A SURVEY ON BROUWER'S CONJECTURE

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by

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CERTIFICATE

This is to certify that the dissertation entitled, **"A SURVEY ON BROUWER'S CONJECTURE"** being submitted by the students with enrollments **21068120002, 21068120037, 21068120046, 21068120060, 21068120061, 21068120067, 21068120069, 21068120073, 21068120076** to the **Department of Mathematics, University of Kashmir, Srinagar,** for the award of Master's degree in Mathematics, is an original project work carried out by the student under my guidance and supervision.

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Abstract

The work in this project work concerns the investigation of eigenvalues of the Laplacian matrix of a graph. The selection of the material is based on the survey conducted regarding the Brouwer's conjecture for Laplacian spectrum of simple graphs. The project thesis consists of two chapters. The introductory chapter deals with the preliminaries required for the understanding of the survey conducted in the subsequent chapters. We include some basic preliminaries like definitions regarding graph theory, a brief history of spectral graph theory and introduction of Brouwer's conjecture.

Chapter 2 deals with the sum of k largest Laplacian eigen values $S_k(G)$ of graph G and Brouwer's conjecture. The upper bounds for $S_k(G)$ for some class of graph's are included and then used to verify the validity of the Brouwer's conjecture for these class of graphs. We present some results regarding Brouwer's conjecture for some class of graphs like connected graphs, disconnected graphs, random graphs, regular graphs, cyclic graphs, bipartite graphs, trees etc. At the end, we provide some references in the bibliography from where the survey has been conducted.

Contents

Chapter 1

Introduction

1.1 Background

Graph theory was first introduced in the 18th century by the Swiss mathematician Leonhard Euler. His work on the famous "Seven Bridges of Königsberg problem," is considered the origin of graph theory.

The city of Königsberg in Prussia (present-day Kaliningrad, Russia) was set on both sides of the Pregel River and included two large islands — Kneiphof and Lomse — that were connected to each other via the two mainland portions of the city by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

Euler, recognizing that the relevant constraints were the four bodies of land and the seven bridges, drew out the first known visual representation of a modern graph. A modern graph, as seen in the image, is represented by a set of points known as vertices or nodes, connected by a set of lines known as edges.

Figure 1.1: Konigsberg Bridge Problem

The "Seven Bridges of Königsberg" problem illustrated in graphs. This abstraction from a concrete problem concerning a city and bridges to a graph makes the problem tractable mathematically, as this abstract representation includes only the information important for solving the problem. Euler actually proved that this specific problem has no solution. However, the challenge he faced was the development of a suitable technique of analysis and subsequent tests established this assertion with mathematical rigor. From there, the branch of math known as graph theory lay dormant for decades. In modern times, however, its applications are exploding fast.

Applications of Graph Theory

Graph Theory is used in vast area of science and technologies. Some of them are given below: In computer science graph theory is used for the study of algorithms like:

• Dijkstra's Algorithm • Prims's Algorithm • Kruskal's Algorithm

Graphs are used to define the flow of computation, networks of communication, data organization. Graph theory is used to find shortest path in road or a network. In Google Maps, various locations are represented as vertices or nodes and the roads are represented as edges and graph theory is used to find the shortest path between two nodes.

• In Electrical Engineering, graph theory is used in designing of circuit connections. These circuit connections are named as topologies. Some topologies are series, bridge, star and parallel topologies.

• In linguistics, graphs are mostly used for parsing of a language tree and grammar of a language tree. Semantics networks are used within lexical semantics, especially as applied to computers, modeling word meaning is easier when a given word is understood in terms of related words.

Methods in phonology (e.g. theory of optimality, which uses lattice graphs) and morphology (e.g. morphology of finite - state, using finite-state transducers) are common in the analysis of language as a graph.

• In physics and chemistry, graph theory is used to study molecules. The 3D structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. Statistical physics also uses graphs. In this field graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems. Graphs are also used to express the micro-scale channels of porous media, in which the vertices represent the pores and the edges represent the smaller channels connecting the pores. Graph is also helpful in constructing the molecular structure as well as lattice of the molecule. It also helps us to show the bond relation in between atoms and molecules, also help in comparing structure of one molecule to other.

• In computer network, the relationships among interconnected computers within the network, follow the principles of graph theory.

Graph theory is also used in network security. We can use the vertex coloring algorithm to find a proper coloring of the map with four colors. Vertex coloring algorithm may be used for assigning at most four different frequencies for any GSM (Grouped Special Mobile) mobile phone networks.

• Graph theory is also used in sociology. For example, to explore rumor spreading, or to measure actor's prestige notably through the use of social network analysis software. Acquaintanceship and friendship graphs describe whether people know each other or not. In influence graphs model, certain people can influence the behavior of others. In collaboration graphs model to check whether two people work together in a particular way, such as acting in a movie together.

• Nodes in biological networks represent bi-molecular such as genes, proteins or metabolites, and edges connecting these nodes indicate functional, physical or chemical interactions between the corresponding bi-molecular.
• Graph theory is used in transcriptional regulation networks. It is also used in Metabolic networks. In PPI (Protein - Protein interaction) networks graph theory is also useful. Characterizing drug - drug target relationships.
• In mathematics, operational research is the important field. Graph theory provides many useful applications in operational research. Like: minimum cost path, A scheduled problem.
• Graphs are used to represent the routes between the cities. With the help of tree that is a type of graph, we can create hierarchical ordered information such as family trees.

1.2 Basic Definitions

Graph: A linear graph (or a simple graph) $G(V, E)$ consists of objects $V = \{v_1, v_2, \ldots v_n\}$ called vertices and another set $E = \{e_1, e_2 \ldots e_m\}$ whose elements are called edges such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices the vertices $v_i v_j$ associated with an edge e_k called end vertices of e_k. The most common representation of a graph is by means of diagram in which the vertices are represented by points and each edge as line segment joining its end vertices.

Multigraph: A multi-graph G is a pair (V, E), where V is a non-void set of vertices and E is a multiset of unordered pairs of distinct elements of V. The number of times an edge occurs in G is called its multiplicity and edges with multiplicity more than one are called multiple edges.

General Graph: A general graph G is a pair (V, E), where V is a non-void set of vertices and E is a multiset of unordered pairs of elements of V. We denote by uv an edge from the vertex u to the vertex v. An edge of the form uu is called loop of G and edges which are not loops are known as proper edges. The cardinalities of V and E are known as order and size of G, respectively.

Sub-graph of a Graph: Let $G = (V, E)$ be a graph, $H = (W, E^{'})$ is the sub-graph of G whenever $W \subset V$ and $E^{'} \subset E$. If $W = V$ the sub-graph is said to be spanning sub-graph of G. An induced sub-graph $< W >$ is the subset of V together with all the edges of G between the vertices in W.

Complement: Let G be a graph, the complement of G, denoted by $\overline{G}$, is a graph on same set of vertices such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

Bipartite Graph: A graph G is said to be bipartite, if its vertex set V can be partitioned into two subsets, say V_1 and V_2 such that each edge has one end in V_1 and other in V_2. It is denoted by $K_{a,b}$, where a are b are the cardinalities of V_1 and V_2, respectively.

Degree: Degree of a vertex v in a graph G is the number of edges incident on v and is denoted by d_v or $d(v)$.

Conjugate Degree: Let d_{v_k} be the degree of vertex $v_k \in V$. Then conjugate degree of v_k is denoted by $d^*_{v_k}$ and is defined as $d^*_{v_k} = |\{v_i : d_{v_i} \geq k\}|$, where $|.|$, denotes cardinality of set.

Degree sequence: Let $d_i, 1 \leq i \leq n$, be the degrees of the vertices v_i of a graph in any order. The sequence $[d_i]_1^n$ is called the degree sequence of the graph.

Regular graph: A graph $G(V, E)$ is said to be regular if all its vertices are of same degree and $r-$regular if all its vertices are of degree r.

Path: A path of length $n-1(n \geq 2)$ denoted by P_n, is a graph with n vertices $v_1, v_2, \ldots, v_n$ and with $n-1$ edges (v_j, v_{j+1}), where $j = 1, 2, \ldots, n-1$.

Cycle: A cycle of length n, denoted by C_n, is the graph with vertex set $v_1, v_2, \ldots, v_n$ having edges $(v_j, v_j+1), j = 1, 2, \ldots n-1$ and (v_n, v_1)

Connectedness in a graph: A graph $G(V, E)$ is said to be connected if for every pair of vertices u and v there is a path from u to v.

Complete graph: A graph in which every pair of distinct vertices is connected by an edge. It is denoted by K_n Complement of a connected graph is an empty graph.

Tree: A Tree T is a connected a-cyclic graph.

Spanning tree: A tree is said to be a spanning tree of a connected graph G, if T is a sub-graph of G and T contains all vertices of G.

Matching : Let G be a graph of order n and size m. A $k-$matching of G is a collection of k independent edges (that is, edges which do not share a vertex) of G. A $k-$matching is called perfect if $n = 2k$

Clique: A clique of a graph is the maximum complete sub-graph of a graph G. The order of the maximum clique is called the clique number of the graph G and is denoted by ω. A subset s of a vertex set $v(G)$ is said to be covering set of G If every edge of G is incident to at least one vertex in s. A covering set with minimum cardinality among all covering sets is called minimum covering set of G and its cardinality is called vertex covering number of G, and is denoted by τ.

Independent set: A vertex subset w of G is said to be independent set of G if the induced sub-graph $< W >$ is a void graph. An independent set with maximum number of vertices is called a maximum independent set of G and the number of vertices in such a set is called as independence number of G and is denoted by $\alpha(G)$

Grith of a graph: The girth of a graph G is the length of smallest cycle of G and is denoted by g.

$c-$**cyclic graphs**: If a connected graph G contains n vertices and $n+c-1$ edges, then G is called a $c-$cyclic graph.

Diameter of a graph: The distance between two vertices in G is the number of edges in a shortest path connecting them. The eccentricity of $v \in V$ is the greatest distance between v and any other vertex. The minimum and maximum eccentricity of any vertex in G are known as radius and diameter, respectively.

Edge graphs: Let $G(V, E)$ be a graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The edge graph $L(G)$ of G has the vertex set E and two vertices e_i and e_j are adjacent in $L(G)$ iff the corresponding edges e_i and e_j of G are adjacent in G.

Digraphs(directed graphs): A digraph D is a pair (V, A), Where V is a nonempty set whose elements are called the vertices and A is the subset of the set of ordered pairs.

Threshold graph: A threshold graph is a graph that can be constructed from a one vertex graph by repeated applications of the following two operations: Addition of a single isolated vertex to the graph Addition of single dominated vertex to the graph, i,e a single vertex that is connected to all other vertices.

Union of graphs: Let $G(V, E)$ and $H(U, F)$ be two graphs with $V \cap U = \phi$. Then union of G and H, denoted by $G \cup H$, is the graph with vertex set $V \cup U$ and the edge set $E \cup F$.

Energy of a Graph: The energy of a Graph G, denoted by $E = E(G)$, is the sum of the absolute values of the eigenvalues of G

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

Definition 1.2.1. Adjacency matrix: The adjacency matrix of a graph G is $n \times n$ matrix whose rows and columns are indexed by vertices and is defined as

$A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from} v_i to v_j \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.2.2. Laplacian Matrix:Given a simple graph G with n vertices $v_!, v_2, \ldots, v_n$ its Laplacian matrix is defined element wise as

$$l_{i,j} := \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

or equivalent by the matrix

$$L = D - A$$

Definition 1.2.3. Laplacian energy:Let G be a graph, the Laplacian energy of G, denoted by $LE(G)$ is defined as

$$LE(G) = \sum_{i=1}^{n-1} |\mu_i - \overline{d}|$$

where, $\overline{d}$ is the average degree of G.

Spectrum of a graph:The collection of all the eigen values of a graph is called a spectrum of graph.

Forest : A forest is an un-directed graph in which any two vertices are connected by atmost one path. Equivalently, a forest is an un-directed a-cyclic graph all of whose connected components are trees.

1.3 Spectral Graph Theory

Spectral graph theory (Algebraic graph theory) is the study of spectral properties of matrices associated to graphs. The spectral properties include the study of characteristic polynomial, eigenvalues and eigenvectors of matrices associated to graphs. This also includes the graphs associated to algebraic structures like groups, rings and vector spaces. The major source of research in spectral graph theory has been the study of relationship between the structural and spectral properties of graphs. Another source has research in mathematical chemistry (theoretical/quantum chemistry). One of the major problems in spectral graph theory lies in finding the spectrum of matrices associated to graphs completely or in terms of spectrum of simpler matrices associated with the structure of the graph. Another problem which is worth to mention is to characterise the extremal graphs among all the graphs or among a special class of graphs with respect to a given graph invariant. Like spectral radius, the second largest eigenvalue, the smallest eigenvalue, the second smallest eigenvalue, the graph energy and multiplicities of the eigenvalues that can be associated with the graph matrix. The main aim is to discuss the principal properties and structure of a graph from its eigenvalues. It has been observed that the eigenvalues of graphs are closely related to all graph parameters, linking one property to another. Spectral graph theory has a wide range of applications to other areas of mathematical science and to other areas of sciences which include Computer Science, Physics, Chemistry, Biology, Statistics, Engineering etc. The study of graph eigen- values has rich connections with many other areas of mathematics. An important development is the interaction between spectral graph theory and differential geometry. There is an interesting connection between spectral Riemannian geometry and spectral graph theory. Graph operations help in partitioning of the embedding space, maximising inter-cluster affinity and minimising inter-cluster proximity. Spectral graph theory plays a major role in deforming the embedding spaces in geometry. Graph spectra helps us in making conclusions that we cannot recognize the shapes of solids by their sounds. Algebraic spectral methods are also useful in studying the groups and the rings in a new light. This new developing field investigates the spectrum of graphs associated with the algebraic structures like groups and rings. The main motive to study these algebraic structures graphically using spectral analysis is to explore several properties of interest. In 2010 monograph 'An Introduction to the Theory of Graph Spectra' by Cvetkovic, Rowlinson and Simic [24] summarised all the results to the date and included the results of 1980 text.

1.4 Brouwer's Conjecture

The Brouwer's conjecture was first proposed by Arthur Brouwer in the 1950s. It was inspired by the Four Color Theorem, which states that any planar graph can be colored with four colors. Brouwer noticed that the maximum degree of a planar graph is always three, and so he conjectured that the chromatic number of any graph is at most one more than the maximum degree.

Statement: The conjecture states that if G is a graph and $L(G)$ be its Laplacian matrix with eigenvalues $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$, then

$$\sum_{i=1}^{k} \lambda_i(L(G)) \leq m(G) + \binom{k+1}{2},$$

$1 \leq k \leq n$ where $m(G)$ is the number of edges.

The Brouwer's conjecture has several implications for graph theory. First, it implies that the chromatic number of any graph is bounded by a linear function of the maximum degree. This means that the chromatic number of a graph can be determined in polynomial time, which is a significant improvement over the exponential time needed to determine the chromatic number of a graph without the Brouwer's conjecture. Second, the Brouwer's conjecture implies that the chromatic number of any graph is at most twice the maximum degree. This means that the chromatic number of a graph can be determined in linear time, which is a significant improvement over the exponential time needed to determine the chromatic number of graph .

Chapter 2

Brouwer's Conjecture for Certain Classes of Graphs

2.1 General Graphs

Theorem 2.1.1. *For any graph G and for any $k, 1 \leq k \leq n$,*

$$S_k(G) \leq \sum_{i=1}^{k} d_i^*(G).$$

Where $d_i^*(G)$ is the conjugate degree of G

This was proved by Hua.Bai and is now called as Grone-Merris theorem. As an analogue to Grone-Merris theorem, Andries Brouwers [5] conjectured the following.

Conjecture 2.1.2. *For any graph G with n vertices and each $k \in \{1, 2, 3, \ldots, n\}$,*

$$S_k(G) \leq e(G) + \binom{k+1}{2} \tag{2.1}$$

with the aid of computer, Brouwer's has confirmed this conjecture for all graphs with at most ten vertices [5]. For $k = n-1$ or n, conjecture 2.1.2 follows trivially from the fact that $S_{n-1}(G) = S_n(G) = 2e(G)$. For $k = 1$, conjecture 2.1.2 follows from the well-known inequality $\mu_1(G) \leq |V(G)|$ (If G is a graph with $n \geq 2$ vertices, then $\mu_1 \leq e(G)+1$, with equality iff $G \cong k_{1,l} \cup (n-l-1)k_1$, where $1 \leq l \leq n-1$.) Haemer et al.[8] showed that 2.1.2 is true for all graphs when $k = 2$.

