the uniqueness of the fixed point of T^m, we have $x = y$.
Hence, $x \in X$ is a unique fixed point of T.

Theorem 2.1.6 (Banach contraction principle for Banach space) *Every contraction mapping T defined on a Banach space X into itself has a unique fixed point $x \in X$.*

Proof:

1). **Existence of a fixed point**:
Let us consider an arbitrary point $x_0 \in X$, and define the iterative sequence $\{x_n\}$ by $x_0,\ x_1 = Tx_0,\ x_2 = Tx_1,\ x_3 = Tx_2, \ldots,\ x_n = Tx_{n-1}$.
Then,

$$\begin{aligned} x_2 &= Tx_1 = T(Tx_0) = T^2x_0, \\ x_3 &= Tx_2 = T(T^2x_0) = T^3x_0, \\ &\vdots \\ x_n &= T^nx_0. \end{aligned}$$

If $m > n$, say $m = n + p$, $p = 1, 2, \cdots$. Then

$$\begin{aligned} \|x_{n+p} - x_n\| &= \|T^{n+p}x_0 - T^nx_0\| \\ &= \|T(T^{n+p-1}x_0 - T^{n-1}x_0)\| \\ &\leq k\|T^{n+p-1}x_0 - T^{n-1}x_0\|, \end{aligned}$$

as T is a contraction mapping, continuing this process $n-1$ times, we have

$$\|x_{n+p} - x_n\| \leq k^n\|T^p(x_0) - x_0\|, \tag{2.1.1}$$

for $n = 0, 1, 2, 3, \ldots$ and for all p.

Now,

$$\begin{aligned}
\|T^p x_0 - x_0\| &= \|T^p x_0 - T^{p-1}x_0 + T^{p-1}x_0 - T^{p-2}x_0 + T^{p-2}x_0 - \ldots + Tx_0 - x_0\|, \\
&\le \|T^P x_0 - T^{p-1}x_0\| + \|T^{p-1}x_0 - T^{p-2}x_0\| + \ldots + \|Tx_0 - x_0\|, \\
&\le \|T^{p-1}x_1 - T^{p-1}x_0\| + \|T^{p-2}x_1 - T^{p-2}x_0\| + \ldots + \|x_1 - x_0\|, \\
&\le k^{p-1}\|x_1 - x_0\| + k^{p-2}\|x_1 - x_0\| + \ldots + \|x_1 - x_0\|, \\
&\le (k^{p-1} + k^{p-2} + \ldots + 1)\|x_1 - x_0\|, \\
&\le \frac{1-k^p}{1-k}\|x_1 - x_0\|. \qquad (2.1.2)
\end{aligned}$$

By the sum of G.P series whose ratio is < 1. Since $0 < k < 1$, so the number $1 - k^p < 1$. Using this result in above inequality 2.1.2, we get

$$\|T^p x_0 - x_0\| \le \frac{1}{1-k}\|x_1 - x_0\|, \qquad (2.1.3)$$

with the help of equation 2.1.1 the result becomes

$$\|x_{n+p} - x_n\| \le \frac{k^n}{1-k}\|x_1 - x_0\|. \qquad (2.1.4)$$

When $n \to \infty$ then $m = n + p \to \infty$, gives

$$\|x_{n+p} - x_n\| \to 0$$

this shows that $\{x_n\}$ is a Cauchy sequence in X. Hence, $\{x_n\}$ must be convergent, say

$$\lim_{n\to\infty} x_n = x.$$

2). **limit x is a fixed point of T:**
Since, T is continuous we have

$$\begin{aligned}
Tx &= T(\lim_{n\to\infty} x_n) \\
&= \lim_{n\to\infty} Tx_n \\
&= \lim_{n\to\infty} x_{n+1} = x,
\end{aligned}$$

since the limit of $\{x_{n+1}\}$ is the same as that of $\{x_n\}$. Thus x is a fixed point of T.

3). **Uniqueness of the fixed point of T:**

Let y be another fixed point of T. Then $Ty = y$, also we have $\|Tx - Ty\| \leq k\|x - y\|$, as T is a contraction mapping.
But $\|Tx - Ty\| \leq \|x - y\|$, because $Tx = x$ and $Ty = y$
therefore $\|x - y\| \leq k\|x - y\|$ that is $k \geq 1$. As $0 < k < 1$, so the above relation is possible only when

$$\|x - y\| = 0$$
$$\Rightarrow x - y = 0$$
$$or\ x = y$$

This proves that fixed point of T is unique.

2.1.1 Applications of Banach's fixed point theorem

Application 1. Let $X = \mathbb{R}$ be the Banach space of real numbers with $\|x\| = |x|$ and $[a, b] \subset R$, $f : [a, b] \to [a, b]$, a differentiable function such that $|f'(x)| \leq k < 1$. Find the solution of the equation $f(x) = x$.
Solution: Let $x, y \in [a, b]$ and $y < z < x$.
Then, by Lagrange's mean value theorem, we have

$$\frac{f(x) - f(y)}{x - y} = f'(z)$$

i.e.,

$$f(x) - f(y) = (x - y)f'(z)$$

or

$$|f(x) - f(y)| = |(x - y)f'(z)|$$

or

$$|f(x) - f(y)| = |x - y||f'(z)|$$

so that,

$$|f(x) - f(y)| \leq k|x - y|.$$

Thus, f is a contraction mapping on $[a, b]$ into itself.

Since $[a, b]$ is a closed subset of $X = R$.
Therefore, by Banach contraction theorem there exists a unique fixed point $x^* \in [a, b]$, such that $f(x^*) = x^*$.

Hence, x^* is the solution of the equation $f(x) = x$.