Theorem 2.1.3. *[8] For any graph G with at least two vertices, $S_2(G) \leq e(G) + 3$.*

Moreover, the authors W.H.Haemers, A.Mohammadian and B.Tayfeh-Rezaie proved that, for any tree T with n vertices, $S_k(T) \leq (n-1) + 2k - 1$ holds for all $k \in \{1, 2, \ldots, n\}$, which implies that conjecture 2.1.2 is true for trees (for all k).In a similar vein, Du and Zhou [7] showed that conjecture 2.1.2 is also true for uni cyclic and bi cyclic graphs.

Theorem 2.1.4. *For $n \geq 3$, let p be an integer with $1 \leq p \leq \frac{(n-1)}{2}$. If the conjecture 2.1.2 holds for all graphs when $k = p$, that is, for any graph G with n vertices,*

$$S_p(G) \leq e(G) + \binom{p+1}{2}, \tag{2.2}$$

Then the conjecture 2.1.2 holds for all graphs when $k = n-p-1$ as well, that is, for any graph G with n vertices,

$$S_{n-p-1}(G) \leq e(G) + \binom{n-p}{2}. \tag{2.3}$$

Moreover, if the equality holds in 2.2 if and only if $G \cong G^*$, then the equality holds in 2.3 if and only if $G \cong \overline{G}^*$.

Corollary 2.1.5. *If G is a graph with $n \geq 3$ vertices,then $S_{n-3}(G) \leq e(G) + \binom{n-1}{2}$, with equality if and only if $G \cong \overline{k_{1,l} \bigcup (n-l-1)k_1}$, where $1 \leq l \leq n-1$.*

Also, the next corollary follows directly from Theorem 2.1.4 and 2.1.3 which indicates that conjecture 2.1.2 holds for all graphs when $k = n-3$.

Corollary 2.1.6. *If G is a graph with $n \geq 5$ vertices, then $S_{n-3}(G) \leq e(G) + \binom{n-2}{2}$*

It is worth mentioning that theorem 2.1.4 suggests that to prove conjecture 2.1.2, it suffices to prove conjecture 2.1.2 for all graphs when $1 \leq k \leq \frac{(n-1)}{2}$ or $\frac{n-1}{2} \leq k \leq n-2$. As an application of this idea we derive the following theorem, which, in some sense, could be viewed as partial solution to conjecture 2.1.2.

Theorem 2.1.7. *Suppose that G is a graph with $n \geq 3$ vertices.*

(i) If $e(G) \geq \frac{1}{8}[2n(n-1) + \sqrt{2}\sqrt{n^2(n-2)^2 - 1}]$, then $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for $1 \leq k \leq \frac{(n-1)}{2}$.

(ii) If $e(G) \leq \frac{1}{8}[2n(n-1) - \sqrt{2}\sqrt{n^2(n-2)^2 - 1}]$, then $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for $\frac{(n-1)}{2} \leq k \leq n-2$.

Theorem 2.1.8. *Let G be a graph with $n \geq 3$ vertices and $\overline{G}$ be its compliment. If $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for all $k \in \{1, 2, \ldots, n-2\}$, then $S_k(\overline{G}) \leq e(\overline{G}) + \binom{k+1}{2}$ also holds for all $k \in \{1, 2, \ldots, n-2\}$.*

Lemma 2.1.9. *For any graph G with at most eight vertices, $S_2(G) \leq e(G) + 3$.*

Lemma 2.1.10. *Let n be a natural number.*

(i) The Laplacian eigenvalues of k_n are n with multiplicity $n-1$, and 0.

(ii) The Laplacian eigenvalues of S_n are $n, 1$ with multiplicity $n-2$, and 0.

The following lemma gives an affirmative answer to 2.1.2 for $k = 1$.

Lemma 2.1.11. *If G is a graph with n vertices, then $\mu_1(G) \leq n$.*

Theorem 2.1.12. *Let G be a graph with n vertices and let G' be a graph obtained from G by inserting a new edge into G. Then the Laplacian eigen values of G and G' interlace, that is,*

$$\mu_1(G') \geq \mu_1(G) \geq \cdots \geq \mu_n(G') = \mu_n(G) = 0.$$

Theorem 2.1.13. *let G be a graph. Then $\mu_1(G) \leq max\{d(v) + m(v)|v \in V(G)\}$, where $m(v)$ is the average of the degrees of the vertices of G adjacent to the vertex v.*

Theorem 2.1.14. *Let G be a graph with n vertices and vertex degrees $d_1 \geq \cdots \geq d_n$. If G is not $k_5 + (n-s)k_2$, then $\mu_s(G) \geq d_s - s + 2$ for $1 \leq s \leq n$.*

The following theorem from the Matrix theory plays a key role in our proofs. We denote the eigen values of a symmetric Matrix M by $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$.

Theorem 2.1.15. *Let A and B be two real symmetric matrices of size n.Then for $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

An immediate consequence of 2.1.15 is the following corollary which will be used frequently.

Corollary 2.1.16. *Let $G_1, \cdots, G_r$ be some edge disjoint graphs. Then $S_k(G_1 \cup \cdots \cup G_r) \leq \sum_{i=1}^{r} S_k(G_i)$ for any k.*

Lemma 2.1.17. *Let G be a graph with n vertices. Suppose that there exists two non adjacent vertices $u, v \in V(G)$ such that $\mu_k(G) \geq d(u) + d(v) + 2$ for some integer $k, 1 \leq k \leq n$. If G' is the graph obtained from G by inserting edge $e = \{u, v\}$ into G, then $S_k(G') \leq S_k(G) + 1$.*

Proof. For $i = 1, 2, \cdots, n$, define $\epsilon_i = \mu_i(G') - \mu_i(G)$. By theorem 2.1.12, $\epsilon_i \geq 0$ for any i. Let $d_1 \geq \cdots \geq d_n$ and $d'_1 \geq \cdots \geq d'_n$ be the vertex degrees of G and G' respectively. Recall that for any graph Γ, considering the trace of the matrix $\mathcal{L}(T)^2$, we have

$$\sum_{i=1}^{|V(\Gamma)|} \mu_i(\Gamma)^2 = \sum_{v \in V(\Gamma)} d(V)^2 + 2e(\Gamma).$$

Applying this fact, we have

$$\begin{aligned}\sum_{i=1}^{n}\mu_i(G')^2 &= \sum_{i=1}^{n} d'_{i^2} + 2e(G') \\ &= \sum_{i=1}^{n} d_i^2 + 2e(G) + 2d(u) + 2d(v) + 4 \\ &= \sum_{i=1}^{n}\mu_i(G)^2 + 2d(u) + d(v) + 2).\end{aligned}$$

this yields that

$$\begin{aligned}2\mu_k(G)\sum_{i=1}^{k}\epsilon_i &\le \sum_{i=1}^{k} 2\epsilon_i\mu_i(G) \\ &\le \sum_{i=1}^{n}\mu_i(G')^2 \\ &= 2(d(u) + d(v) + 2).\end{aligned}$$

since

$\mu_k(G) \ge d(u) + d(v) + 2$, that is, $S_k(G') - S_k(G) = \sum_{i=1}^{k}\epsilon_i \le 1$ and the assertion follows. $\square$

Theorem 2.1.18. *Let G be a graph with n vertices and $1 \le k \le n-2$. If $S_k(G) \le e(G) + \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$, then $S_{n-k-1}(\overline{G}) \le e(\overline{G}) + \begin{pmatrix} n-k \\ 2 \end{pmatrix}$, where $\overline{G}$ is the complement of G.*

Proof. We have $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ for $i = 1, \cdots, n-1$ therefore,

$$\begin{aligned}
S_{n-k-1}(\overline{G}) &= n(n-k-1) - (\mu_{k+1}(G) + \cdots + \mu_n - 1(G)) \\
&= n(n-k-1) - 2e(G) + (\mu_1(G) + \cdots + \mu_k(G)) \\
&= n(n-k-1) - \begin{pmatrix} n \\ 2 \end{pmatrix} + e(\overline{G}) + (\mu_1(G) + \cdots + \mu_k(G)) - e(G) \\
&\leq e(\overline{G}) + n(n-k-1) - \begin{pmatrix} n \\ 2 \end{pmatrix} + \begin{pmatrix} k+1 \\ 2 \end{pmatrix} \\
&= e(\overline{G}) + \begin{pmatrix} n-k \\ 2 \end{pmatrix}
\end{aligned}$$

as desired. □

The case $k = 2$, in this section, we prove conjecture 2.1.2 for $k = 2$. First we establish the conjecture for graphs with matching number at most three and then we conclude the assertion using lemma 2.2.2.

Lemma 2.1.19. *Let G be a graph with $e(G) = 1$. Then $S_2(G) \leq e(G) + 3$.*

Proof. Let $n = |V(G)|$. Since $e(G) = 1$, it is easily checked that either $G = S_m + (n-m)k_1$ for some $m, 1 \leq m \leq n$ or $G = k_3 + (n-3)k_1$. By lemma 2.1.10, the assertion holds.

we say that a connected has the form $\triangle$ if it has a sub graph H is isomorphic to k_3 such that every edge is incident with some vertex of H. □

Lemma 2.1.20. *Let G be a graph of the form $\triangle$. Then $S_2(G) \leq e(G) + 3$.*

Proof. Let $n = |V(G)|$ and $d_1^T \geq \cdots \geq d_n^T$ be the conjugate degrees of G. If the t is the number of vertices of degree 1 in G, then it is not hard to see that $2(n-$

$t-3) \leq e(G)-t-3$. This implies that $d_2^T = n-t \leq t \leq e(G)-n+3$. Since $d_1^T = n, d_1^T + d_2^T \leq e(G)+3$. By [6][Theorem 7.1], the Grone-Merris conjecture is true for $k=2$. Therefore, $S_2(G) \leq d_1^T + d_2^T \leq e(G)+3$. $\square$

Lemma 2.1.21. *Let G be a graph with $e(G)=2$. Then $S_2(G) \leq e(G)+3$.*

Proof. By lemma 2.1.9 and lemma 2.2.1, we may assume that G is connected graph with at least seven vertices. First suppose that G has a sub-graph $H = k_3$ with $V(H) = \{u,v,w\}$. If every edge of G at least one endpoint in $V(H)$, then by lemma 2.1.20, we are done. Hence assume that there exists an edge $e = \{a,b\}$ whose endpoints are in $V(G)\backslash V(H)$. Let $M = V(G)\backslash\{a,b,u,v,w\}$. Since $m(G)=2$, there are no edges between $V(H)$ and M. Since $|M| \geq 2$, it is easily seen that all vertices in M are adjacent to one of the endpoints of e say a. Hence there are no edges between b and $V(H)$. Now by ignoring the edges between a and $V(H)$, we find a sub-graph K of G which is a disjoint union of k_3 and a star with the centre a. Since the graph $L = G - E(k)$ is a star, corollary 2.1.16 yields that $S_2(G) \leq S_2(k) + S_2(L) \leq (e(k)+1) + (e(L)+2) = e(G)+3$, as required.

Next assume that G has no k_3 as a sub-graph. Suppose that $e_1 = \{a_1,b_1\}$ and $e_2 = \{a_2,b_2\}$ are two independent edges in G. Since G contains no $3k_2$ and k_3 as sub-graphs, $M = V(G)\backslash\{a_1,b_1,a_2,b_2\}$ is an independent set and at least one of the two endpoints of e_i has no neighborhood in M for $i=1,2$. Assume those endpoints to be b_1 and b_2. If b_1 and b_2 are adjacent, then $|M| \geq 2$ yields that all vertices in M are adjacent to only one of the two vertices a_1 and a_2, say a_1. This implies that G is a bipartite graph with the vertex set partition $\{\{a_1,b_2\}, V(G)\backslash\{a_1,b_2\}$ and so lemma 2.2.4 yields the assertion. Now assume that b_1 and b_2 are not adjacent. If a_1 and a_2 are adjacent, then G is a tree and we are done by theorem 2.6.1. Otherwise, G is a bipartite graph with vertex set partition $\{\{a_1,a_2\}, V(G)\backslash\{a_1,a_2\}\}$ and using lemma 2.2.4, the proof is complete. $\square$

Lemma 2.1.22. *Let G be a graph with $e(G) = 3$. Then $S_2(G) \leq e(G) + 3$.*

Proof. By lemma 2.1.9 and lemma 2.2.1, we may assume that G is connected graph with at least nine vertices. using lemma 2.2.2, we may suppose that G has no sub-graph H with $S_2(H) \leq e(H)$. In particular, lemma 2.1.10 implies that G has no sub-graph $3S_3$. Suppose that G has a sub-graph $k = k_3 + 2k_2$. Let $x \in V(G) \backslash V(k)$. Since $e(G) = 3$, the vertex x is not incident with the sub-graph k_3 of k and so G has a sub-graph $H = k_3 + S_3 + k_2$. Now by lemma 2.1.10, we have $S_2(H) = e(H)$ and therefore G has no sub-graph $K_3 + 2K_2$. Let $e_1 = \{a_1, b_1\}, e_2 = \{a_2, b_2\}$ and $e_3 = \{a_3, b_3\}$ be three independent edges in G. Since $e(G) = 3, M = V(G) \backslash V(\{e_1, e_2, e_3\})$ is an independent set. Since G has no $4K_2$ and $k_3 + 2K_2$ as sub-graphs, either $N(a_i) \cap M = \{\phi\}$ or $N(b_i) \cap M = \{\phi\}$, for $i = 1, 2, 3$. With no loss of generality, we may assume that $N(M) \subseteq \{a_1 a_2, a_3, \}$. We consider the following cases.

Case 1. $|N(M)| = 3$. We have $N(M) = \{a_1, a_2, a_3\}$. Since G has no $3S_3$ the bipartite sub-graph $G - \{b_1, b_2, b_3\}$ has no perfect matching. By Hall's theorem, there exists a subset of $\{a_1, a_2, a_3\}$ with 2 elements, say $\{a_2, a_3\}$, such that $|N(\{a_2, a_3\}) \cap M| = 1$. This means that there exists exactly one vertex $y \in M$ which is adjacent to both a_2 and a_3. If $d(b_1) \geq 2$, then we clearly find a sub-graph isomorphic to $3S_3$ in G, a contradiction. Therefore $d(b_1) = 1$. Suppose that H is the star with centre a_1 and $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3, y\}$. Then $G - E(H)$ is a disjoint union of a star S with center a_1 and a graph K containing P_5 with the vertex set $\{a_2, a_3, b_2, b_3, y\}$. Using Theorem 2.1.13, we have $\mu_1(P_5) \leq 4$ and by lemma 2.1.10, we obtain that $\mu_1(k) \leq e(k)$. This yields that $S_2(G - E(H)) + 1$. Thus $S_2(G) \leq S_2(H) + S_2(G - E(H)) \leq e(G) + 3$, as desired.

Case 2. $|N(M)| = 2$ without loss of generality, assume that $N(M) = \{a_1, a_2\}$. Since $m(G) = 3, b_1$ is not adjacent to b_2. if b_1 is adjacent to a_3 or b_3, then changing the role of e_1, e_2, e_3 by three independent edges $\{a_1, z\}, e_2, e_3$ for some vertex $z \in M \cap N(a_1)$, we have case 1. Therefore, we may assume that b_1, similarly b_2 is adjacent to none of the vertices a_3 and b_3. Let H be the induced sub-graph on $\{a_1, a_2, a_3, b_3\}$.

First assume that H has a sub-graph $L = k_3$. if $\{a_1, a_2\}$ is an edge of L, then clearly any edge of G is incident with L and by lemma 2.1.20, there is nothing to prove. Now assume that exactly one of the two vertices a_1 and a_2, say a_1, is a vertex in L. Let k be the disjoint union of L and the induced sub-graph of G on $\{a_2, b_2\} \cup (N(a_2) \cap M)$ which is a star with at least three vertices. Note that $G - E(k)$ is a star or a disjoint union of two stars. Now by lemma 2.1.10 and corollary 2.1.16, $S_2(G) \leq S_2(k) + S_2(G - E(k)) = (e(k)+1) + (e(G-E(k))+2) = e(G) + 3$ as required.

Next suppose that H has no k_3 as a sub-graph. Let $t = d(a_3) + d(b_3)$. we have $t \in \{3, 4\}$. It is not hard to see that $G - e_3$ contains two disjoint stars S_t with centers a_1 and a_2. Therefore, by theorem 2.1.12, $\mu_2(G - e_3) \geq \mu_2(S_t) = t$ using lemma 2.1.17, and lemma 2.1.21, we find that $S_2(G) \leq S_2(G - e_3) + 1 \leq (e(G - e_3) + 3) + 1 = e(G) + 3$, as required.

Case 3. $|N(M)| = 1$. Without loss of generality, assume that $N(M) = \{a_1\}$. If $d(b_1) \geq 2$, then we clearly find three independent edges e'_1, e'_2, e'_3 in G such that the set $M' = V(G) \backslash V(\{e'_1, e'_2, e'_3\})$ is an independent set and $|N(M')| \geq 2$ which is dealt with the previous cases. Hence we assume that $d(b_1) = 1$. Suppose that H is a star with centre a_1 and the vertex set $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3\}$. Then $G - E(H)$ is a disjoint union of a star S with centre a_1 and a graph L containing $2k_2$ with $V(L) = \{a_2, a_3, b_2, b_3\}$. First assume that $L \neq P_4$. Using lemma 2.1.10(ii)and lemma 2.1.11, we have $\mu_1(L) \leq e(L)$. This yields that $S_2(G - E(H) \leq \mu_1(s) + \mu_1(L) \leq e(G - E(H)) + 1$. Thus $S_2(G) \leq S_2(H) + S_2(G - E(H)) \leq e(G) + 3$, as desired. Next assume that $L = P_4$.

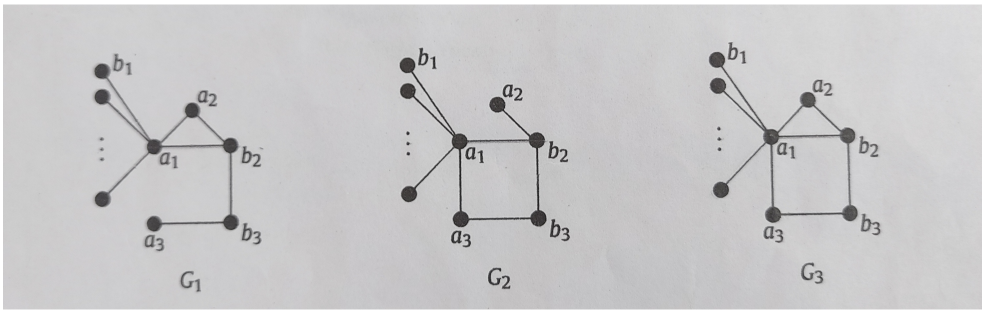


Figure 2.1:

With no loss of generality, suppose that L is a path $a_2 - b_2 - b_3 - a_3$. If $|N(a) \bigcap L| = 1$, then G is a tree and the assertion follows from theorem 2.6.1. If a_1 is adjacent to both b_2 and b_3, then by lemma 2.1.20, there is nothing to prove. Suppose that a_1 is adjacent to none of b_2 and b_3. If we let k be a disjoint union of star $G - V(L)$ and the edges $\{a_2, b_2\}$ and $\{a_3, b_3\}$, then the graph $G - E(K)$ is a disjoint union of a star with the centre a_1 and the edge $\{b_2, b_3\}$. Now, by lemma 2.1.10 and corollary 2.1.16, we have $S_2(G) \leq S_2(K) + S_2(G - E(K)) \leq (e(K) + 1) + (e(G - E(K) + 2) = e(G) + 3$. If none of the above cases occurs, then G is one of the following forms:

If $G = G_1$, then by theorem 2.1.14, we have $\mu_2(G) \geq 3$. Since $d(a_3) + d(b_3) = 3$, applying lemma 2.1.17 for the graph $G - e_3$ and using lemma 2.1.21, we find that $S_2(G) \leq S_2(G - e_3) + 1 \leq (e(G - e_3) + 3) + 1 = e(G) + 3$, as required. Hence assume that $G = G_2$ or $G = G_3$. First suppose that $\mu_2 \geq 4$. Since $d(a_3) + d(b_3) = 4$, applying lemma 2.1.17 for the graph $G - e_3$ and using lemma 2.1.21, the result follows. Now suppose that $\mu_2 < 4$. By theorem 2.1.13, we have $\mu_1(G_2) \leq |V(G_2)| - 1 = e(G_2) - 1$ and by lemma 2.1.11, $\mu_1(G_3) \leq |V(G_3)| = e(G_3) - 1$. Therefore, $S_2(G) < (e(G) - 1) + 4 = e(G) + 3$. this completes the proof. □

Theorem 2.1.23. *If $\mu_n \in (0, 1 - \gamma]$ for some $\gamma \in (0, 1)$, then there are $\epsilon > 0, \delta > 0$, and n_0 such that for all $n \geq n_0$, we have*

$$k(1+\epsilon)\mu_n n < \mu_n(1-\delta)\binom{n}{2} + \binom{k+1}{2},$$

for all $k \in R$.

Lemma 2.1.24. *Assume* $\{G_1, G_2, G_3, ...\}$*are independent random graphs as in condition (Let* $\{G_1, G_2, G_3, \dots\}$ *be a sequence of random weighted graph with* n *vertices and Laplacian matrix given by* $L_n =$

$$\begin{pmatrix} \sum\limits_{j\neq 1} \xi_{1j}^{(n)} & -\xi_{12}^{(n)} & \dots & \xi_{1n}^{(n)} \\ -\xi_{21}^{(n)} & \sum\limits_{j\neq 2} \xi 2j^{(n)} & \dots & -\xi_{2n}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ -\xi_{n1}^{(n)} & \dots & & \sum\limits_{j\neq n} \xi_{nj}^{(n} \end{pmatrix}$$

where for $i < j$ *we have that* $[\xi_{ij}^{(n)}]$ *are bounded random variables on the same probability space and independent for each* n *with* $\xi_{ij}^{(n)} = \xi_{ji}^{(n)}, E[\xi_{ij}^{(n)}] = \mu_n$ *and* $var[\xi_{ij}^{(n)}] = \sigma_n^2$, *and*

$$\sup_{i}^{,j,n} E[|(\xi_{ij}^{(n)} - \mu_n)/\sigma_n|^p] \leq \infty)$$

if $lim_{n\to\infty} \frac{\mu_n}{\sigma_n}(\frac{n}{logn})^{\frac{1}{2}} = \infty$ *and* $\mu_n > 0$, *then*

if $\sigma_n^2 \frac{logn}{\mu_{nn}} \longrightarrow 0$ *as* $n \longrightarrow \infty$, *then there are* $\epsilon > 0, \delta > 0$, *and* n_0 *such that for all* $n \geq n_0$ *we have*

$$k(2+\epsilon)\sigma_n\sqrt{nlogn}\mu_n(1-\delta)\binom{n}{2} + \binom{k+1}{2}$$

Lemma 2.1.25. *If* G *is a graph with* n *vertices and* $H_1, H_2, \dots, H_t$ *are its edge disjoint sub-graphs with* $E(G) = \bigcup\limits_{i=1}^{t} E(H_i)$, *then for each integer* k *with* $1 \leq k \leq n$,

$$S_k(G) \leq \sum_{i=1}^{t} S_k(H_i), \text{ where } S_k(H_i) = S_{n_i}(H_i) \text{ if } k > |V(H_i)| = n_i.$$

Theorem 2.1.26. *Let G be a graph of order n. If $\alpha(G) \geq \frac{(3n-1)}{4}$, then $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for $\frac{(n-1)}{2} \leq k \leq n$.*

Corollary 2.1.27. *Let G be a graph of order n with p pendent vertices. if $p \geq \frac{(3n-1)}{4}$, then $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for $\frac{(n-1)}{2} \leq k \leq n$.*

Corollary 2.1.28. *Let G be a graph of order n. If $\omega(G) \geq \frac{(3n-1)}{4}$, then $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for $1 \leq k \leq \frac{(n-1)}{2}$.*

Proof. Let $\overline{G}$ be the complement of a graph G. Noting that

$$\alpha(\overline{G}) = \omega(G) \geq \frac{(3n-1)}{4},$$

and using theorem 2.1.26, we get

$$S_j(\overline{G}) \leq e(\overline{G}) + \binom{j+1}{2} \tag{2.4}$$

holds for $\frac{(n-1)}{2} \leq j \leq n$. Now for $1 \leq k \leq \frac{(n-1)}{2}$, we have $\frac{(n-1)}{2} \leq n-k-1 \leq n-2$, and consequently,

$$\begin{aligned} S_k(G) &= \sum_{i=1}^{k} \mu_i(G) \\ &= \sum_{i=1}^{k} (n - \mu_{n-i}(\overline{G})) \end{aligned}$$

If G is a graph with n vertices and $\overline{G}$ is its compliment, then $\mu_n(\overline{G}) = 0$ and $\mu_i(\overline{G}) = n - \mu_i(G), i = 1, 2, \ldots, n-1$

$$\begin{aligned} \mu_i(\overline{G}) &= nk - 2e(\overline{G}) + S_{n-k-1}(\overline{G}) \\ &\leq nk - e(\overline{G}) + \binom{n-k}{2} \\ &= e(G) - \binom{n}{2} + nk + \binom{n-k}{2} \\ &= e(G) + \binom{k+1}{2}, \end{aligned}$$

□

Theorem 2.1.29. *Split graphs are spectrally threshold dominated.*

Theorem 2.1.30. *Disjoint unions and compliments of spectrally threshold dominated graphs are spectrally threshold dominated.*

Corollary 2.1.31. *Co-graphs are spectrally threshold dominated.*

Theorem 2.1.32. *A graph G satisfies Brouwer's conjecture if and only if it is spectrally threshold dominated.*

Quite likely this relation to threshold graphs has been motivation for Brouwer's conjuncture. Recognizing this equivalence also opens the door to previous, rather different proofs of theorem 2.1.29, theorem 2.1.30 and corollary 2.1.31, by [12], who proved that Brouwer's conjecture holds in these cases. Establishing Brouwer's conjucture would prove the Laplacian energy conjucture in the non connected case.The requirement of spectral threshold dominance is, however, stronger than needed for the Laplacian energy conjecture. A counter example for Brouwer's conjecture might not be sufficient to disprove the Laplacian energy conjecture. On the other hand, a counter example for the non connected Laplacian energy conjecture would immediately disprove the spectral threshold dominance conjecture and thus also Brouwer's conjecture.

Proof of theorem 2.1.32: Note that by the Grone-Merris Bai-theorem Brouwer's conjecture is equivalent to $\sum\limits_{i=1}^{k} \lambda_i(G) \leq min\{kn, m + \frac{k(k+1)}{2}, 2m\}$ holding for $k \in \{1 \ldots n\}$, because no conjugate degree exceeds n and the sum of all eigenvalues is $2m$. Thus the equivalence is proven if for arbitrary $k \in 1 \ldots n$ we show $min\{kn, m + \frac{k(k+1)}{2}, 2m\} = max\{\sum\limits_{i=1}^{k} d_i^*(T)$: T threshold graph on n nodes and m edges.$\}$ Depending on the relation between k, n and m, we discern the following cases.

Case 1: $\min\{kn, m+\frac{k(k+1)}{2}, 2m\} = kn$: Consider the threshold graph T constructed by filling up Ferrers diagram below the diagonal in column-wise order (on and above the diagonal in corresponding row-wise order).The first k columns below the diagonal are fully filled because they require $kn - \frac{k(k+1)}{2} \leq m$ boxes. Hence T satisfies $d_i^* = n$ for $i = 1, \ldots, k$ and $\sum_{i=1}^{k} \lambda_i(T) = \sum_{i=1}^{k} d_i^*(T) = kn$. This is the maximum attainable over all threshold graphs on n nodes.

Case 2: $\min\{kn, m+\frac{k(k+1)}{2}, 2m\} = m+\frac{k(k+1)}{2}$: In this case put $h := \lfloor \frac{m}{k} + \frac{k+1}{2} < n \rfloor$ and $r := \lfloor m + \frac{k(k+1)}{2} - kh < k \rfloor$. Note that this implies $h \geq k+1$. Define a threshold graph T on n nodes with m edges of trace k by the conjugate degrees $d_i^*(T) = \begin{cases} h+1 & i \leq r \\ h & r < i \leq k \end{cases}$ then $\sum_{i=1}^{k} \lambda_i(T) = \sum_{i} d_i^*(T) = m + \frac{k(k+1)}{2}$. This value cannot be exceeded by any threshold graph on n nodes and m edges by Grone-Merris -Bai Majorization theorem, because in the Ferrers diagram of the conjugate degrees up to column k all boxes are used on and above the diagonal, while all possible m boxes are included below the diagonal.

Case 3: $\min\{kn, m+\frac{k(k+1)}{2}, 2m\} = 2m$: put $h := \max h \in \{1, \ldots, n\} : h(h+1) \leq 2m < k$ and $r := 2m - \frac{h(h+1)}{2}, 2m \leq h+1$, then the threshold graph T of trace h with conjugate degrees

$$d_i^*(T) = \begin{cases} h+2 & i \leq r \\ h+1 & r < i \leq h \\ r & i = h+1 \\ 0 & h+1 < i \end{cases}$$

satisfies $\sum_{i=1}^{k} \lambda_i(T) = \sum_{i=1}^{k} d_i^* = 2m$ and this is the maximum attainable over all threshold graphs with m edges.

Theorem 2.1.33. *Let $G \in \Gamma_2$ be a graph of order $n \geq 24$ having clique number $\omega \geq n - \frac{n}{8} = \frac{7n}{8}$. Then*

$$S_k(G) \leq m + \frac{k(k+1)}{2},$$

for all $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Proof. Proof follows by using $\tau \leq n-1$ and $\omega \geq \frac{7n}{8}$. $\square$

Proposition 2.1.34. *$BC_k(G)$ holds for any graph with m edges and n vertices if either $k \geq \frac{2m^{\frac{1}{3}}}{(1-t)^{\frac{2}{3}}}$ or $m \geq \frac{(2n)^{\frac{3}{2}}}{(1-t)}$, where*

$$t = t(G) := \max_{\substack{S \subset V(G) \\ S \neq \emptyset}} \frac{\rho(G[S])}{|S|} < 1$$

is the maximum sub-graph spectral density of G. Thus, any graph G for which $BC_k(G)$ fails to hold has $k < 2m^{\frac{1}{3}}/(1-t)^{\frac{2}{3}}$, $m < (2n)^{\frac{3}{2}}/(1-t)$, and $k > \frac{m}{n}$, so

$$\frac{m}{n} < k < \frac{2((2n)^{\frac{3}{2}})/(1-t)^{\frac{1}{3}}}{(1-t)^{\frac{2}{3}}} = \frac{\sqrt{8n}}{1-t}$$

Recall that the arboricity $\gamma(G)$ is defined to be the smallest r so that G is a union of r forests .

Proposition 2.1.35. *For any graph G with arboricity $\gamma, BC_k(G)$ holds for $k \geq 4\gamma - 1$.*

Proof. Note that, if a graph G with arboricity γ, then decomposing it into forests $T_1, \ldots, T_\gamma$ yields. By theorem 5 of [17]

$$\begin{aligned} \|L(G)\|_k &\leq \sum_i^{\gamma} \|L(T_i)\| \\ &\leq \sum_i^{\gamma} [m_i + (2k-1)] \\ &= m + \gamma(2k-1). \end{aligned}$$

Thus, if $\gamma \leq k(k+1)/(4k-2)$, then the conjectured bound is satisfied.This holds if $k \geq 4\gamma - 1$. For example, planar graphs have arboricity at most 3, whence $BC_k(G)$ holds for $k \geq 11$. $\square$

Corollary 2.1.36. *For any graph G with maximum degree $\triangle, BC_k(G)$ holds for any $k \geq 2\triangle + 3$*

Proof. Since $\gamma \leq [\triangle/2]+1$, we also have that bc holds for $k \geq 2\triangle+3$ by Proposition 2.1.35. $\square$

Lemma 2.1.37. *$BC_k(G)$ holds for any graph G with*

$$\sum_v {d_v}^2 \leq (1 - \frac{2m}{n^2} + \frac{2m^2}{n^4})\frac{8m^2}{n} - \frac{2m}{n^3}.$$

Below, when we refer to the variance of a sequence $\{a_i\}_{i=1}^N$, we mean the variance of the random variable aX, where X takes a uniformly random value in $[N]$.