Application 2. Find the solution of the system of n linear algebraic equations with n unknowns:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \tag{2.1.5}$$

Solution: The system 2.1.5 can be written as

$$\left.\begin{array}{ll} x_1 & = (1 - a_{11})x_1 - a_{12}x_2 - \ldots - a_{1n}x_n + b_1 \\ x_2 & = -a_{21}x_1 + (1 - a_{22})x_2 - \ldots - a_{2n}x_n + b_2 \\ \vdots & \qquad\qquad\vdots \\ x_n & = -a_{n1}x_1 - a_{n2}x_2 - \ldots + (1 - a_{nn})x_n + b_n \end{array}\right\} \tag{2.1.6}$$

Let $a_{ij} = -a_{ij} + \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Then the equation 2.1.6 can be written in the following equivalent form

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + b_i, i = 1, 2, \ldots, n \tag{2.1.7}$$

If $x = (x_1, x_2, \ldots, x_n) \in R^n$, then equation 2.1.7 can be written in the form $Tx = x$, where T is defined by

$$Tx = y, \tag{2.1.8}$$

where $y = (y_1, y_2, \ldots, y_n)$ and $y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i$. Here $T : R^n \to R^n$ and (a_{ij}) is an $(n \times n)$ matrix. Finding solutions of the system 2.1.5 or 2.1.6 is thus equivalent to find the fixed point of operator 2.1.8 in order to find a unique fixed point of T. That is, a unique solution of equation 2.1.8, We apply the Banach contraction principle, Equation 2.1.8 has a unique solution if

$$\sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |-a_{ij} + \delta_{ij}| \leq k < 1, i = 1, 2, \ldots, n$$

For
$x = (x_1, x_2, \dots, x_n) \in R^n$
$x' = (x'_1, x'_2, \dots, x'_n) \in R^n$
$y = (y_1, y_2, \dots, y_n) \in R^n$
$y' = (y'_1, y'_2, \dots, y'_n) \in R^n$
We have

$$||Tx - Tx'|| = ||y - y'||$$

$$y'_i = \sum_{j=1}^{n} a_{ij} x'_j + b_i, \; i = 1, 2, \dots, n$$

Also if $y = (y_1, y_2, \dots, y_n) \in R^n$, then $\|y\| = \sup_{1 \le i \le n} |y_i|$.
Therefore,

$$\begin{aligned}
\|Tx - Tx'\| &= \|y - y'\| \\
&= \sup_{1 \le i \le n} |y_i - y'_i| \\
&= \sup_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j + b_i - \sum_{j=1}^{n} a_{ij} x'_j - b_i \right| \\
&= \sup_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} (x_j - x'_j) \right|
\end{aligned}$$

Using the triangle inequality we get

$$\begin{aligned}
\|Tx - Tx'\| &\le \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}||x_j - x'_j| \\
\Rightarrow \|Tx - Tx'\| &\le \sup_{1 \le i \le n} \sum_{j=1}^{n} |x_j - x'_j| \sup_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \\
&\le k \sup_{1 \le i \le n} |x_j - x'_j| \quad \text{where } \sum_{j=1}^{n} |a_{ij}| \le k < 1
\end{aligned}$$

$$\text{gives} \quad \|Tx - Tx'\| \le k\|x - x'\|, \quad \text{where } \|x - x'\| = \sup_{1 \le i \le n} |x_j - x'_j|.$$

This shows that T is a contraction mapping of the Banach space into itself.

Hence, by Banach contraction principle, there exists a unique fixed point x^* of T in $\mathbb{R}^n$, that is, x^* is a solution of 2.1.5.

Application 3. Show that the Fredholm integral equation

$$x(s) = y(s) + \mu \int_a^b K(s,t)x(t)dt,$$

has a unique solution on $[a, b]$.

Solution: We assume that $K(s,t)$ is continuous in both variables $a \leq s \leq b$ and $a \leq t \leq b$. Let $y \in C[a, b]$. Hence, $|K(s,t)| \leq \lambda$ for all $(s,t) \in [a,b] \times [a,b]$. We first consider the integral equation on $C[a, b]$, the space of all continuous functions defined on the interval $[a, b]$ with the metric

$$d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|.$$

Write the given integral equation in the form $x = Tx$, where

$$Tx(s) = y(s) + \mu \int_a^b K(s,t)x(t)dt. \tag{2.1.9}$$

Since the kernel K and the function y are continuous,

It follows that equation 2.1.9 defines an operator

$$T : C[a, b] \to C[a, b],$$

such that

$$\begin{aligned} d(Tx, Ty) &= \max_{t\in[a,b]} |Tx(t) - Ty(t)| \\ &= \max_{t\in[a,b]} \left| y(t) + \mu \int_a^b K(s,t)x(t)dt - y(t) - \mu \int_a^b K(s,t)y(t)dt \right| \\ &= |\mu| \max_{t\in[a,b]} \left| \int_a^b K(s,t)[x(t) - y(t)]dt \right|. \end{aligned}$$

Using triangle inequality for integrals gives

$$\begin{aligned} d(Tx, Ty) &\leq |\mu| \max_{t\in[a,b]} \int_a^b |K(s,t)|\ |x(t) - y(t)|dt \\ &\leq |\mu|\lambda d(x, y)(b - a), \\ \Rightarrow d(Tx, Ty) &\leq Kd(x, y), \qquad \text{where} \quad K = |\mu|\lambda(b - a). \end{aligned}$$

If $K < 1 \Rightarrow |\mu|\lambda(b-a) \ < 1 \Rightarrow |\mu| < \frac{1}{\lambda(b-a)}$.
Then T becomes contraction.

Under this condition, we conclude that T has a unique solution x on $[a, b]$.

Application 4. Let the function $K(x, y)$ be defined and measurable in the square $A = \{(x, y) : a \leq x \leq b, a \leq y \leq b\}$.

Further, let $\int_a^b \int_a^b |K(x,y)|^2 dxdy \ < \infty$ and $g(x) \in L_2(a, b)$ where $L_2(a, b)$ is the space of square-Lebesgue integrable functions on the interval (a, b). Then the integral equation

$$f(x) = g(x) + \lambda \int_a^b K(x, y) f(y) dy$$

has a unique solution $f(x) \in L_2(a, b)$ for every sufficiently small value of the parameter λ.

Proof: Let $x = L_2$ and Consider the mapping $T : L_2(a, b) \to L_2(a, b)$ defined as

$$Tf = h,$$

where $h(x) = g(x) + \lambda \int_a^b K(x, y) f(y) dy$.
This definition is valid for each $f \in L_2(a, b), h \in L_2(a, b)$. Since $g \in L_2(a, b)$ and λ is a scalar, it is sufficient to show that

$$\psi(x) = \lambda \int_a^b K(x, y) f(y) dy \in L_2(a, b).$$

Since,

$$|\psi(x)| = \left|\lambda \int_a^b K(x, y) f(y) dy\right| \leq \int_a^b |K(x, y) f(y)| \, dy.$$

By Cauchy-Schwartz inequality, we have

$$|\psi(x)| \leq \left(\int_a^b |K(x,y)|^2 \, dy\right)^{\frac{1}{2}} \left(\int_a^b |f(y)|^2 \, dy\right)^{\frac{1}{2}}$$

$$\Rightarrow |\psi(x)|^2 \leq \left(\int_a^b |K(x,y)|^2 \, dy\right) \left(\int_a^b |f(y)|^2 \, dy\right)$$

$$\text{this gives, } \int_a^b |\psi(x)|^2 \, dx \leq \int_a^b \left(\int_a^b |K(x,y)|^2 \, dy\right) dx \int_a^b \left(\int_a^b |f(y)|^2 \, dy\right) dx.$$

By the hypothesis $\int_a^b \int_a^b |K(x,y)|^2 \, dxdy \, < \infty$ and $\int_a^b \left(\int_a^b |f(y)|^2 \, dy \right) dx \, < \infty$.

$$\therefore \int_a^b |\psi(x)|^2 \, dx \, < \infty.$$

Thus, $\psi(x) = \int_a^b K(x,y) f(y) dy \in L_2(a,b)$.
We know that $L_2(a,b)$ is a Banach space with norm

$$\|f\| = \left(\int_a^b |f(y)|^2 \, dy \right)^{\frac{1}{2}}.$$

We now show that T is a contraction mapping. We have,

$$\|Tf - Tf_1\| = \|h - h_1\|.$$

Where $h_1(x) = g_1(x) + \lambda \int_a^b K(x,y) f_1(y) dy$. Now,

$$\begin{aligned}
\|h - h_1\| &= \left\| g(x) + \lambda \int_a^b K(x,y) f(y) dy - g_1(x) - \lambda \int_a^b K(x,y) f_1(y) dy \right\| \\
&= \left\| [g(x) - g_1(x)] + \lambda \int_a^b [K(x,y)\{f(y) - f_1(y)\}] \, dy \right\| \\
&\leq \|g(x) - g_1(x)\| + \left\| \lambda \int_a^b [K(x,y)\{f(y) - f_1(y)\}] \, dy \right\| \\
&\leq \left\| \lambda \int_a^b [K(x,y)\{f(y) - f_1(y)\}] \, dy \right\| \\
&\leq |\lambda| \left(\int_a^b \left| \left[\int_a^b K(x,y)\{f(y) - f_1(y)\} dy \right] \right|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

By Cauchy-Schwartz-Bunyakowski inequality, we get

$$\begin{aligned}
\|h - h_1\| &\leq |\lambda| \left(\int_a^b \int_a^b |K(x,y)|^2 \, dxdy \right)^{\frac{1}{2}} \left(\int_a^b |f(y) - f_1(y)|^2 \, dy \right)^{\frac{1}{2}} \\
&\leq |\lambda| \left(\int_a^b \int_a^b |K(x,y)|^2 \, dxdy \right)^{\frac{1}{2}} \|f - f_1\|.
\end{aligned}$$

Hence, $\|Tf - Tf_1\| \leq |\lambda| \left(\int_a^b \int_a^b |K(x,y)|^2 \, dxdy \right)^{\frac{1}{2}} \|f - f_1\|$.

If

$$|\lambda| < \frac{1}{\left(\int_a^b \int_a^b |K(x,y)|^2 \, dxdy\right)^{\frac{1}{2}}}$$

then,

$$\|Tf - Tf_1\| \le K \|f - f_1\|,$$

where $K = |\lambda| \left(\int_a^b \int_a^b |K(x,y)|^2 \, dxdy\right)^{\frac{1}{2}} < 1$.
Thus T is a contraction and so T has a unique fixed point. That is, there exists a unique $f^* \in L_2(a,b)$ such that $Tf^* = f^*$.
This fixed point f^* is a unique solution of the given equation.

Application 5. Application of Banach contraction theorem to differential equations. We give an application of Banach contraction theorem to prove the existence and uniqueness of the following ordinary differential equation with an intial condition:

$$\frac{dy}{dx} = f(x,y) \ , \ y(x_0) = y_0.$$

Theorem 2.1.7 ***(Picard's theorem)****: Let $f(x,y)$ be a continuous function of two variables in a rectangle, $A = \{(x,y) : a \le x \le b, c \le y \le d\}$ and satisfy the Lipschitz condition in the second variable y.*
Further, let (x_0, y_0) be any interior point of A. Then the differential equation $\frac{dy}{dx} = f(x,y)$. has a unique solution, say $y = g(x)$ which passes through (x_0, y_0).

Proof: Given that the differential equation is

$$\frac{dy}{dx} = f(x,y) \tag{2.1.10}$$

Let $y = g(x)$ satisfy 2.1.10 and the property that $g(x_0) = y_0$.

Integrating 2.1.10 from x_0 to x we get

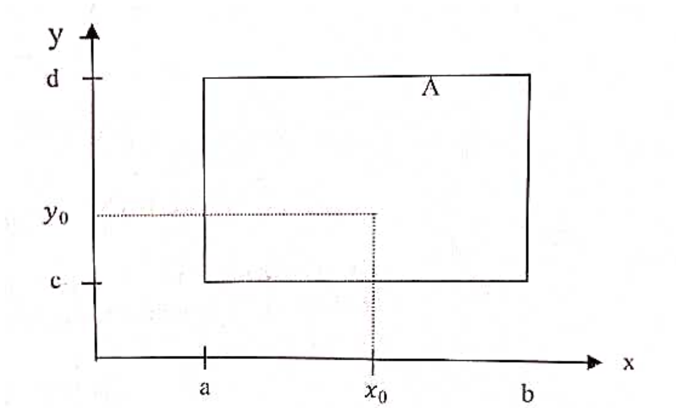
$$[y]_{x_0}^x = \int_{x_0}^x f\left(t, g(t)\right) dt$$

$$\text{thus} \quad g(x) - g(x_0) = \int_{x_0}^x f\left(t, g(t)\right) dt \qquad \because y = g(x).$$

Therefore,

$$g(x) = y_0 + \int_{x_0}^x f\left(t, g(t)\right) dt. \tag{2.1.11}$$

Thus a unique solution of 2.1.10 is equivalent to a unique solution of 2.1.11. Since, $f(x,y)$ satisfies the Lipschitz condition in y, there exists a constant $q > 0$, such that $|f(x,y_1) - f(x,y_2)| \leq q|y_1 - y_2|$ where $(x,y_1),(x,y_2) \in A$.