Theorem 2.1.38. *$BC_k(G)$ holds for any graph G whose degree sequence has variance at most $[\beta(1-\beta)n]^2 - \frac{\beta}{n^2}$, where $\beta = \frac{2m}{n^2}$ is the edge density.*

Proof. The hypothesis yields

$$\begin{aligned}
[\beta^2(1-\beta)n]^2 - \frac{\beta}{n^2} &\geq Var(\{d_v\}) \\
&= \frac{1}{n}\sum_v {d_v}^2 - \frac{1}{n^2}(\sum_v d_v)^2 \\
&= \frac{1}{n}\sum_v {d_v}^2 - \frac{4m^2}{n^2} = \frac{1}{n}\sum_v {d_v}^2 - \beta^2 n^2
\end{aligned}$$

so that

$$\begin{aligned}
\sum_v {d_v}^2 &\leq \beta^2(1-\beta)^2 n^3 + \beta^2 n^3 - \frac{\beta}{n} \\
&= n^3(2 - 2\beta + \beta^2)\beta^2 - \frac{\beta}{n} \\
&= (1 - \frac{2m}{n^2} + \frac{2m^2}{n^4})\frac{8m^2}{n} - \frac{2m}{n^3},
\end{aligned}$$

and the result follows by applying Lemma 2.1.37 . $\square$

Corollary 2.1.39. *Denoting the maximum, average, and minimum degrees by* $\triangle, \overline{d}$, *and* δ, *respectively*

$$\triangle - \delta \leq \frac{2\overline{d}(n-\overline{d})}{n} - 1,$$

for any graph G, *then Brouwer is true for* G.

Proof. Popoviciu's inequality states that $var(\{d_v\}) \leq (\triangle - \delta)^2/4$. Thus, by theorem 2.1.38, since $2\overline{d}(1-\frac{\overline{d}}{n}) - 1 \leq 2\overline{d}(1-\frac{\overline{d}}{n}) - \frac{2}{n^2} = 2\beta(1-\beta)n - \frac{2}{n^2}$,

$$\begin{aligned} var(\{d_v\}) &\leq \frac{(2\beta(1-\beta)n - \frac{2}{n^2})^2}{4} \\ &\leq \beta^2(1-\beta)^2n^2.(1 - \frac{1}{\beta(1-\beta)n^3})^2 \end{aligned}$$

as long as $\beta(1-\beta)n^3 \geq 1$, which is satisfied if the graph is non-empty. Continuing,

$$\begin{aligned} var(\{d_v\}) &\leq \beta^2(1-\beta)^2n^2 - \frac{\beta^2(1-\beta)^2n^2}{\beta(1-\beta)n^3} \\ &\leq \beta^2(1-\beta)^2n^2 - \beta(1-\beta)/n^2 \leq \beta^2(1-\beta)^2n^2 - \frac{\beta}{n^2} \end{aligned}$$

since,$1-\beta \geq \frac{1}{n}$, from which the result follows per theorem 2.1.38. □

Corollary 2.1.40. *If* G *belongs to a class of graphs with* $\triangle + 1 < (2-\epsilon)\overline{d}$ *for any fixed* $\epsilon \geq 0$ *and* $m = o(n^2)$, *then* $BC_k(G)$ *holds for all sufficiently large* n.

Proof.

$$\begin{aligned} \frac{2\overline{d}(n-\overline{d})}{n} - 1 &= 2\overline{d}(1 - \frac{2m}{n^2}) - 1 \\ &= 2\overline{d}(1 - o(1)) - 1 \\ &\leq \overline{d}(2-\epsilon) - 1 > \triangle \geq \triangle - \delta. \end{aligned}$$

□

Proposition 2.1.41. *Suppose G violates $BC_k(G)$. Then*

$$\mid \binom{S_k}{2} / E(G) \mid \leq k\sqrt{2n} - k$$

and

$$\mid \overline{S_k} \cap E(G) \mid \leq k\sqrt{2n} - k$$

That is, there is a split graph G' with blocks S_k and $\overline{S_k}$ which differs from G on a set of at most $k\sqrt{8n} - 2k$ edges, and in particular, the splittance of G satisfies $\sigma(G) < k\sqrt{8n}$.

Proof. Suppose G violates $BC_k(G)$.Then, by corollary 2.1.40,

$$m + \binom{k+1}{2} < ||L(G)||_k \leq m + e(S_k) - e(\overline{S_k}) + ||A(G)||_k,$$

and, when combined with $||A||_k \leq \sqrt{2km}$ and $k > \frac{m}{n}$, we obtain

$$e(S_k) > \binom{k}{2} + k - k\sqrt{2n}$$

and

$$e(\overline{S_k}) < k\sqrt{2n} - k.$$

In particular, G is at most $2k\sqrt{2n} - 2k < k\sqrt{8n}$ edges away from being a split graph with bi-partition $(S_k, \overline{S_k})$. □

Corollary 2.1.42. *$BC_k(G)$ holds if $k \leq \sigma(G)/\sqrt{8n}$, and so in particular holds for all k if $\sigma(G) \geq /\sqrt{2}n^{\frac{3}{2}}$.*

Proof. This is just an application of proposition 2.1.41, with the observation that, by corollary 2.1.39 we can assume that $k \leq \frac{n}{2}$. □

Theorem 2.1.43. *For any graph G,*

$$||L(G)||_k - (m + \binom{k+1}{2}) \leq 2^{\frac{3}{4}} n^{\frac{1}{4}} \sqrt{k(n-k)}$$
$$\leq 2^{\frac{3}{4}} n^{\frac{3}{4}} \sqrt{k} = 0(n^{\frac{5}{4}}).$$

If $t = t(G)$, then

$$||L(G)||_k \leq m + \binom{k+1}{2} + \frac{2^{\frac{3}{2}}}{\sqrt{1-t}}.n,$$

and for bipartite G,

$$||L(G)||_k \leq m + \binom{k+1}{2} + 4n.$$

Proof. Let $k' = min\{k, n-k\}$. Note that, by proposition 2.1.41 and corollary 2.1.39, $m_1, m_2 < k'\sqrt{2n}$. Therefore,

$$\begin{aligned} ||L(G)||_k &\leq m + \binom{k+1}{2} + \sqrt{max\{2(n-k)k'\sqrt{2n}, 2kk'\sqrt{2n}\}} \\ &= m + \binom{k+1}{2} + 2^{\frac{3}{4}} n^{\frac{1}{4}} \sqrt{max\{(n-k)k', kk'\}} \\ &\leq m + \binom{k+1}{2} + 2^{\frac{3}{4}} n^{\frac{1}{4}} \sqrt{k'(n-k')} \\ &= m + \binom{k+1}{2} + 2^{\frac{3}{4}} n^{\frac{1}{4}} \sqrt{k(n-k)} \\ &\leq m + \binom{k+1}{2} + 2^{\frac{-1}{4}} n^{\frac{5}{4}}. \end{aligned}$$

Then, using proposition 2.1.34, $k \leq \sqrt{8n}/(1-t)$ for any G not satisfying $BC_k(G)$, with $t = \frac{1}{2}$ for bipartite graphs . □

Theorem 2.1.44. *If $BC_k(G)$ and $BC_l(G)$ fail to hold ,then*

$$|l-k| < (2n)^{\frac{1}{4}} max\{k, l\}^{\frac{1}{4}} < 2^{\frac{1}{4}} n^{\frac{3}{4}}$$

Proof. By proposition 2.1.41, $m_1, m_2 \leq k\sqrt{2n}$ if $BC_k(G)$ fails to hold. Suppose that G also violates the conjecture $l > k$.Then, since $\frac{s_l}{S_k} \subset \overline{S_k}$,

$$\begin{aligned} m + \binom{l+1}{2} &< ||L||_l \\ &\leq m + e(S_1) - e(\overline{s_l}) + l\sqrt{2n}. \\ &\leq m + e(S_k) + e(S_k, \frac{S_l}{S_k}) + e(\frac{S_l}{S_k}) - e(\overline{S_l}) + l\sqrt{2n} \\ &\leq m + \binom{l}{2} - \binom{l-k}{2} + e(\frac{S_l}{S_k}) - 0 + l\sqrt{2n} \\ &\leq m + \binom{l}{2} - \binom{l-k}{2} + k\sqrt{2n} + l\sqrt{2n} \end{aligned}$$

by proposition 2.1.41

$$= m + \binom{l}{2} - \binom{l-k}{2} + (k+l)\sqrt{2n}.$$

Thus,

$$\begin{aligned} (k+l)\sqrt{2n} &> \binom{l+1}{2} - \binom{l}{2} + \binom{l-k}{2} \\ &= \frac{(l-k)^2 + (l+k)}{2} \end{aligned}$$

So $\sqrt{2n} > 2[(l-k)^2 + (l-k)]/(k+l) > (l-k)^2/l$ i.e., $l - k < (2n)^{\frac{1}{4}} l^{\frac{1}{2}}$.

similarly, suppose that G violates Brouwer's conjecture at $l < k$, Then, since $\frac{\overline{S_l}}{\overline{S_k}} \subset S_k$,

$$\begin{aligned} m + \binom{l+1}{2} &< ||L||_l \\ &\leq m + e(S_l) - e(\overline{S_l}) + l\sqrt{2n} \\ &\leq m + \binom{l}{2} - e(\frac{\overline{S_l}}{\overline{S_k}}) + l\sqrt{2n} \\ &\leq m + \binom{l}{2} - (\binom{k-l}{2} - k\sqrt{2n}) + l\sqrt{2n} \\ &= m + \binom{l}{2} - \binom{k-l}{2} + (k+l)\sqrt{2n}, \end{aligned}$$

from which it follows that $k - l < (2n)^{\frac{1}{4}} k^{\frac{1}{2}}$.Thus, $|l - k| < (2n)^{\frac{1}{4}} max\{k, l\}^{\frac{1}{2}} < 2^{\frac{1}{4}} n^{\frac{3}{4}}$. $\square$

Corollary 2.1.45. *Consider G in the family $s_\omega(H_1, H_2, \dots, H_\omega)$. If $t = 3$ then Brouwer's conjecture holds for all k, except $k = \omega - 1$ or ω. If $t = 4, 5$, then Brouwer's conjecture holds for all k, except $k = \omega - 2$ or $\omega - 1$ or ω or $\omega + 1$.*

Theorem 2.1.46. *Let G belongs to a family $s_\omega(H_1, H_2, \dots, H_\omega), H_i = k_{1,a}$. Then Brouwer's conjecture holds for $k \in [1, \omega - 0.5 - u]$ and $k \in [\omega - 0.5 + u, n$, where $u = \sqrt{2t - 3.75}$*

Proof. Let G be a connected graph as in the hypothesis.Then $n = (a+1)(\omega - 1) + a - 1 + t - 1 = \omega(a+1) + t - 3$ and $m = \frac{\omega(\omega-1)}{2} + a\omega + t - 2$. By definition of conjugated degrees, we have $d_1^* = \omega(a+1) + t - 3, d_2^* \leq \omega + t - 1, d_3^* \leq \omega \dots, d_\omega + a - 1 \leq \omega$ and $d_i^* = 0$, for $i = \omega + a, \dots, n$. Since the Brouwer's conjecture is always true for $k \leq 2$, we assume that $k \geq 3$. For $k \geq 3$, by Grone-Merris-Bai theorem, it follows that

$$
{}_k(G) \leq \sum_{i=1}^{k} d_i^*(G) \leq a\omega + 2t - 4 + k\omega
$$
$$
\leq m + \frac{k(k+1)}{2} = \frac{\omega(\omega-1)}{2} + a\omega + t - 2 + \frac{k(k+1}{2},
$$

provided

$$
k^2 - (2\omega - 1)k + \omega(\omega - 1) - 2t + 4 \geq 0 \tag{2.5}
$$

Consider the function $f(k) = k^2 - (2\omega - 1)k + \omega(\omega - 1) - 2t + 4$, for $k \in [3, n - 1]$. Since $t \geq 3$, it follows that the discriminant $\nabla = 8t - 15$ of the polynomial $f(k)$ is always positive. The roots of this polynomial are

$\alpha = \frac{2\omega-1-\sqrt{8t-15}}{2}$ and $\beta = \frac{2\omega-1+\sqrt{8t-15}}{2}$

This shows that 2.5 holds for all $k \in [3, \alpha]$ and for all $k \in [\beta, n-1]$. This completes the proof. In particular,if $t = 3, 4, 5$, we have the following consequences of this theorem 2.1.46 □

Theorem 2.1.47. *Let G be a graph with n vertices and let $G - e$ be a sub-graph of G obtained by deleting the edge $e \in E(G)$. Then*

$$\mu_1(G) \geq \mu_1(G-e) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) \geq \mu_n(G-e).$$

The graph obtained from G deleting the vertex $v \in V(G)$, denoted by $G - v$, is the graph with $V(G - v) = V(G) - \{v\}$ and $E(G - v) = E(G) - \{uv|uv \in E(G)\}$. We enunciate the interlacing theorem for the Laplacian eigenvalues of G and $G - v$ below [21]

Theorem 2.1.48. *If the $j-th$ inequality of the Brouwers conjecture is not valid for a graph G with n vertices and $\mu_1(G) \leq j + 1$. Then the Brouwer's conjecture is not valid for $G - e$, where $e \in E(G)$.*

Proof. Suppose that the $j - th$ inequality of the Brouwer conjecture is not valid for a graph G. Specifically

$$\sum_{i=1}^{j} \mu_i(G) > e(G) + \binom{j+1}{2} \tag{2.6}$$

As the Brouwer's conjecture is proved for $k = 1, 2, \ldots, n-1$ and n, we assume that $3 \leq j \leq n-2$. Summing the $j-1$ largest Laplacian eigenvalues of $G-e$ and using the inequality of theorem 2.1.48, we have

$$\sum_{i=1}^{j-1} \mu_i(G-e) \geq \sum_{i=2}^{j} \mu_i(G) = \sum_{i=1}^{j} \mu_i(G) - \mu_1(G). \tag{2.7}$$

From inequality 2.6 it follows

$$\sum_{i=1}^{j-1} \mu_i(G-e) > e(G) + \binom{j+1}{2} - \mu_1(G)$$
$$= e(G-e) + \binom{j}{2} + [j+1-\mu_1(G)].$$

From the hypothesis we have $j+1-\mu_1(G) \geq 0$, and we conclude that the Brouwer's conjecture is not valid of $G-e$ for the sum of the eigenvalues upto $j-1$. Below we present the result associated with the deletion of a vertex with any degree. $\square$

Theorem 2.1.49. *Let G be a graph with n vertices and v a vertex with $d_v \leq n-5$. If the $(j+d_v)-th$ inequality of the Brouwer's conjecture is not valid for G and*

$$\mu_1(G) + \cdots + \mu_{d_(v)}(G) \leq \frac{d_v(2j+d_v+3)}{2}.$$

Then the Brouwer's conjecture is not valid for $G-v$

Corollary 2.1.50. *Let G be a graph with n vertices and v a vertex with $d_v = 1$. If the $j-th$ inequality of the Brouwer's conjecture is not valid for G and $\mu_1(G) \leq j+1$. Then the Brouwer's conjecture is not valid for $G-v$.*

Theorem 2.1.51. *For the graph $G \in C_\omega(a, a, \ldots, a), a \geq 1$, Brouwer's conjecture holds for all $k, C = 0$. If $C = 1$, Brouwer's conjecture holds for all $k, k \neq \omega+1$, provided $a \leq \omega+1$. If $C = 2$, Brouwer's conjecture holds for all $K, k \notin \{\omega+1, \omega+2\}$, holds for $k = \omega+1$, provided $a \leq \omega+2$ and holds $k = \omega+2$, provided $a \leq 2\omega + \frac{1}{2}$. If $C \geq 3$, Brouwer's conjecture holds for all k, provided $a \leq \omega - \frac{1}{2} + \sqrt{(2c-1)\omega}$.*

Example 2.1.52. *Let $G = S_5$ be the star with $n = 5$ vertices .The Laplacian eigenvalues of S_5 are $\{5,1,1,1,0\}$. The degree sequence d(G) is $\{1,1,1,1,4\}$, it is clear that the conjugate degree sequence $d^*(G)$ of this graph is $\{5,1,1,1,0\}$. So the equality holds for all $k = 1, ..., 5$ in the Grone- Merris conjecture (theorem).*

2.2 Connected Graphs

Lemma 2.2.1. *Let G be a graph.Then either $S_2(G) = S_2(H)$ for a connected component H of G or $S_2(G) \leq e(G) + 2$.*

Proof. If the first statement does not hold, then G has two connected components H_1 and H_2 such that $\mu_1(G) = \mu_1(H_1)$ and $\mu_2(G) = \mu_1(H_2)$. By lemma 2.1.11, we have $\mu_1(H_i) \leq |V(H_i)| \leq e(H_i) + 1$ for $i = 1, 2$. Therefore, $S_2(G) \leq (e(H_1) + 1) + (e(H_2) + 1) \leq e(G) + 2$. □

The next lemma is the key to our approach. Because of this result, it suffices to consider only a very restrictive class of graphs.

Lemma 2.2.2. *If conjecture 2.1.2 is false for $k = 2$, then there exists a counterexample G for which $S_2(H) > e(H)$ for every sub-graph H of G.*

Proof. Let G be a counterexample for conjecture 2.1.2 with $k = 2$ having a minimum number of edges. If G has a sub-graph H that satisfies $S_2(H) \leq e(H)$, then corollary 2.1.16 gives $e(G)+3 < S_2(G) \leq S_2(H)+S_2(G-H)$. This implies that $S_2(G-H) > e(G-H) + 3$, which contradicts the minimality of $e(G)$. □

Corollary 2.2.3. *Let G be a connected graph with n vertices. Then for an integer k with*

$$\frac{(3n-4) + \sqrt{8n^2 e(G) - 8n^3 + 9n^2 - 8n + 16}}{2n} \leq k \leq n$$

we have

$$S_k(G) \leq e(G) + \binom{k+1}{2}$$

Proof. Note that

$$\frac{(3n-4) + \sqrt{8n^2 e(G) - 8n^3 + 9n^2 - 8n + 16}}{2n} \leq k \leq n$$

which implies that

$$e(G) \leq n + \frac{2k-2}{n} + \frac{k^2-3k}{2}.$$

Then by theorem(Let G be a connected graph with n vertices, and let K be an integer with $1 \leq k \leq n$. then we have

$$S_k(G) \leq e(G) + 2k - n - \frac{2k-2}{n}.)$$

it is not difficult to obtain that

$$S_k(G) \leq 2e(G) + 2k - n - \frac{2k-2}{n}$$

$$\leq e(G) + \binom{k+1}{2}$$

this completes the proof. □

Lemma 2.2.4. *Let $n \geq 3$ and let G be a connected spanning sub-graph of $K_{2,n-2}$. Then $S_2(G) \leq e(G) + 3$.*

Proof. Assume that $\{\{v, w\}, B\}$ is the partition of $V(G)$. For simplicity, we write $\mu_i(G) = \mu_i$ for $1 \leq i \leq n$. Let $d_1 \geq \cdots \geq d_n$ be the vertex degrees of G and let r and s be the number of vertices of degree 1 and 2 respectively. By theorem 2.6.1 we can assume that G is not a tree. Hence $s \geq 2$ and the degrees $d_1, d_2 \geq 2$ are the degrees of v and w. It is easily seen that s rows of $2I - \mathcal{L}$ are identical and therefore the multiplicity of 2 as an eigen value of $\mathcal{L}(G)$ is at least $s-1$. Similarly, the multiplicity of 1 as eigen values of $\mathcal{L}(G)$ is at least $s-1$. Similarly, the multiplicity of 1 as eigen values of $\mathcal{L}(G)$ is at least $r-2$. If $\mu_2 \leq 2$, then lemma 2.1.11 implies that $\mu_1 + \mu_2 \leq n + 2 < e(G) + 3$. Hence we may assume that $\mu_2 > 2$ and so $\mu_1 \geq \mu_2 \geq \mu_a \geq \mu_b \geq \mu_n = 0$ are the five remaining eigen values. By trace $\mathcal{L}(G) = \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} d_i$, we have $\mu_1 + \mu_2 + \mu_a + \mu_b \leq d_1 + d_2 + 4$. Finally, by the interlacing theorem [7][p.193] for the $(n-3)(n-2)$ sub-matrix $D = diag(1, \cdots, 1, 2, \cdots, 2)$ of $\mathcal{L}(G)$, we find that $\mu_a \geq \mu_{n-2} \geq \lambda_{n-2}(D) \geq 1$, hence $\mu_1 + mu_2 \leq d_1 + d_2 + 4 - \mu_a - \mu_b \leq d_1 + d_2 + 3 = e(G) + 3$. □

Theorem 2.2.5. *Let G be a connected graph of order $n \geq 2$ with m edges and let ω be the clique number and τ be the vertex covering number of G.Then*

$$S_k(G) \leq K(\tau+1) + m - \frac{\omega(\omega-1)}{2}, \tag{2.8}$$

with equality if and only if $G \cong K_n$.

Proof. let G be a graph with clique number ω and vertex covering number τ and let $c = \{v_1, v_2, \ldots, v_r\}$ be a minimum vertex covering set in G. Since the clique number of G is ω, it follows that K_ω is a sub-graph of G. Also, the vertex covering number of a complete graph on ω vertices is $\omega - 1$. Let $v_1, v_2, \ldots, v_\omega - 1$ be a the vertices in c, which belongs to $v(K_w)$. The Laplacian spectrum of K_w is $\{\omega^{[\omega-1]}, 0\}$, therefore by lemma (let A and B be two real symmetric matrices of order n. Then for any $1 \leq K \leq n$,

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B),$$

where $\lambda_i(X)$ is the $I-th$ eigenvalue of X) we have

$$\begin{aligned} S_k(G) = \sum_{i=1}^{k} \mu_i(G) &\leq \sum_{i=1}^{k} \mu_i(K_\omega) + \sum_{i=1}^{k} \mu(G \backslash K_\omega) \\ &= S_K(K_\omega) S_K(G \backslash K_\omega) \leq K(\omega) + S_k(G \backslash K_\omega), \end{aligned}$$

where $G \backslash K_\omega$ is a graph obtained from G by removing the edge of K_ω. in order to establish the result, we need to estimate $S_K(G \backslash K_\omega)$. let $G_\omega, G_\omega + 1, \ldots, G_\tau$ be the spanning sub graph of $H = G \backslash K_\omega$ corresponding to the vertices $v_\omega, v_\omega + 1, \ldots, v_\tau$ of c, having the vertex set same as H and the edge set defined as follows.

$$\begin{aligned} E(G_\omega) &= v_\omega(v_t) : v_t \in N(v_\omega) \backslash \{v_1, v_2, \ldots, v_\omega - 1\} \\ E(G_\omega + 1) &= v_\omega + 1(v_t) : v_t \in N(v_\omega + 1) \backslash \{v_1, v_2, \ldots, v_\omega - 1\}, \end{aligned}$$

and in general

$$E(G_i) = v_i v_t : v_t \in N(v_i) \backslash \{v_1, v_2, \ldots, v_i - 1\}, i = \omega, \omega + 1, \ldots, \tau$$

For $i = \omega, \omega+1, \ldots, \tau$, let $m_i = |E(G_i)|$. clearly $E(H) = E(G_\omega) \bigcup E(G-\omega+1) \bigcup \cdots \bigcup E(G_\tau)$ and $G_i = k_{1,m_i} \bigcup (n(H)-m_i-1)k_1$, for all $i = \omega, \omega+1, \ldots, \tau$. Also it is clear that

$$L(H) = L(G_\omega) + L(G_\omega + 1) + \cdots + L(G_\tau) \tag{2.9}$$

The Laplacian spectrum of $G_i = k_{1,m_i} \bigcup (n(H)-m_i-1)k_1$ is $\{m_i+1, 1^{[n(G_i)-2]}, 0^{[n(H)-m_i]}\}$. Therefore,

$$S_k(G_i) \leq m_i + k, forall i = \omega, \omega+1, \ldots, \tau. \tag{2.10}$$

Now, applying theorem 2.1.15 to equation 2.9 and 2.10 and the fact that $\sum\limits_{i=\omega}^{\tau} m_i = m(H) = m - \frac{\omega(\omega-2)}{2}$, we have

$$\begin{aligned}
S_k(H) &= \sum_{j=1}^{k} \mu_j(H) \leq \sum_{i=\omega}^{\tau} \sum_{j=1}^{k} \mu_j(G_i) = \sum_{i=\omega}^{\tau} s_k(G_i) \\
&\leq \sum_{i=\omega}^{\tau} m_i + k = m - \frac{\omega(\omega-1)}{2} + (\tau - \omega + 1)k.
\end{aligned}$$

This shows that

$$S_K(G \backslash k_\omega) = S_K(H) \leq m - \frac{\omega(\omega-1)}{2} + (\tau - \omega + 1)k.$$

Therefore, it follows that

$$\begin{aligned}
S_k(G) &\leq K(\omega) + S_K(G \backslash K_\omega) \leq K\omega + m - \frac{\omega(\omega-1)}{2} + (\tau - \omega + 1)k \\
&= K(\tau+1) + m - \frac{\omega(\omega-1}{2}.
\end{aligned}$$

Equality occurs in 2.8 if all the inequalities above occurs as equalities. As a graph G is connected, the equality in inequality $S_K(G) \leq S_K(K_\omega) + S_K(G \backslash K_\omega)$ can only occur if and only if $G \cong K_n$.

□

Theorem 2.2.6. *Let G be a connected graph of order n with m edges having diameter d and vertex covering τ. Then*

$$S_k(G) \leq (\tau - \lfloor \frac{d}{2} \rfloor + 2)k + m - d + cos(\frac{k\pi}{d}) + \frac{cos(\frac{\pi}{d})sin(\frac{k\pi}{d}) + sin(\frac{k\pi}{d})}{sin(\frac{\pi}{d})} \tag{2.11}$$

with equality if and only if $G \cong P_n$.

Proof. Let G be a graph with diameter d and vertex covering number τ and let $C = \{v_1, v_2, \ldots, v_\tau\}$ be a minimum vertex covering set in G. Since the diameter of G is d, it follows that P_d is a sub graph of G. Also, the vertex covering number of a path graph P_d is $\lfloor \frac{d}{2} \rfloor$. Let $v_1, v_2, \ldots, v_{\lfloor \frac{d}{2} \rfloor}$ be the vertices in C, which belong to $V(P_d)$. The Laplacian spectrum of p_d is $\{2 - 2cos(\frac{\pi j}{d}), 0 : j = 1, 2, \ldots, d-1\}$. If we remove the edges of p_d from G, then the Laplacian matrix of G can be decomposed as $L(G) = L(P_d \bigcup (n-d-1)k_1) + L(G \backslash P_d)$, where $G \backslash P_d$ is a graph obtained from G by removing the edges of P_d. Applying lemma (let A and B be two real symmetric matrices of order n. Then for any $1 \leq k \leq n$,

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B),$$

where $\lambda_i(X)$ is the $i - th$ eigenvalue of X) and using the fact that $S_k(P_d \bigcup (n-d-1)k_1) = S_k(P_d)$, we have

$$\begin{aligned}
S_k(G) = \sum_{i=1}^{k} \mu_i(G) &\leq \sum_{i=1}^{k} \mu_i(P_d) + \sum_{i=1}^{k} \mu_i(G \backslash P_d) = S_k(P_d) + S_k(G \backslash P_d) \\
&= \sum_{j=0}^{k-1} (2 - 2cos(\frac{\pi(d-j-1)}{d})) + S_k(G \backslash p_d) \\
&= 2k + cos(\frac{k\pi}{d}) + \frac{cos(\frac{\pi}{d}) sin(\frac{k\pi}{d}) + sin(\frac{k\pi}{d})}{sin(\frac{\pi}{d})} - 1 + S_k(G \backslash P_d),
\end{aligned}$$

where we have used the well known inequality

$$\sum_{j=0}^{k-1} cos(nj) = \frac{sin(nk)cos(n) + sin(nk)}{2sin(n)} - \frac{1}{2} cos(nk) + \frac{1}{2}.$$

In order to establish the result, we need to estimate $S_k(G \backslash P_d)$. Let $G_{\lfloor \frac{d}{2} \rfloor} + 1, G_{\lfloor \frac{d}{2} \rfloor} + 2, \ldots, G_\tau$ be a spanning sub-graph of $H = G \backslash p_d$ corresponding to vertices $v_{\lfloor \frac{d}{2} \rfloor} +$

$1, v_{\lfloor \frac{d}{2} \rfloor + 2}, \ldots, v_\tau$ of C, having vertex set same as H and edge set defined as follows as

$$E(G_i) = \{v_i v_t : v_t \in N(v_i) \backslash \{v_1, v_2, \ldots, v_i - 1\}\}, i = \lfloor \frac{d}{2} \rfloor + 1, \lfloor \frac{d}{2} \rfloor + 2, \ldots, \tau.$$

Now proceeding as in Theorem 2.2.5, we have we obtain

$$S_k(G \backslash P_d) \leq k(\tau - \lfloor \frac{d}{2} \rfloor) + m - d + 1$$

Therefore, from above we have

$$\begin{aligned} S_k(G) &\leq 2k + cos(\frac{k\pi}{d}) + \frac{cos(\frac{\pi}{d})sin(\frac{k\pi}{d}) + sin(\frac{k\pi}{d})}{sin(\frac{\pi}{d})} - 1 + s_k(G \backslash P_d) \\ &\leq (\tau - \lfloor \frac{d}{2} \rfloor + 2)k + m - d + cos(\frac{k\pi}{d}) + \frac{cos(\frac{\pi}{d})sin(\frac{k\pi}{d}) + sin(\frac{k\pi}{d})}{sin(\frac{\pi}{d})}, \end{aligned}$$

and the result follows. Equality occurs in 2.11 if all the inequalities above occurs as equalities. since G is connected, the equality in the inequality $S_k(G) \leq S_k(P_d) + s_k(G \backslash P_d)$ can only occur if and only if $G \cong P_n$. □

Theorem 2.2.7. *Let G be a connected graph of order $n \geq 2$ with m edges having vertex covering number τ. If $k_{s_1,s_2}(s_1 \leq s_2)$ is the maximal complete bipartite subgraph of G, then*

$$S_k(G) \leq k(\tau + s_2 - s_1) + m - s_1(s_2 - 1) \tag{2.12}$$

with equality if and only if $G \cong k_{s_1,s_2}$ with $s_1 + s_2 = n$.
If $s_1 = s_2$, it is easy to see that the upper bound 2.12 is always better than the upper bound ($S_k(G) \leq m + k\tau$.)

Theorem 2.2.8. *For a connected graph G of order $n \geq 24$ having clique number $\omega \geq \frac{7n}{8}$, we have*

$$S_k(G) \leq m + \frac{k(k+1)}{2},$$

for all $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Proof. Let G be a graph of order n having the clique number ω and vertex covering number τ. Then $\tau \leq n-1$. So from 2.8, we have

$$S_k(G) \leq kn + m - \frac{\omega(\omega-1)}{2}. \tag{2.13}$$

Clearly the right hand side of 2.13 is a decreasing function of ω. Therefore, to prove the result it suffices to consider $\omega = \frac{7n}{8}$. For this value of ω, we have from 2.13, $s_k(G) \leq kn + m - \frac{7n(7n-8)}{128} \leq m + \frac{k(k+1)}{2}$, if $2kn \leq k^2 + k + \frac{7n(7n-8)}{64}$, that is,

$$k^2 - (2n-1)k + \frac{7n(7n-8)}{64} \geq 0 \tag{2.14}$$

Now, consider the polynomial

$$f(k) = k^2 - (2n-1)k + \frac{7n(7n-8)}{64}.$$

The roots of this polynomial are

$$\alpha = \frac{(2n-1) + \frac{1}{4}\sqrt{15n^2 - 8n + 16}}{2}$$

and

$$\beta = \frac{(2n-1) - \frac{1}{4}\sqrt{15n^2 - 8n + 16}}{2}$$

This shows that $f(k) \geq 0$, for all $k \geq \alpha$ and $f(k) \geq 0$, for all $k \leq \beta$. Since $\alpha = \frac{(2n-1)+\frac{1}{4}\sqrt{15n^2-8n+16}}{2} > n-1$ and $k \leq n-1$, it follows that $k \geq \alpha$ is not possible. Therefore, we only need to consider $k \leq \beta$. we have

$$15(n - \frac{4}{15})^2 \leq 15((n - \frac{4}{15})^2 + \frac{224}{225}) = 15n^2 - 8n + 16,$$

giving

$$\beta = \frac{(2n-1) - \frac{1}{4}\sqrt{15n^2 - 8n + 16}}{2} \leq \frac{(2n-1) - \frac{\sqrt{15}}{4}(n - \frac{4}{15})}{2}$$
$$= 0.5159n - 0.3708$$

Similarly,

$$15n^2 - 8n + 16 = 15((n - \frac{4}{15})^2 + \frac{224}{225}) < 15((n - \frac{4}{15})^2 + 1),$$

giving

$$\beta = \frac{(2n-1) - \frac{1}{4}\sqrt{15n^2 - 8n + 16}}{2} > \frac{(2n-1) - \frac{1}{4}\sqrt{15((n-\frac{4}{15})^2 + 1)}}{2}$$

$$\geq \frac{(2n-1) - \frac{1}{4}\sqrt{15(n-\frac{4}{15})^2} - \frac{1}{4}\sqrt{15}}{2} = 0.5159n - 0.3708,$$