Since $f(x,y)$ is continuous on a compact subset A of R^2, it is bounded. So, there exists a positive constant m such that $|f(x,y)| \leq m, \forall\ (x,y) \in A$.
Let us choose a positive constant p such that $pq < 1$ and the rectangle

$$B = \{(x,y)|x_0 - p \leq x \leq x_0 + p, y_0 - pm \leq y \leq y_0 + pm\}$$

is contained in A.
Let X be the set of all real-valued continuous functions $y = g(x)$ defined on $[x_0 - p, x_0 + p]$ such that $\|g(x) - y_0\| \leq mp$ i.e, X is a closed subset of the Banach space $C[x_0 - p, x_0 + p]$ with the sup norm.
Let $T : X \to X$ be defined as $Tg = h$ where $h(x) = y_0 + \int_{x_0}^{x} f\left(t, g(t)\right) dt$.
Here

$$\begin{aligned}
\|h(x) - y_0\| &= \left\| \int_{x_0}^{x} f\left(t, g(t)\right) dt \right\| \\
&\leq \int_{x_0}^{x} \left|f\left(t, g(t)\right)\right| dt \\
&\leq m \int_{x_0}^{x} dt \\
&\leq m(x - x_0) \leq mp
\end{aligned}$$

therefore $h(x) \in X$ and so T is well defined.

Let $g, g_1 \in X$, then

$$\begin{aligned}
\|Tg - Tg_1\| &= \|h - h_1\| \\
&= \left\| y_0 + \int_{x_0}^{x} f\left(t, g(t)\right) dt - y_0 - \int_{x_0}^{x} f\left(t, g_1(t)\right) dt \right\| \\
&= \left\| \int_{x_0}^{x} \left(f\left(t, g(t)\right) - f\left(t, g_1(t)\right) \right) dt \right\| \\
&\leq \int_{x_0}^{x} \left\| f\left(t, g(t)\right) - f\left(t, g_1(t)\right) \right\| dt \\
&\leq q \int_{x_0}^{x} \|g(t) - g_1(t)\| \, dt \\
&= q(x - x_0) \|g - g_1\| \\
&\leq pq \|g - g_1\| \\
&\|Tg - Tg_1\| \leq k \|g - g_1\|,
\end{aligned}$$

where $0 < k = pq < 1$.
Hence, T is a contraction mapping of X onto itself. Therefore, by Banach contraction theorem, T has a unique fixed point $g^* \in X$. This unique fixed point g^* is the unique solution of 2.1.11.

In 1969, Boyd and Wong [18] obtained the following generalization of Banach contraction theorem.

Theorem 2.1.8 *Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semicontinuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$. If $T : X \to X$ satisfies*

$$d\left(T(x), T(y)\right) \leq \psi\left(d(x, y)\right) \quad \forall \quad x, y \in X, \tag{2.1.12}$$

then it has a unique fixed point $\overline{x} \in X$ and $\{T^n(x)\}$ converges to $\overline{x}$ for all $x \in X$.

Proof. For any fixed $x \in X$, let $x_n = T^n(x), n = 1, 2, \ldots$ and $a_n = d(x_n, x_{n+1}) = d\left(T^n(x), T^{n+1}(x)\right)$. We show that a_n is convergent.
We may assume that $a_n > 0$ for all $n > 0$.

Then, for all $n > 1$,

$$\begin{aligned} a_n &= d\left(T^n(x), T^{n+1}(x)\right) \\ &= d\left(T(x_{n-1}), T(x_n)\right) \\ &\leq \psi\left(d(x_{n-1}, x_n)\right) \\ &= \psi(a_{n-1}) \\ &< a_{n-1}. \end{aligned}$$

Thus, the sequence $\{a_n\}$ is monotonically decreasing and bounded below so it is convergent.
Let $\lim_{n\to\infty} a_n = a$. We show that $a = 0$. If $a > 0$, then $a_{n+1} \leq \psi(a_n)$.
By the upper semicontinuity from the right of the function ψ, we obtain $a \leq \psi(a)$ which is a contradiction with the property of ψ.
Thus $a = 0$ and $a_n \to 0$ as $n \to \infty$.
We claim that $\{x_n\}$ is a Cauchy sequence.
Assume to the contrary that the sequence $\{x_n\}$ is not Cauchy.
Then there exists $\varepsilon > 0$, such that for any $k \in \mathbb{N}$, there exist $m_k > n_k \geq k$, such that

$$d\left(x_{m_k}, x_{n_k}\right) \geq \varepsilon \tag{2.1.13}$$

Furthermore, assume that for each k , m_k is the smallest number greater than n_k for which 2.1.13 holds.
Let $a_k = d(x_{m_k}, x_{n_k})$. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} a_n = 0$, there exists k_0 such that $d(x_k, x_{k+1}) \leq \varepsilon$ for all $k \geq k_0$.
For such k, we have

$$\begin{aligned} \varepsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \varepsilon \\ &\leq d(x_k, x_{k-1}) + \varepsilon. \end{aligned}$$

This proves $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \lim\limits_{k\to\infty} a_k = \varepsilon$.
On the other hand, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq a_{m_k} + \psi\left(d(x_{m_k}, x_{n_k})\right) + a_{n_k} \qquad \textit{where} \quad a_{m_k} = d(x_{m_k}, x_{m_k+1}) \\ &\leq 2a_k + \psi\left(d(x_{m_k}, x_{n_k})\right). \end{aligned}$$

As $k \to \infty$, we obtain

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \\ &\leq \lim_{k\to\infty} \left(2a_k + \psi\left(d(x_{m_k}, x_{n_k})\right)\right) \\ &= \psi(\varepsilon). \end{aligned}$$

Thus $\varepsilon \leq \psi(\varepsilon)$ which is a contradiction.
Hence $\{T^n(x)\} = \{x_n\}$ is a Cauchy sequence.
Since, $\{T^n(x)\}$ is a Cauchy sequence and X is complete, $\lim\limits_{n\to\infty} T^n(x) = \overline{x} \in X$. Since T is continuous, $T(\overline{x}) = \overline{x}$. Uniqueness of $\overline{x}$ follows from condition 2.1.12.