This shows that $0.5159n - 0.8549 < \beta < 0.5159n - 0,3708$, and the result follows. Let Ω_n be a family of connected graphs for which the clique number ω is one more the vertex covering number τ, that is $\Omega_n = \{G : G$ is a connected of order n with $\omega = \tau + 1\}$. For the family of graphs Ω_n, we have the following observation. □

Theorem 2.2.9. *If $G \in \Omega_n$, then for all $k, 1 \leq k \leq n$,*

$$S_k(G) \leq m + \frac{k(k+1)}{2}.$$

Proof. If $G \in \Omega_n$, then $\omega = \tau + 1$.therefore from (2.62) we have

$$S_k(G) \leq k(\omega) + \frac{\omega(\omega-1)}{2} \leq m + \frac{k(k+1)}{2},$$

if $2k\omega \leq k^2 + k + \omega^2 - \omega$, that is

$$k^2 - (2\omega - 1)k + \omega^2 - \omega \geq 0 \tag{2.15}$$

Now, for the polynomial $f(k) = k^2 - (2\omega - 1)k + \omega^2 - \omega$, the roots are $\omega - 1$ and ω. It follows that $f(k) < 0$, for all $k \in (\omega - 1, \omega)$. since k and ω are integers and there is no integer between $\omega - 1$ and ω, it follows that $f(k) \geq 0$ for all k, that is 2.15 always holds. Thus, the result follows. □

Theorem 2.2.10. *Let G be a connected graph of order $n \geq 2$ with m edges having vertex covering number $\tau \leq 1.3s_1$. If k_{s_1,s_2} is the maximal complete bipartite subgraph of the graph G, then for all $k = 1, 2, \ldots, n$*

$$S_k(G) \leq m + \frac{k(k+1)}{2}.$$

Proof. Using $s_1 = s_2$ in 2.12, we have

$$S_k(G) \leq k\tau + m - s_1(s_1 - 1) \leq m + \frac{k(k+1)}{2},$$

if

$$k^2 - (2\tau - 1)k + 2s_1(s_1 - 1) \geq 0. \tag{2.16}$$

The discriminant of the polynomial $f(k) = k^2 - (2\tau - 1)k + 2s_1(s_1 - 1)$ is $D = (2\tau - 1) - 8s_1(s_1)$. If $\tau \leq 1.3s_1$, it is easy to see that $D < 0$ and so 2.16 holds for k. Thus the result follows. For connected bipartite graphs of order $n \geq 2$, the vertex covering number $\tau \leq \frac{n}{2}$. For the bipartite graphs, we have the following observations. $\square$

Theorem 2.2.11. *Let G be a connected bipartite graph of order $n \geq 2$ with m edges having vertex covering number τ. If k_{s_1,s_1} with $s_1 \geq \frac{n}{4}$ is the maximal bipartite sub-graph of G, then*

$$s_k(G) \leq m + \frac{k(k+1)}{2}.$$

For all $k \leq \frac{n}{7} - 1$ and $k \geq \frac{6n}{7}$.

Proof. Using $s_1 = s_2$ in 2.12 and the fact that $\tau \leq \frac{n}{2}$ for bipartite graphs, we have

$$s_k(G) \leq k\tau + m - s_1(s_1 - 1) \leq k(\frac{n}{2}) + m - s_1(s_1 - 1) \leq m + \frac{k(k+1)}{2},$$

If

$$kn \leq k(k+1) + 2s_1(s_1 - 1). \tag{2.17}$$

The right hand side of 2.17 is an increasing function of s_1. Therefore to prove the assertion, it suffices to consider $s_1 = \frac{n}{4}$. With this value of s_1, from 2.17, we have

$$k^2 - (n-1)k + \frac{n(n-4)}{8} \geq 0. \tag{2.18}$$

The roots of the polynomial $f(k) = k^2 - (n-1)k + \frac{n(n-4)}{8}$ are

$$\alpha = \frac{n - 1 + \sqrt{0.5n^2 + 1}}{2}, \beta = \frac{n - 1 - \sqrt{0.5n^2 + 1}}{2}.$$

This shows that $f(k) \geq 0$, for all $k \geq \alpha$, and $f(k) \geq 0$, for all $k \leq \beta$. Proceeding similarly as in theorem 2.2.8, it can be seen that $\alpha < 0.8535n$ and $\beta > 0.1464n - 1$. From these the result follows □

Theorem 2.2.12. *Let G be a connected graph of order n. If*

$$g(G) \geq 6 + 2\sqrt{8(e(G) - n + 1) + 1}, \tag{2.19}$$

then $S_k(G) \leq e(G) + \binom{k+1}{2}$ holds for $1 \leq k \leq n$

Theorem 2.2.13. *For a split graph S_{n,n_1} and for any $1 \leq k \leq n$,*

$$S_k(S_{n,n_1}) \leq kn_1 + e(S_{n,n_1}) - \frac{n_1(n_1 - 1)}{2}, \tag{2.20}$$

with equality if $1 \leq k \leq n_1 - 1$ and $S_{n,n_1} \cong kS_{n,n_1}$, or if $n_1 \leq k \leq n - 1$ and $S_{n,n_1} \cong CS_{n,n_1}$.

Theorem 2.2.14. *Let $G \in \Gamma_3$ be a connected graph of order $n \geq 2$ with m edges having the vertex covering number $\tau \leq 1.3s_1$. If K_{S_1,S_1} is the maximal complete bipartite sub-graph of the graph G, then for all $k = 1, 2, \ldots, n$*

$$S_K(G) \leq m + \frac{k(k+1)}{2}.$$

Theorem 2.2.15. *Let $G \in \Gamma_3$ be a connected bipartite graph of order $n \geq 2$ with m edges having vertex covering number τ. If K_{S_1,S_1} with $S_1 \geq \frac{n}{4}$ is the maximal complete bipartite sub-graph of graph G, then*

$$S_K(G) \leq m + \frac{k(k+1)}{2},$$

For all $k \leq \frac{n}{7} - 1$ and $k \geq \frac{6n}{7}$.

Theorem 2.2.16. *Let G be a connected graph of order $n \geq 4$ and size m and let $K_{s,s}, S \geq 2$ be the maximal complete bipartite sub graph of graph G. If $H = G \backslash k_{s,s}$ is a graph having r non-trivial components $C_1, C_2, \dots C_r$, each of which is a c - cyclic graph and $p \geq 0$ trivial components, then for $s \geq \frac{5+\sqrt{8r(c-1)+34}}{2}$, Brouwers conjecture holds for all k; and for $s < \frac{5+\sqrt{8r(c-1)+34}}{2}$, Brouwer's conjecture holds for all $k \in [1, y_1]$, where $x_1 = \frac{2s+3+\sqrt{20s-4s^2+8r(c-1)+9}}{2}$ and $y_1 = \frac{2s+3-\sqrt{20s-4s^2+8r(c-1)+9}}{2}$*

Proof. Let G be a connected graph of order n and size m. If $H = G \backslash k_{s_s}$ is graph having r non-trivial components $C_1, C_2, \dots C_r$, each of which is a c-cyclic graph and $p \geq 0$ trivial components, then $m = s^2 + n - p + (c-1)r$. For $1 \leq k \leq n$, from corollary, we have

$$\begin{aligned} S_k(G) &\leq s + k(s+2) + n - p + 2r(c-1) - \alpha < s + k(s+2) + n - p + 2r(c-1) \\ &\leq m + \frac{k(k+1)}{2} \\ &= s^2 + n - p + r(c-1) + \frac{k(k+1)}{2} \end{aligned}$$

if

$$k^2 - (2s+3)k - (2s + 2r(c-1) - 2s^2) \geq 0 \tag{2.21}$$

Now consider the polynomial

$$f(k) = k^2 - (2s+3)k - (2s + 2r(c-1) - 2s^2)$$

The discriminant of this polynomial is $d = 20s - 4s^2 + 8r(c-1) + 9$. We have $d \leq 0$. If $20s - 4s^2 + 8r(c-1) + 9 \leq 0$, which gives $s \geq \frac{5+\sqrt{8r(c-1)+34}}{2}$. This shows that for $s \geq \frac{5+\sqrt{8r(c-1)+34}}{2}$ the Inequality 2.21 and so the Brouwer's conjecture always holds. For $s < \frac{5+\sqrt{8r(c-1)+34}}{2}$ the roots of the polynomial $f(k)$ are $x_1 = \frac{2s+3+\sqrt{20s-4s^2+8r(c-1)+9}}{2}$ and $y_1 = \frac{2s+3-\sqrt{20s-4s^2+8r(c-1)+9}}{2}$ which implies that $f(k) \geq 0$ for all $k \leq y_1$. So, for $s < \frac{5+\sqrt{8r(c-1)+34}}{2}$, the Inequality 2.21 and Brouwer's conjecture holds for all $k \in [x_1, n]$ and for all $k \in [1, y_1]$. $\square$

Corollary 2.2.17. *Let G be connected graph of order $n \geq 4$ and size m and let $k_{s,s}$, $s \geq 2$, be the maximal complete bipartite subgraph of graph G. If $H = G/K_{s,s}$ is a graph having r non-trivial components $C_1, C_2, \ldots C_r$, each of which is a $c-$cyclic graph and $p \geq 0$ trivial components, then*

$$S_K(G) \leq \begin{cases} 2s^2 + n - p + 2r(c-1) + 2k, & if k \geq 2s - 1 \\ s + k(s+2) + n - p + 2r(c-1) + 2k, & If k \leq 2s - 2 \end{cases}$$

Theorem 2.2.18. *Let G be a connected graph with $n \geq 4$ vertices and m edges having $p \geq 1$ pendent vertices.*

(i) If $p < \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1}{8} - \frac{(n-3)}{2}p$, then Brouwer's conjecture holds for $k \in [1, \frac{(n-1)}{2}]$

(ii) If $p > \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-3)}{2}p$, then Brouwer's conjecture holds for $k \in [\frac{n-2}{2}, n]$. Now let the graph G have p vertices each of degree r. The following theorem verifies Brouwers conjecture under certain restrictions on the size m of G

Theorem 2.2.19. *Let G be a connected graph with $n \geq 4$ vertices and m edges having $p \geq 1$ vertices of degree $r \geq 1$.*

(i) If $m \geq \frac{(2n-r-1)r}{2}$, then Brouwer's conjecture holds for $k \in [1, r]$

(ii) If $p \leq \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-1-2r)p}{2}$, then Brouwer's conjecture holds for $k \in [r+1, \frac{n-1}{2}]$

(iii) If $p > \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-1-2r)p}{2}$, then Brouwer's conjecture holds for $k \in [\frac{n-1}{2}, n]$

Proof. Let G be connected graph with n vertices and m edges having $p \geq 1$ vertices of degree $r \geq 1$. By definition of conjugate degree, We have $d_1^* \leq n, d_2^* \leq n, \ldots, d_r^* \leq$

$n, d_i^* \leq n-p$, where $i = r+1 \dots n$ we consider the case $k \leq r$, By Gone-Merris Bai theorem if follows that $S_k(G) \leq \sum_{i=1}^{k} d_i^*(G) \leq kn \leq m + \frac{k(k+1)}{2}$ If $k^2-(2n-1)k+2m \geq 0$. This shows that Brouwer's conjecture holds for all $k \leq \frac{2n-1-\sqrt{(2n-1)^2-8m}}{2}$, as $\frac{2n-1+\sqrt{(2n-1)^2-8m}}{2} \geq n-1$ Thus, $\frac{2n-1-\sqrt{(2n-1)^2-8m}}{2} \geq r$ implies that $m \geq \frac{(2n-r-1)r}{2}$, completing the proof in this case. For $r+1 \leq k \leq n-1$, by Grone-Merris Bai theorem, it follows that $S_k(G) \leq \sum_{i=1}^{k} d_i^*(G) \leq k(n-p)+pr \leq m + \frac{k(k+1)}{2}$ Provided that $k^2 - k(2n-2p-1) + 2(m-pr) \geq 0$. Consider the polynomial $f(x) = K^2 - (2n-2p-1)k + 2(m-pr)$, for $k \in [r+1, n-1]$. The discriminant of this polynomial is $\bigtriangledown = (2n-2p-1)^2 - 8(m-pr)$. Since G has p vertices of degree r, we have $m \leq \binom{n-p}{2} + pr < \frac{(2n-2p-1)^2}{2} + pr$, which implies that $\bigtriangledown > 0$. The roots of this polynomial are $x = \frac{(2n-2p-1)-\sqrt{\bigtriangledown}}{2}$, $y = \frac{(2n-2p-1)+\sqrt{\bigtriangledown}}{2}$. This shows that $f(x) \geq 0$ is true for $k \in [r+1, x]$ and $k \in [y, n-1]$. If $p < \frac{n}{2}$, then $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-1-2r)p}{2}$ implies that $x = \frac{(2n-2p-1)-\sqrt{\bigtriangledown}}{2} \geq \frac{n-1}{2}$ similarly, If $p\frac{n}{2}$, then $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-1-2r)p}{2}$ implies that $y = \frac{(2n-2p-1)+\sqrt{\bigtriangledown}}{2} \leq \frac{n-1}{2}$, completing the proof. □

Theorem 2.2.20. *Let G be a connected graph of order $n \geq 4$ and size m having clique number $\omega \geq 2$. If $H = G \backslash k_\omega$ is a graph having r non-trivial components $c_1, c_2, \dots, c_r$ each of which is a $c-$cyclic graph and $p \geq 0$ trivial components, then*
$$S_k(G) \leq \begin{cases} \omega(\omega-1) + n - p + 2r(c-1) + 2k & if k \geq \omega - 1, \\ k(\omega+2) + n - p + 2r(c-1), & if k \leq \omega - 2. \end{cases}$$

Theorem 2.2.21. *Let G be a connected graph of order $n \geq 4$ and size m. Let $k_{s_1,s_2}, s_1 \leq s_2 \geq 2$ be the maximal complete bipartite sub-graph of the graph G. If $H = G \backslash K_{s_1,s_2}$ is a graph having r non-trivial components $c_1, c_2, \dots c_r$, each of which is a c-cyclic graph and $p \geq 0$ trivial components, then*

$$S_k(G) \leq \begin{cases} 2s_1s_2 + n - p + 2r(c-1) + 2k, & if k \geq s_1 + s_2 - 1 \\ s_2 + ks_1 + n - p + 2r(c-1) + 2k, & if \leq s_1 + s_2 - 2 \end{cases}$$

Proof. Consider a connected graph G with $k_{s_1,s_2}, (s_1 \geq s_2)$, as its maximal complete bipartite sub-graph. If we remove the edges of k_{s_1,s_2} from G, the Laplacian matrix can be decomposed as $L(G) = L(k_{s_1,s_2} \bigcup (n-s_1-s_2)k_1) + L(H)$, where $H = G \backslash k_{s_1,s_2}$ is a graph obtained from G by removing the edges of k_{s_1,s_2}. By applying the lemma (Let A and B be a two real symmetric matrices both of order n. If $k, 1 \leq k \leq n$, is a positive integer,then $\sum\limits_{i=1}^{k} \lambda_i(A+B) \leq \sum\limits_{i=1}^{k} \lambda_i(A) + \sum\limits_{i=1}^{k} \lambda_i(B)$, where $\lambda_i(X)$ is the $i-th$ eigen value of X) and using the fact that $s_k(k_{s_1,s_2} \bigcup (n-s_1-s_2)k_1) = s_k(k_{s_1,s_2})$ for $1 \leq k \leq n$, we have

$$s_k(G) \leq \sum_{i=1}^{k} \mu_i(k_{s_1,s_2}) + \sum_{i=1}^{k} \mu_i(H) = S_K(k_{s_1,s_2}) + s_k(H).$$

Now proceeding similar as in theorem 2.2.20 and using the fact that the Laplacian spectrum of k_{s_1,s_2} is $\{s_1+s_2, s_1^{[s_2-1]}, s_2^{[s_1-1]}, 0\}$, the result follows in particular, if $s_1 = s_2$, we have the following consequence of the theorem 2.2.21 $\square$

Corollary 2.2.22. *Let G be a connected graph of order $n \geq 4$ and size m and let $k_{s,s}, s \geq 2$, be the maximal bipartite sub-graph of graph G.If $H = G \backslash k_{s,s}$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$ each is a $c-$cyclic graph and $p \geq 0$ trivial components, then*

$$s_k(G) \leq \begin{cases} 2s^2 + n - p + 2r(c-1) + 2k, & if k \geq 2s-1 \\ s + k(s+2) + n - p + 2r(c-1) + 2k, & if k \leq 2s-2 \end{cases}$$