Remark 2.1.9 If we replace the condition $\psi(t) < t$ by the condition $\psi(t_0) < t_0$ for at least one value to t_0, then theorem 2.1.8 may fail. In this case, T may have no fixed point or else more than one fixed point.

Chapter 3

Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem is a fundamental result in topology and mathematics, first proven by the Dutch mathematician Luitzen Egbertus Jan Brouwer in 1910. This theorem is a pivotal concept in the field of algebraic topology and plays a crucial role in various areas of mathematics, economics, and the natural sciences. The theorem essentially states that every continuous function from a closed, bounded interval in Euclidean space to itself has at least one fixed point.

The core idea of Brouwer's Fixed Point Theorem is the existence of stationary points in continuous transformations. The Theorem, with its elegance and generality, has made a profound impact on various areas of mathematics, and it continues to be a fundamental result in the field of topology and mathematical analysis. It underscores the concept of invariance in mathematical transformations and has inspired the development of many other fixed-point theorems and related concepts in mathematics and its applications.

1. Key Assumptions

i. The set X must be closed and bounded. In one dimension, this could be an interval $[a, b]$, while in higher dimensions, it could be a closed ball or a closed and bounded region.

ii. The function f defined on X must be continuous, meaning that small changes in the input should lead to small changes in the output.

2. Intuitive Example

Consider a sheet of rubber with some ink dots on it. You can stretch and deform the rubber sheet, but you can't tear it. Brouwer's Fixed Point Theorem tells us that no matter how you deform the sheet, at least one dot will end up exactly where it started, even though the distances between the dots may have changed.

3. Significance

Brouwer's Fixed Point Theorem has broad implications and applications in mathematics, economics and the sciences. It forms the foundation of various mathematical theories and algorithms, such as the Kakutani fixed-point theorem in game theory [9], [10] and the topological degree theory. Additionally, it has applications in fields like game theory, physics and computer science.

3.1 Results

Theorem 3.1.1 Brouwer's fixed point theorem (for unit disc B^2):
If $f : B^2 \to B^2$ is a continuous map, then there exists a point $x \in B^2$ such that $f(x) = x$.

Proof: We proceed by contradiction. Suppose that $f(x) \neq x$ for any $x \in B^2$. Then $v(x) = f(x) - x$ gives a non-vanishing vector field $(x, v(x))$ on B^2.
Therefore, there exists a point x of S^1 the boundary of B^2 where the vector field points directly outward that is
$v(x) = ax$, where $a > 0$
or $f(x) - x = ax$, this implies $f(x) = (1 + a)x$, this gives a contradiction, since $(1 + a)x$ lies outside the unit disc B^2.

Remark 3.1.2 Intermediate value theorem: Suppose $f(x)$ is a continuous function on $[a, b]$ and l is a number that lies between $f(a)$ and $f(b)$, then there exists at least one c such that $c \in (a, b)$ and $f(c) = l$.

Theorem 3.1.3 Brouwer's fixed point theorem for $[0, 1]$:
If $f : [0, 1] \to [0, 1]$ is a continuous function, then there is $x \in [0, 1]$ such that $f(x) = x$ i.e, x is a fixed point of $f(x)$.

Proof: We draw any graph of continuous function $f(x)$ from $[0, 1] \to [0, 1]$, as shown in figure 3.1

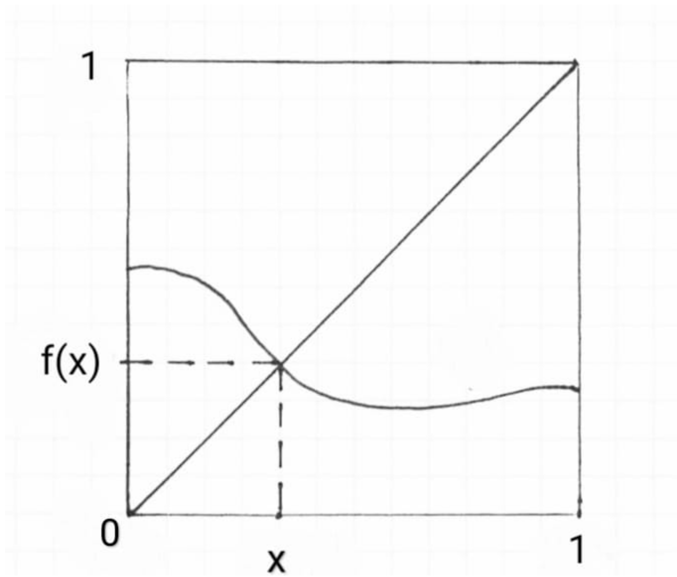


Figure 3.1: figure

Define

$$g(x) = x - f(x),\ x \in [0,1],$$

Clearly g is continuous.
Also

$$g(0) = 0 - f(0) = -f(0) \leq 0$$

$$g(1) = 1 - f(1) \geq 1 - 1 = 0.$$

So, $g(0) \leq 0$ whereas $g(1) \geq 0$.
Therefore, by Intermediate value theorem, there exists $x' \in [0,1]$ with $g(x') = 0$ which gives

$$x' - f(x') = 0$$
$$\implies f(x') = x'.$$

By definition, x' is a fixed point of $f(x)$.
Lemma: The set $S^{n-1} = \{x \in R^n : \|x\| = 1\}$ is not a retract of D^n.
Proof: The lemma can be easily proved by means of algebraic topology tools. Indeed, a retraction r induces a homomorphism $r_\star : H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$, where H_{n-1} denotes the $(n-1)$-dimensional homology group.