Theorem 2.2.23. *Let G be a connected graph of order $n \geq 4$ and size m having clique number $\omega \geq 2$. If $H = G \backslash K_\omega$ is a graph having r non-trivial components $C_1, C_2, \ldots, c_r$, each of which is a $c-$cyclic graph, then $S_k(G) \leq m + \frac{k(k+1)}{2}$, for all $k \in [1, \triangle_1]$ and $k \in [\beta_1, n]$, where $\triangle_1 = min\{\omega-2, \gamma_1\}, \gamma_1 = \frac{2\omega+3-\sqrt{16\omega+8r(c-1)+9}}{2}$ and $\beta_1 = \frac{3+\sqrt{4\omega^2-4\omega+8r(c-1)+9}}{2}$.*

Proof. Let G be a connected graph of order n and size m having clique number $\omega \geq 2$. If $H = G \backslash k_\omega$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$, each of which is a $c-$cyclic graph and $p \geq 0$ trivial components, then $m = \frac{\omega(\omega-1)}{2} + n - p + r(c-1)$. For $k \geq \omega - 1$, from theorem 2.2.20, we have

$$\begin{aligned} S_k(G) &\leq \omega(\omega-1) + n - p + 2r(c-1) + 2k \\ &\leq m + \frac{k(k+1)}{2} = \frac{\omega(\omega-1)}{2} + n - p + r(c-1) + \frac{k(k+1)}{2}, \end{aligned}$$

If

$$k^2 - 3k - (\omega(\omega-1) + 2r(c-1)) \geq 0 \tag{2.22}$$

Now considering the polynomial $f(k) = k^2 - 3k - (\omega(\omega-1) + 2r(c-1))$. The roots of this polynomial are $\beta_1 = \frac{3+\sqrt{4\omega^2-4\omega+8r(c-1)+9}}{2}$ and $\beta_2 = \frac{3-\sqrt{4\omega^2-4\omega+8r(c-1)+9}}{2}$. This shows that $f(k) \geq 0$, for all $k \geq \beta_1$ and for all $k \leq \beta_2$. Since $1 \leq r \leq \omega$, it can be seen that $2 - \omega \leq \beta_2 \leq 3 - \omega$. Thus it follows that 2.22 holds for all $k \geq \beta_1$. It is easy to see that $\beta_1 \geq \omega - 1$. This completes the proof in this case. For $k \leq \omega - 2$, from theorem 2.2.20 we have

$$\begin{aligned} S_k(G) &\leq (\omega+2)k + n - p + 2r(c-1) \leq m + \frac{k(k+1)}{2} \\ &= \frac{\omega(\omega-1)}{2} + n - p + r(c-1) + \frac{k(k+1)}{2}, \end{aligned}$$

If

$$k^2 - (2\omega+3)k - 2r(c-1) + \omega(\omega-1) \geq 0. \tag{2.23}$$

Proceeding, similarly as above, it can be seen that 2.23 holds for all $k \leq \gamma_1 = \dfrac{2\omega + 3\sqrt{16\omega + 9 + 8r(c-1)}}{2}$. Indeed, $\gamma_1 \leq \omega - 2$, holds for any $c \geq 1$, completing the proof in this case as well Evidently, if $C = 0$, then $\triangle_1 = \omega - 2$ for $2 \leq \omega \leq 5$; and $\triangle_1 = \gamma_1$ for $\omega \geq 6$. Further, if $C = 1$, then $\triangle_1 = \omega - 2$, for $\omega = 2$; and $\triangle_1 = \gamma_1$ for $\omega \geq 3$. For $C \geq 2$, clearly $\triangle_1 = \gamma_1$. From this theorem we have the following observations □

Corollary 2.2.24. *Let G be a connected graph of order $n \geq 4$ and size m having clique number $\omega \geq 2$. Let $H = G \backslash K_\omega$ be a graph having r non-trivial components $C_1, C_2 \ldots, C_r$, each of which is a $c-$cyclic graph and $\alpha = \sum\limits_{i=1}^{r} \frac{2k_1 - 2}{n_i}$.*

(i) If $C = 0$, that is, each C_i is a tree,then Brouwer's conjecture holds for all $k \in [1, \triangle_1]$ and $k \in [\omega+1, n]$, where $\triangle_1 = min\{\omega - 2, \gamma_1\}$ and $\gamma_1 = \frac{2\omega+3-\sqrt{16\omega+9-8r}}{2}$.

(ii) If $c = 1$, that is, each C_i is a uni-cyclic graph then the Brouwer's conjecture holds for all $k \in [\omega + 2, n]$ and $k \in [1, \triangle_1]$, where $\triangle_1 = min\{\omega - 2, \gamma_1\}$ and $\gamma_1 = \frac{2\omega+3\sqrt{16\omega+9}}{2}$.

(iii) If $C = 2$, that is, each C_i is a bi-cyclic graph, then the Brouwer's conjecture holds for all $k \in [\omega + 3, n]$ and $k \in [1, \frac{2\omega+3-\sqrt{16\omega+9+8r}}{2}]$.

(iv) If $C \geq 3$, that is each C_i is a $c-$cyclic graph, then the Brouwer's conjecture holds for all $k \in [\omega + c, n]$ and $k \in [1, \frac{2\omega+3-\sqrt{16\omega+9+8r(c-1)}}{2}]$

Proof. (i) If $C = 0$, then $\beta_1 = \frac{3+\sqrt{4\omega^2-4\omega-8r+9}}{2}$ and $\gamma_1 = \frac{2\omega+3-\sqrt{16\omega+9-8r}}{2}$. Using the fact that $1 \leq r \leq \omega$

$$\beta_1 = \frac{3 + \sqrt{4\omega^2 - 4\omega - 8r + 9}}{2} \leq \frac{3 + \sqrt{4\omega^2 - 4\omega + 1}}{2} = \omega + 1$$

(ii) If $C = 1$, then $\beta_1 = \frac{3+\sqrt{4\omega^2-4\omega+9}}{2}$ and $\gamma_1 = \frac{2\omega+3-\sqrt{8\omega+9}}{2}$. we have

$$\beta_1 = \frac{3 + \sqrt{4\omega^2 - 4\omega + 9}}{2} \leq \frac{3 + \sqrt{4\omega^2 - 4\omega + 1}}{2} = \omega + 2$$

(iii) Proceeding as in part (i) and (ii), we can prove the part (iii)

(iv) If $C \geq 3$, then using $r \leq \omega$, we have

$$\beta_1 = \frac{3 + \sqrt{4\omega^2 - 4\omega + 8r(c-1) + 9}}{2} \leq \frac{3 + \sqrt{4\omega^2 + 4\omega(2C - 3) + 9}}{2}$$
$$= \frac{3 + \sqrt{(2\omega + (2C - 3))^2 - 4C(C - 3)}}{2} \leq \omega + C.$$

□

Now considering the special classes of graphs satisfying the hypothesis of 2.2.23 Let $C_\omega(a,a,\ldots,a), a \geq 1$, be the family of connected graphs of order $n = \omega(a+1)$ and size m obtained by identifying one of the vertex of $c-$cyclic graph C of order $a+1$ to each vertex of a clique k_ω. For the family of graphs $C_\omega(a,a,\ldots,a)$, we see that Brouwer's conjecture is true for various subfamilies depending upon the value of C, the order of a $c-$cyclic graphs and the clique number of the graph.

Theorem 2.2.25. *Let G be a connected graph of order $n \geq 4$ and size m and let $K_(s,s), s \geq 2$ be a maximal bipartite sub-graph of graph G. If $H = G \backslash K_s, s$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$, each of which is a $c-$cyclic graph and $p \geq 0$ trivial components, then for $S \geq \frac{5+\sqrt{8r(c-1)+34}}{2}$, Brouwer's conjecture holds for all k; and for $S < \frac{5+\sqrt{8r(c-1)+34}}{2}$, Brouwer's conjecture holds for all $k \in [x_1, n]$ and for all $k \in [1, y_1]$, where $x_1 = \frac{2s+3+\sqrt{20s-4s^2+8r(c-1)+9}}{2}$ and $y_1 = \frac{2s+3+\sqrt{20s-4s^2+8r(c-1)+9}}{2}$*

Proof. Let G be a connected graph of order n and size m. If $H = G \backslash k_s, s$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$ each of which is a $c-$cyclic graph and $p \geq 0$ trivial components,then $m = s^2 + n - p + (c-1)r$. For $1 \leq k \leq n$, then from corollary 2.2.22 we have

$$\begin{aligned} S_k(G) &\leq s + k(s+2) + n - p + 2r(c-1) - \alpha \leq s + k(s+2) + n - p + 2r(c-1) \\ &\leq m + \frac{k(k+1)}{2} = s^2 + n - p + r(c-1) + \frac{k(k+1)}{2}, \end{aligned}$$

If

$$k^2 - (2s+3)k - (2s + 2r(c-1) - 2s^2) \geq 0 \tag{2.24}$$

Now, consider the polynomial

$$f(k) = k^2 - (2s+3)k - (2s + 2r(c-1) - 2s^2).$$

The discriminant of this polynomial is $d = 20s - 4s^2 + 8r(c-1) + 9$. we have $d \leq 0$ if $20s - 42^2 + 8r(c-1) + 9 \leq 0$, which gives $s \geq \frac{5+\sqrt{8r(c-1)+34}}{2}$. This shows that for

$s \geq \frac{5+\sqrt{8r(c-1)}}{2}$ the inequality 2.24 and so the Brouwer's conjecture always holds. For $s < \frac{5+\sqrt{8r(c-1)+34}}{2}$ the roots of the polynomial $f(k)$ are $x_1 = \frac{2s+3+\sqrt{20s-4s^2+8r(c-1)+9}}{2}$ and $y_1 = \frac{2s+3-\sqrt{20s-4s^2+8r(c-1)+9}}{2}$, which implies that $f(k) \geq 0$ for all $k \geq x_1$ and $f(k) \geq 0$ for all $k \leq y_1$. So for $s < \frac{5+\sqrt{8r(c-1)+34}}{2}$, the inequality 2.24 and Brouwer's conjecture holds for all $k \in [1, y_1]$. The following result explores some interesting families of graphs given by theorem 2.2.25 for which Brouwer's conjecture holds. we will use the fact that Brouwer's conjecture is always true for all $k \leq 2$. □

Corollary 2.2.26. *Let G be a connected graph of order $n \geq 4$ and size m having clique number $\omega \geq 2$. Let $H = G \backslash K_{s,s}, S \geq 2$ be a graph having r non-trivial components $c_1, c_2, \dot{;} c_r$ each of which is a $c-$cyclic graph.*

(i) If $c = 0$, that is, each C_i is a tree, then Brouwer's conjecture holds for all k, if $s \geq 5$ and holds for all k, $k \notin [3, 7]$, if $2 \leq s \leq 4$.

(ii) If $c = 1$, that is,each C_i is a uni cyclic graph, then Brouwer's conjecture holds for all k, if $s \geq 6$ and holds for all k, if $s \geq 6$ and holds for all k, $k \notin [3, 7]$, if $2 \leq s \leq 5$.

(iii) If $c = 2$, that is, each C_i is a bi cyclic graph, then Brouwer's conjecture holds for all k if $S \geq 10$ and holds for all $k, k \notin [3, 12]$, if $2 \leq s \leq 9$.

(iv) If $c = 3$, that is, each C_i is a tri-cyclic graph, then Brouwer's conjecture holds for all k, if $s \geq 14$ and holds for all $k, k \notin [3, 17]$, if $2 \leq s \leq 13$.

(v) If $c = 4$, that is, each C_i is a tetra cyclic graph, then Brouwer's conjecture holds for all k, if $s \geq 18$ and holds for all $k, k \notin [3, 20]$, if $2 \leq s \leq 17$.

Proof. (i) If $c = 0$, then each of the r non-trivial components of H and so Brouwer's conjecture holds for all k, if $s \geq 5$; and for all $k, k \neq 2, 3, 4, 5, 6, 7$ if $2 \leq s \leq 4$ see theorem 3.9 in [10]

(ii) If $c = 1$, then each of the r non-trivial components of H are uni cyclic graphs and so the discriminant of the polynomial given by the left hand side of 2.24 becomes $d = 20s - 4s^2 + 9$. clearly, for $s \geq 6$, the discriminant $d < 0$ and therefore for such s, Brouwer's conjecture always holds. For $s = 5$, we have $x_1 = \frac{13+\sqrt{9}}{2} = 8$ and $y_1 = \frac{13-\sqrt{9}}{2} = 5$, implies that Brouwer's conjecture holds for all $k, k \neq 6, 7$. for $s = 4$, we have $x_1 = \frac{11+\sqrt{25}}{2} = 8$ and $y_1 = \frac{11-\sqrt{20}}{2} = 3$, implies that Brouwer's conjecture holds for all $k, k \neq 4, 5, 6, 7$. for $s = 3$, we have $x_1 = \frac{9+\sqrt{33}}{2} = 7.3722$ and $y_1 = \frac{13-\sqrt{33}}{2} = 1.6277$, implying that Brouwer's conjecture holds for all $k, k \neq 2, 3, 4, 5, 6, 7$. (iii),(iv),(v). These follow by proceeding similar to above cases. We have the following observation for a connected graph G of order $n \geq 4$ size m and having $k_{s,s}, s \geq 2$ as it's maximal bipartite sub graph. Let $H \backslash k_{s,s}, s \geq 2$ be a graph having r non-trivial components $c_1, C_2, \ldots, C_r$, each of which is a $c-$cyclic graph. If $c \leq \frac{s}{2}, r \leq \frac{s}{2}$, then Brouwer's conjecture holds for all K, if $s = 7, s \geq 9$. if $s = 8$, then Brouwer's conjecture holds for all k for $c \leq 4, r \leq 3$ holds for all $k, k \neq 9, 10$ for $c = r = 4$. If $s = 6$ then Brouwer's conjecture holds for all k for $c \leq 2, r \leq 1$; holds for all k, for $c = r = 2$; holds for all $k, k \neq 7, 8$ for $c = 2, r = 3$; holds for all $k, k \notin [6, 9]$ for $c = 3, r = 2$; and holds for all $k, k \notin [5, 10]$, for $c = r = 3$. If $s = 5$, then Brouwer's conjecture holds for all $k, k \neq 6, 7$ for $c \leq 1, r \leq 2$; holds for all $k, k \notin [5, 8]$, for $c = 2, r \leq 2$. If $s = 4$, then Brouwer's conjecture holds for all $k, k \notin [4, 7]$, for $c \leq 1, r \leq 2$, holds for all $k, k \notin [3, 8]$, for $c = 2, r \leq 2$. If $s = 3$, then Browser's conjecture holds for all $k, k \neq 3, 4, 5, 6, 7$. If $s = 2$, then Brouwer's conjecture holds for all $k \notin [3, 6]$.

□

Corollary 2.2.27. *Let G be a connected graph with $n \geq 4$ vertices and m edges having $p \geq 1$ pendent vertices and $q \geq 1$ vertices of degree 2* *(i)* *if* $p+q < \frac{n}{2}$ *and* $m \geq \frac{(n-1)(3n-1)}{8} - \frac{n-3}{2}p - \frac{n-5}{2}q$*, then the Brouwer's conjecture holds for all* $k, 3 \leq k \leq \frac{n-1}{2}$.

(ii) *if* $p+q > \frac{n}{2}$ *and* $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-3)}{2}p - \frac{(n-5)}{2}q$*, then the Brouwer's conjecture holds for all* $k, \frac{n-1}{2} \leq k \leq n$.

Theorem 2.2.28. *Let G be a connected graph with $n \geq 4$ vertices, m edges and having $p \geq 1$ and $q \geq 1(q \neq p$ vertices of degree r and $s(s > r \geq 1)$, respectively.*

(i) *if* $m \geq \frac{(2n-r-1)r}{2}$*, then the Brouwer's conjecture holds for* $k \in [1, r]$.

(ii) *If* $n > p + s + \frac{1}{2}$ *and* $m \geq \frac{s(2s-2p-s-1)}{2} + pr$*; or* $n < p + r + \frac{3}{2}$ *and* $m \geq \frac{(r+1)(2n-2p-r-2)}{2} + pr$*, then the Brouwer's conjecture holds for* $k \in [r+1, s]$.

(iii) *If* $p + q < \frac{n}{2}$ *and* $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-2r-1)}{2}p - \frac{(n-2s-1)}{2}q$*, then Brouwer's conjecture holds for all* $k, s+1 \leq k \leq (n-1)/2$.

(iv) *If* $p + q > \frac{n}{2}$ *and* $m \geq \frac{(n-1)(3n-1)}{2} - \frac{(n-2r-1)}{2}p - \frac{(n-2s-1)}{2}q$*, then the Brouwer's conjecture holds for all* $k, (n-1)/2 \leq k \leq n$.