The natural injection $j : S^{n-1} \to D^n$ induces in turn a homomorphism

$$j_\star : H_{n-1}(S^{n-1}) \to H_{n-1}(D^n),$$

and the composition $r \circ j$ is the identity map on S^{n-1}. Hence $(r \circ j)_\star = r_\star \circ j_\star$ is the identity map on $H_{n-1}(S^{n-1})$.
But since $H_{n-1}(D^n) = 0$, $j_\star$ is the null map.
On the other hand, $H_{n-1}(S^{n-1}) = Z$ if $n \neq 1$ and $H_0(S^0) = Z \oplus Z$, leading to a contradiction.
The analytic proof reported below is less evident and makes use of exterior forms. Moreover, it provides a weaker result, namely, it shows that there exists no retraction of class C^2 from unit disc D^n to S^{n-1} in $\mathbb{R}^n$. This will be however enough for our scopes.
The proof associate to a C^2 function $h : D^n \to D^n$ the exterior from

$$w_h = h_1 dh_2 \wedge \ldots \wedge dh_n.$$

Theorem 3.1.4 (Brouwer) Let $f : D^n \to D^n$ be a continuous function. Then f has a fixed point $\bar{x} \in D^n$.

Proof: Since we want to rely on analytic proof, let $f : D^n \to D^n$ be a class of C^2. If f has no fixed point, then

$$r(x) = t(x)f(x) + (1 - t(x))x,$$

where

$$t(x) = \frac{||x||^2 - \langle x, f(x)\rangle - \sqrt{\left(||x||^2 - \langle x, f(x)\rangle\right)^2 + (1 - ||x||^2)||x - f(x)||^2}}{||x - f(x)||^2}$$

is a retraction of class C^2 from D^n to S^{n-1}, against the conclusion of lemma. Graphically, $r(x)$ is the intersection with S^{n-1} of the line obtained extending the segment connecting $f(x)$ to x. Hence such an f has a fixed point. Finally, let $f : D^n \to D^n$ be continuous. Appealing to the Stone-Weierstrass theorem, we find a sequence $f_j : D^n \to D^n$ of functions of class C^2 converging uniformly to f on D^n. Denote $\bar{x}_j$ the fixed point of f_j. Then there is $\bar{x} \in D^n$ such that, up to a subsequence, $\bar{x}_j \to \bar{x}$. Therefore

$$||f(\bar{x}) - \bar{x}|| \leq ||f(\bar{x}) - f(\bar{x}_j)|| + ||f(\bar{x}_j) - f_j(\bar{x}_j)|| + ||\bar{x}_j - \bar{x}|| \to 0$$

as $j \to \infty$, which yields $f(\bar{x}) = \bar{x}$.

3.1.1 Applications of Brouwer's Theorem

A fundamental result that underpins a great deal of mathematics is the intermediate value theorem (IVT). In the n-dimensional case, the IVT is the following:

Theorem 3.1.5 (Intermediate value theorem)

Suppose that $f : D^n \to R^n$ is continuous and suppose that when $|x| = 1$ $(x \in \delta D^n)$ we have

$$\langle f(x), x \rangle > 0$$

where $\langle , \rangle$ is an Euclidean inner product.
There exists an $x \in D^n$ such that

$$f(x) = 0.$$

Proof: Let $f : D^n \to R^n$ be a continuous map that satisfies the above criteria. To prove that there exists an $x \in D^n$ such that $f(x) = 0$, we can construct a new map $g : D^n \to D^n$ using f such that, when we apply the Brouwer's fixed point theorem to g, the result simplifies to $f(x) = 0$. Take $g(x) = \alpha f(x) + x$, for some $\alpha > 0$. Then by Brouwer's fixed point theorem, there exists some $x \in D^n$ such that

$$g(x) = x$$

$$\text{or} \quad \alpha f(x) + x = x,$$

$$\text{or} \quad \alpha f(x) = 0,$$

$$\text{implies} \quad f(x) = 0$$

So, all that needs to be established is that for any continuous map $f : D^n \to R^n$ with the condition that $x \in \delta D^n$.
$\Rightarrow \langle f(x), x \rangle > 0$, there exists an $\alpha > 0$ such that

$$g(x) = \alpha f(x) + x$$

is a continuous function $g : D^n \to D^n$.

To prove this, suppose that such a function g does not exist. We can write any $\alpha > 0$ as $\alpha = \frac{1}{m}$, $m \in R$. Then for all m where $g(x) = \frac{1}{m} f(x) + x$, there is some $x_m \in D^n$ such that $|g(x_m)| > 1$ $(x_m \notin D^n)$.

As the choice of α is arbitrary, we have

$$|0f(x_0) + x_0| \geq 1$$

$$\Rightarrow |x_0| \geq 1.$$

But we know that $x_0 \in D^n$, so we deduce that $|x_0| = 1$.
By assumption and the continuity of f,

$$\langle f(x), x \rangle > 0$$

$$\Rightarrow \langle f(x_0), x_0 \rangle > 0.$$

Again we deduce that there is some $\delta > 0$ and an $M \in N$, such that $m \geq M$, we have that

$$\langle f(x_m), x_m \rangle \leq \delta > 0$$

Now

$$\left| \frac{1}{m} f(x_m) + x_m \right|^2$$

$$= \frac{1}{m^2}|f(x_m)|^2 + \frac{2}{m}|(x_m)f(x_m)|^2 + |x_m|^2$$

$$= \frac{1}{m^2}|f(x_m)|^2 + \frac{2}{m}\langle f(x_m), x_m \rangle^2 + |x_m|^2$$

By definition of supremum, and as $|x_m| \leq 1$,

$$\left| \frac{1}{m} f(x_m) + x_m \right|^2 \leq \frac{1}{m^2}||f||_\infty^2 + \frac{2}{m}|(x_m)f(x_m)|^2 + |x_m|^2$$

$$\leq \frac{1}{m^2}||f||_\infty^2 + \frac{2}{m}|(x_m)f(x_m)|^2 + 1$$

When $m \geq M$. We can now set up a contradiction.
Suppose that $m > \frac{||f||_\infty^2}{\delta}$.
Then

$$\frac{\delta}{m} > \frac{||f||_\infty^2}{m^2},$$

and we would have that

$$\left| \frac{1}{m} f(x_m) + x_m \right|^2 < \frac{\delta}{m} - \frac{2\delta}{m} + 1 < 1 - \frac{\delta}{m},$$

now as $m, \delta > 0$, this implies $\left|\frac{1}{m}f(x_m) + x_m\right|^2 < 1$, Which is a contradiction.

Thus we can conclude that for any continuous $f : D^n \to R^n$, there exists an $\alpha > 0$ such that $g(x) = \alpha f(x) + x$ is a continuous function $g : D^n \to D^n$. This concludes the proof.

Remark 3.1.6 The intermediate value theorem (IVT) and the Brouwers fixed point theorem (BFPT) are actually equivalent theorems. We have already seen that Brouwers fixed point theorem $\Rightarrow$ intermediate value theorem.

Now let us prove the converse

Proposition: IVT $\Rightarrow$ BFPT.

Proof: Suppose $f : D^n \to D^n$ is continuous, then by the BFPT, there is some $x_0 \in D^n$ such that $f(x_0) = x_0$. Now set $g(x) = f(x) + x$,
we know that $g : D^n \to R^n$. To check the two criteria of IVT holds, consider that

$$\langle g(x), x\rangle = \langle f(x) + x, x\rangle$$

$$= |xf(x)| - |x|^2$$

by the Cauchy-Schwarz inequality

$$\langle g(x), x\rangle \leq |f(x)||x| - |x|^2.$$

As $|f(x)| \leq 1$, so

$$\Rightarrow \langle g(x), x\rangle \leq |x| - |x|^2$$

under the condition that $|x| = 1$, it holds that $\langle g(x), x\rangle \leq 0$.
Note that

$$g(x_0) = f(x_0) - x_0$$

$$= x_0 - x_0 = 0.$$

Hence, the IVT holds for g.

Topological Invariance of Domain and dimension

Theorem 3.1.7 (Invariance of Domain):

Let U be an open subset of R^n and let $f : U \to R^n$ be continuous, injective function. Then $f(U)$ is open.

Proof: We wish to show that every point $f(x) \in f(U)$ is an interior point of $f(U)$. Then, by definition $f(U)$ will be open. Consider that, for all $x \in U$, there is some $r > 0$ such that $\overline{B_r(x)} \subseteq U$. But $x \in \overline{B_r(x)} \subseteq U$

$$\Rightarrow f(x) \in f(\overline{B_r(x)}) \subseteq f(U)$$

by continuity of f.

So, to show that $f(U)$ is an open subset of R^n, it suffices to show that $f(x)$ is an interior point of $f(\overline{B_r(x)})$.

Further, by rescaling and translation, it suffices to show that, if $f : D^n \to R^n$ is continuous and injective, then $f(0)$ is an interior point of $f(D^n)$.

As D^n is compact, it follows that $f : D^n \to f(D^n)$ is a homeomorphism. We define an inverse $g : f(D^n) \to D^n$.

By Tietze Extension Theorem, we extend g to $g : R^n \to R^n$. This function has a zero, that is of course, $f(0)$. We wish to show that any function sufficiently close to g has a zero. This will prove the above.

More formally, let $\tilde{g} : f(D^n) \to R^n$ be continuous, with condition that

$$|\tilde{g}(y) - g(y)| \leq 1,$$

for all $y \in f(D^n)$. Then $\tilde{g}$ has a zero in $f(D^n)$. We define a function

$$h(x) : D^n \to D^n,$$

such that

$$h(x) = x - \tilde{g}(f(x)).$$

By the Brouwer's fixed point theorem, it follows that there exists $x \in D^n$ such that

$$h(x) = x$$

$$\text{implies} \;\; x - \tilde{g}(f(x)) = x$$

$$\text{or} \;\; \tilde{g}(f(x)) = 0.$$

As $\tilde{g} : f(D^n) \to R^n$ is continuous, so it follows that $\tilde{g}$ has a zero, namlely $f(x)$.

Suppose that $f(0)$ is not an interior point of $f(D^n)$. Then $f(0)$ must lie on the boundary and so a function has a zero on the boundary. We wish to construct some function $\tilde{g}$ from g which will contradict the fact that any function sufficiently close to g has a zero. By the definition of continuity, there exists a $\delta > 0$ such that for all $y \in R^n$,

$$|y - f(0)| < 2\delta$$

$$\Rightarrow |g(y) - g(f(0))| < \frac{1}{4}.$$

But of course we already have that $g(f(0)) = 0$,
So,

$$|y - f(0)| \Rightarrow |g(y)| < \frac{1}{4}.$$

By assumption, there exists $\alpha \notin f(D^n)$ such that $|\alpha - f(0)| < \delta$.
Assume $\alpha = 0$. If not, we can translate it so that α gets moved to the origin.
So we have that $0 \notin f(D^n)$, $|f(0)| < \delta$, and by the triangle inequality $|y| < \delta$

$$\Rightarrow |y - f(0)| < \delta.$$

Define

$$L = L_1 \cup L_2 = (f(D^n) \cap \{|y| \geq \delta\}) \cup \{y \in R^n - f(D^n) : |y| = \delta.\} \tag{3.1.1}$$

Notice that L_2 is the boundary of the ball of radius δ centered on the origin and L_1 is the part of $f(D^n)$ that lies outside that ball. By the compactness of $f(D^n)$, L_1 and L_2 are also compact. Further, there are also no zeros on L_1.
We define a continuous function

$$\phi : f(D^n) \to L$$

by

$$\phi(y) = max\left\{\frac{\delta}{|y|}, 1\right\} y.$$

This is well defined and continuous. When $y \in L_1$, then $\phi(y) = y$. When $y \in f(D^n)$ with $|y| < \delta$,

$$\phi(y) = \frac{\delta y}{|y|} \in L_2.$$

Note that these are the boundary points of the ball. Now take

$$\tilde{g} = g \circ \phi : f(D^n) \to R^n.$$

When $y \in L_1$, $\tilde{g}(y) = g(y) \neq 0$ the only place for a zero then may be when $y \in L_2$.
To prove that there is no zero when $y \in L_2$, if necessary we can pertrub $\tilde{g}$ to be a better suited function, by means of the Weierstrass Approximation Theorem or perhaps by a more direct topological approach. This ensures that the function does not vanish in L_2.

After establishing this, we can conclude the proof by showing that $\tilde{g}$ is sufficiently close to g,

1. if $|y| \geq \delta$, then $\phi(y) = y$, and $|g(y) - \tilde{g}(y)| = |g(y) - go\phi(y)|$

$$= |g(y) - g(y)|$$

$$= 0 < 1.$$

2. if $|y| \leq \delta$, then we have that $|g(y)| < \frac{1}{4}$ and $|\tilde{g}(y)| < \frac{1}{4}$ so, $|g(y) - \tilde{g}(y)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$.

In both the cases $|g(y) - \tilde{g}(y)| < 1$, $\tilde{g}$ should have a zero in $f(D^n)$. But we have proven it doesn't, this is a contradiction and our claim holds.

Theorem 3.1.8 (Invariance of Dimension). If U is an open subset of R^m and V an open subset of R^n, and U is an homeomorphic to V, then $m = n$.

Proof: To prove invariance of dimension, We will use invariance of domain. Suppose there exists continuous, injective map $f : U \to R^n$, where U is an open subset of R^m.

We will first prove that it must be that $m \leq n$. We will proceed by contradiction. Suppose that $m > n$, then take some linear injection $P : R^n \to R^m$. The image, $P(R^n)$ is a proper subspace of R^m. Then $Pf : U \to R^m$ is linear injection whose image is contained within a proper subspace of R^m.
Lemma: Every proper subspace of R^m has an empty interior.
Proof: Let $S \subset R^m$ be a proper subspace. Suppose S has a non-empty interior. Let $x \in S$, then it contains some ball $B_r(x)$. Take some $y \in R^m$, let

$$z = x + \frac{r}{2||y||} y$$

then $z \in B_r(x) \subset S$ this gives

$$y = (z - x)\frac{2||y||}{r},$$
$$\Rightarrow y \in S.$$

As as $y \in R^m$ is arbitrary, it follows that $S = R^m$. This is a contradiction, as we assumed S is a proper subspace of R^m.
By the theorem(3.1.7), this is a contradiction, as $Pf(U)$ is assumed to be an open set.

Remark 3.1.9 In topology, the Tietze extension theorem (also known as Tietze-Ursohn-Brouwers extension theorem or Uryshon-Brouwer lemma) [3] states that if X is a normal space and $f : A \to R$ is a continuous map from a closed subset A of X into the real numbers R carrying the standard topology, then there exists a continuous extension of f to X ; that is, there exists a map

$$F : X \to R$$

continuous on all of X with $F(a) = f(a)$ for all $a \in A$. Moreover, F may be choosen such that

$$\sup\{|f(a)| : a \in A\} = \sup\{|F(x)| : x \in X\}$$

that is, if f is bounded then F may be choosen, to be bounded (with the same bound as f).

Remark 3.1.10 (Weierstrass Approximation theorem). Let I be closed and bounded interval. Suppose $f : I \to R$ is a continuous function, then for each $\varepsilon > 0$, there exists a polynomial function $P_\varepsilon : I \to R$ such that

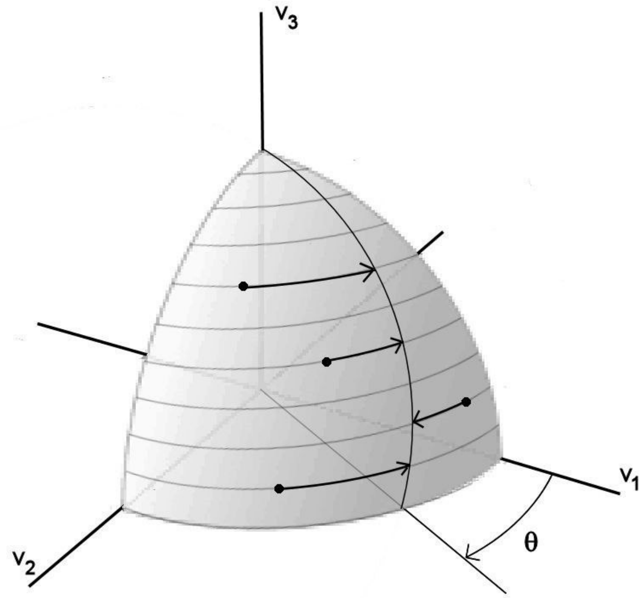
$$|f(x) - P_\varepsilon(x)| < \varepsilon,$$

for all $x \in I$ or equivalently

$$sup\{|f(x) - P_\varepsilon(x)| : x \in I\} < \varepsilon.$$

Theorem 3.1.11 Let A be a 3×3 matrix of positive real numbers. Then A has a positive real eigen value.

Proof: Let $T : R^3 \to R^3$ be a linear transformation, whose matrix relative to the standard basis of R^3 is A.
Let $B = S^2 \cap \{(v_1, v_2, v_3) \in R^3 | v_i \geq 0, \forall\ i = 1, 2, 3\}$



Clearly, B is homeomorphic to a unit disc B^2. So the fixed point theorem holds for continuous maps of B into itself. Now if $v = (v_1, v_2, v_3) \in B$, then all the components of v are non-negative and atleast one is positive. Since A has all positive entries, therefore $T(v)$ is a vector whose all components are positive. Now define $\phi : B \to B$ by

$$\phi : y \to \frac{y}{||y||}.$$

Clearly ϕ is a continuous map and hence there exists a point $x_0 \in B$, such that

$$\phi(x_0) = x_0.$$

This gives

$$x_0 = \frac{T(x_0)}{||T(x_0)||}$$

or

$$T(x_0) = ||T(x_0)||x_0.$$

Hence T has a positive eigen value namely $||T(x_0)||$.

Example 3.1.12 The theorem has some several real world illustrations. Here are some examples:

1. Given two similar maps of a country of different sizes resting on top of each other, there always exists a point that represents the same place on both maps.

2. Consider a map of a country, if that map is placed anywhere in that country, there will always be a point on the map that represents the exact point in that country.

3. Recently [11] provide an alternative proof using Brouwer's fixed point theorem of Browder's theorem, which states that for every continuous mapping $f : [0, 1] \times X \to X$, where X is a nonempty, compact, and convex set in a Euclidean space, the set of fixed points of f, namely the set $\{(t, x) \in [0, 1] \times X : f(t, x) = x\}$, has a connected component whose projection onto the first coordinate is $[0, 1]$.

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