Corollary 2.2.29. *Let G be a connected graph with $n \geq 7$ vertices and m edges having $p \geq 1$ pendent vertices and $q \geq 1$ vertices of degree 3.*

(i) *if* $n > p + \frac{7}{2}$ *and* $m \geq 3n - 2p - 6$*; or* $n < p + \frac{3}{2}$ *and* $m \geq 2n - p - 3$*, then brouwer's conjecture holds for* $k = 3$.

(ii) *If* $p+q < \frac{n}{2}$ *and* $m \geq \frac{(n-1)(3n-1)}{2} - \frac{(n-3)}{2}p - \frac{(n-5)}{2}q$*, then the brouwer's conjecture holds for all* $k, 4 \leq k \leq \frac{(n-1)}{2}$.

(iii) *If* $p+q > \frac{n}{2}$ *and* $m > \frac{(n-1)(3n-1)}{8} - \frac{(n-3)}{2}p - \frac{(n-5)}{2}q$*, then the brouwer's conjecture holds for all* $k, \frac{(n-1)}{2} < k \leq n$.

Proof. This follows from theorem 2.2.28, by taking $r = 1$ and $s = 3$. We note that part (i) of corolarry 2.2.29 imposes conditions on the number of edges and the number of vertices in terms of pendent vertices of graph G for brouwer's conjecture to hold for $k = 3$. This information will be helpfull for further investigations, as one can investigate the case $k = 3$, just as the case $k = 2$ has been discussed for any graph G. In fact, part (i) of corolarry 2.2.29 guarantees that for a graph G with $n \geq p+4$ vertices Brouwer's conjecture holds for $k = 3$, provided that $m \geq 3n - 2p - 3$. For $n \leq p + 2$ vertices Brouwer's conjecture holds for $k = 3$, provided $m \geq 2n - p - 3$. If in particular $p = \frac{n}{2}$, then the Brouwer's conjecture holds for $k = 3$, provided that $m \geq 2n - 6$. The following upper bounds for $s_k(G)$, in terms of order n, the size m, the maximum degree $\triangle$ and the number of pendent vertices p can be found in [18]:

$$s_k(G) \leq 2m - n + 3k - \triangle + p. \tag{2.25}$$

with $k = 3$, upper bound 2.25 implies that $s_3(G) \leq 2m - n + 10 - \triangle + p \leq m + 6$, provided that $m \leq n + \triangle - p - 4$. Thus we have the following observations. $\square$

Corollary 2.2.30. *Let G be a connected graph of order $n \geq 7$ having size m and maximum degree $\triangle$. Let p be the number of pendent vertices of G. If $3 \leq p \leq n-4$, then the Brouwer's conjecture holds for $k = 3$, provided that*

$$m \geq n + \triangle - p - 4$$

or

$$m \geq 3n - 2p - 6.$$

For regular graphs, it is well known that Brouwer's conjecture always holds [12]. For bi-regular graphs (a graph in which degree of vertices is either r or s is said to be an (r, s)-regular graph or a bi-regular graph)no such result can be found in literature. However we have the following observation for a connected (r, s)-regular graph G

Corollary 2.2.31. *Let G be a connected (r, s)-regular graph with $n \geq 7$ vertices and m edges having $p \geq 1$ vertices of degree r and $q \geq 1$ vertices of degree $s, s > r$, with $p + q = n$.*

(i) If $pr + 2qs \leq qs + r(r + 1)$, then Brouwer's conjecture holds for $k \in [1, r]$.

(ii) If $2qs+pr \leq s(s+1)$, then the Brouwer's conjecture holds for all $k, k \in [r+1, s]$.

(iii) If $(p + q)^2 \geq 4(pr + qs) + 1$, then Brouwer's conjecture holds for all $k \in [(n - 1)/2, n]$.

Proof. This follows from theorem 2.2.28 by using the fact that $n = p + q$ and $m = \frac{pr+qs}{2}$. To see the strength of corollary 2.2.31 we consider examples. For a (2,3)-regular graph G of order $n \geq 7$, part (iii) of corollary 2.2.31 implies that $(p + q)^2 \geq 4(2p + 3s) + 1$, that is,

$$p(p - 6) + q^2 - 2p - 1 + 2q(p - 6) \geq 0. \tag{2.26}$$

Is is easy to see that 2.26 is true for $p \geq 6$ and $q \geq p$. For $p = 4, 5$, it can be seen that 2.26 holds for $q \geq 8$. Thus it follows if $p = 4, 5$ and $q \geq 6$, Brouwer's conjecture holds for all $k \in [(n - 1)/2, n]$ Let $s_\omega(H_1, H_2, \ldots, H_\omega)$, where H_i is a graph of order $a_i, 0 \leq a_i < \omega, 1 \leq i \leq \omega$, be a family of connected graphs of order $n = \sum\limits_{i=1}^{\omega} a_i$ and size m obtained by identifying a vertex of a graph H_i at the i^th vertex of a clique k_ω. If $H_i = k_1, a$,for $i = 1, 2, \ldots, \omega - 1, H_\omega = H^*$ is a graph obtained by identifying a vertex of cycle c_t with root vertex of $k_{a-2}, 1$ and vertex of H_i to be identified at $i^(th)$ vertex of k_ω is the root vertex, then any graph in the family $s_\omega(H_1, H_2, \ldots, H_\omega)$, can be obtained from a split graph (a graph whose vertex set $V(G)$ can be partitioned into two parts V_1 and V_2 such that the sub-graph induced by V_1 is empty graph and the sub-graph induced by V_2 is a clique) by fusing a vertex of a cycle c_t at a vertex of the clique of degree $\omega + a - 3$. For this family of graphs we have the following observations. $\square$

2.3 Graphs with cycles

Theorem 2.3.1. *Let G be uni-cyclic graph of order n. Then*

$$S_k(G) \leq e(G) + \binom{k+1}{2}$$

where $k = 1, 2, \ldots n$.

Proof. The case of $k = 1$ is trivial, and the case $k = 2$ has been proved in [8]. If $k \geq 3$, since G is a uni-cyclic graph, then $e(G) = n$ and

$$\frac{(3n-4) + \sqrt{9n^2 - 8n + 13}}{2n} \leq 3 \leq k.$$

Hence by corollary 2.2.3, we immediately the desired result $\square$

Theorem 2.3.2. *Let G be a bi-cyclic graph of order n. Then*

$$S_k(G) \leq e(G) + \binom{k+1}{2}$$

where k is an integer with $1 \leq k \leq n$ and $k \neq 3$

Proof. The case of $k = 1$ is trivial, and the case of $k = 2$ has been proved in [3]. If $k \geq 3$, since G is a uni-cyclic graph then $e(G) = n$ and

$$\frac{(3n-4) + \sqrt{17n^2 - 8n + 16}}{2n} \leq \frac{(3n-4) + \sqrt{25n^2 - 8n + 16}}{2n} \leq 4 \leq k$$

hence by corollary 2.2.3, theorem 2.3.2 is established

Let $C(n_1, n_2, 1)$ denote the bi-cyclic graph obtained from two cycles C_{n_1} & C_{n_2} by identifying a vertex of c_{n_1} with a vertex of C_{n_2}, where $n_1 \geq n_2 \geq 3$. Let $C(n_1, n_2, t)$ denotes the bi-cyclic graph obtained from a path p_t and two cycles c_{n_1} and c_{n_2} by identifying a vertex of C_{n_1} with one end vertex of p_t (and a vertex of C_{n_2} with the other end vertex of p_t), where $n_1 \geq n_2 \geq 3t$ and $t \geq 2$. A bi-cyclic graph is called

$\infty-$type if it is $C(n_1, n_2, t)$ or it can be obtained by attaching some hanging trees to $C(n_1, n_2, t)$, where $n_1 \geq n_2 \geq 3$ and $t \geq 1$. Let $G - e$ denote the graph obtained from G by deleting an edge $e \in E(G)$. $\square$

Theorem 2.3.3. *Let G be an $\infty-$type bi-cyclic graph of order n. Then*

$$S_k(G) \leq e(G) + \binom{k+1}{2}$$

where $k = 1, 2, \ldots, n$

Lemma 2.3.4. *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. If $e(G_i) \geq 1$ and $s_{k_i}(G_i) \leq e(G_i) + \binom{k+1}{2}$ for $k_i = 1, 2, \ldots, n_i$ and $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$,*

$$s_k(G_1 \sim G_2) \leq e(G_1 \sim G_2) + \binom{k+1}{2}$$

Lemma 2.3.5. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. If $e(G) \geq 2$ and $S_{k_i}(G_i) \leq e(G_i) + \binom{k_i+1}{2}$ for $k_i = 1, 2, 3 \ldots n_i$ and $i = 1, 2$ then for $1 \leq k \leq n_1 + n_2$,*

$$S_k(G_1 \approx G_2) \leq e(G_1 \approx G_2) + \binom{k+1}{2}.$$

Lemma 2.3.6. *Let G be a graph of order $n \geq 4$ such that there is a hanging tree T of G with atleast two vertices. Suppose $G - T$ has n^* vertices, where $2 \leq n^* \leq n - 2$. Let k be an integer with $1 \leq k \leq n$. If $e(G - T) \geq 1$ and*

$$S_k^*(G - T) + \binom{k+1}{2}$$

for $k^* = 1, 2, \ldots, n^*$, then

$$s_(G) \leq e(G) + \binom{k+1}{2}.$$

Proof. Since G is an $\infty-$type bi-cyclic graph, then G is obtained by attaching some hanging trees to $C(n_1, n_2, t)$, where $n_1 \geq n_2 \geq 3$ and $t \geq 1$. By Lemma 2.3.6, it will suffice to consider the $\infty-$type bi-cyclic graph G which is obtained by attaching some pendent vertices to $C(n_1, n_2, t)$ or $G = C(n_1, n_2, t)$, where $n_1 \geq n_2 \geq 3$ and $t \geq 1$.

Case 1. $t \geq 2$. Let e_1 be an edge of P_t , then $G - e_1$ is the union of two uni-cyclic graphs. Then by Lemma 2.3.4 and Theorem 2.3.1, we get the desired result.

Case 2. $t = 1$. Let $C(3, 3, 1)^*$ be the graph obtained from $C(3, 3, 1)$ by attaching $n5$ pendent vertices to the vertex of degree 4 in $C(3, 3, 1)$. If $G \cong C(3, 3, 1)^*$, then by directly computing, we have

$$S_3(C(3, 3, 1)^*) = n + 6 < (n + 1) + \binom{3+1}{2}$$

Combining this with Theorem 2.3.2, the result follows. Otherwise, there exist two edges e_2 and e_3 of a cycle such that $G - e_2 - e_3$ is the union of a uni-cyclic graph and a tree with at least two edges. Combining Inequality (1) and Theorem 2.3.1 with Lemma 2.3.5, the result is obtained as desired. □

2.4 Regular Graphs

Lemma 2.4.1. *Let G be an $r-$regular graph on n vertices. If either*

(i) $4kr + n^2 + n \leq 2nr + 2kn + 2k$, *or*

(ii) $4kr + 2n^2 \leq 3nr + 2kn + k^2 + k$,

then $S_k(G) \leq e(G) + \binom{k+1}{2}$.

Lemma 2.4.2. *Let G be an $r-$regular graph on n vertices and suppose that $4k \leq n + 2r + 3$. If $S_k(G) \leq e(G) + \binom{k+1}{2}$, then $S_{n-k+1}(\bar{G}) \leq e(\bar{G}) + \binom{n-k+1}{2}$.*

Lemma 2.4.3. *[14, 16], Let A be a $(0,1)-$symmetric matrix with eigen values $\theta_1 \geq \cdots \geq \theta_n$, then $\theta_1 + \cdots + \theta_k \leq \frac{n}{2}\left(1+\sqrt{k}\right)$, for $k = 1, \ldots, n$. If G is an $r-$regular graph, then $Q(G) = (r-1)I + (A(G)+I)$. If $\theta_1 \geq \cdots \geq \theta_n$ are eigen values of $A(G)+I$ then by 2.4.3*

$$q_1 + \cdots + q_k = k(r-1) + \theta_1 + \cdots + \theta_k \leq k(r-1) + \frac{n}{2}(1+\sqrt{k}). \tag{2.27}$$

Theorem 2.4.4. *Conjecture (For any graph with n vertices and any $k = 1 \ldots n$, $S_k(G) \leq e(G) + \binom{k+1}{2}$) holds for regular graphs.*

Proof. Let the adjacency eigenvalues of $r-$regular graph G be $\theta_1 \geq \cdots \geq \theta_n$. Then $q_i = r + \theta_i$ for $i = 1, \ldots, n$. Using Chauchy-Schwarz inequality and the fact that $\sum_{i=1}^{n} \theta_i^2 = 2e = nr$, we observe that

$$\sum_{i=1}^{k} q_i = kr + \sum_{i=1}^{k} \theta_i \leq kr + \left(k \sum_{i=1}^{k} \theta_i^2\right)^{1/2} \leq kr + \sqrt{knr}.$$

If we show that the right hand side is at most $\frac{nr}{2} + \binom{k+1}{2}$, then the proof is complete. So it suffices to show that

$$nr + k^2 + k - 2kr - 2\sqrt{nkr} \geq 0.$$

The left hand side is quadratic function in $\sqrt{r}$. Substituting $\sqrt{r} = x$ we may write it as

$$\begin{aligned} f(x) &= (n-2k)x^2 - 2\sqrt{nkx} + k^2 + k \\ &= (n-2k)(x - \frac{\sqrt{nk}}{n-2k})^2 + k^2 + k - \frac{nk}{n-2k}. \end{aligned} \tag{2.28}$$

Now we consider three cases.

Case 1. $n \geq 2k+2$. In this case, 2.28 is non-negative,as desired.

Case 2. $n = 2k+1$. If $r \geq \frac{n}{2}$, then the result follows from lemma 2.4.1(i). Suppose that $r < \frac{n}{2}$. The roots of $f(x)$ are $\sqrt{\frac{n(n-1)}{2}} \pm \frac{n-1}{2}$. Both the roots are greater than $\sqrt{\frac{n}{2}}$ for $n \geq 11$. So for $n \geq 11$, we have $f(\sqrt{r}) > 0$, as desired. Since in this case n is odd, the assertion for the remaining values of n follows from lemma(conjecture 2.1.2 is true for all graphs on at most 10 vertices.)

Case 3. $n \leq 2k$.

The result for $k \geq \frac{3n}{4}$ follows in view of lemma 2.4.2 and the fact that the theorem is true for $k \leq \frac{n}{4}$ by case2. So we only need to prove the theorem for $\frac{n}{2} \leq k < \frac{3n}{4}$. First assume that $r \leq \frac{3n}{4}$. By lemmma 2.4.3, we have $s_k(G) \leq k(r-1) + \frac{n}{2}(1+\sqrt{k})$. So it is sufficient to show that

$$g(r) \colon = 2k(r-1) + n(1+\sqrt{k}) - rn - k(k+1) \leq 0.$$

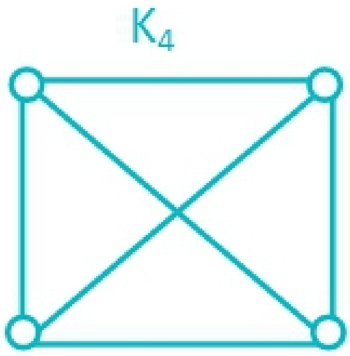
As $\frac{n}{2} \leq k$, g is increasing with respect to r. Thus $g(r) \leq g(\frac{3n}{4}) = \frac{-3n^2}{4} + n(\frac{3k}{2} + \sqrt{k} + 1) - k(k+3)$. Now, $g(\frac{3n}{4})$ as a quadratic form in n has negative discriminant, and thus it is negative. Finally assume that $r > \frac{3n}{4}$. In view of lemma 2.4.1(ii) it suffices to show that

$$2n^2 \leq (3n-4k)r + 2kn + k^2 + k \tag{2.29}$$

Since $k < \frac{3n}{4}$, the right hand side of 2.29 is increasing in r, so it is enough to show that 2.29 holds for $r = \frac{3n}{4}$ but this amounts to show that $\frac{n^2}{4} - kn + k^2 + k \geq 0$ which always holds. This completes the proof. □

Theorem 2.4.5. *The Brouwer Conjecture holds for regular graph .*

Example 2.4.6. *Consider the following* $3-$*regular graph* G *below. We will prove* *that the Brouwer's Conjecture holds without equality .*



The adjacency matrix of the graph is $A(G)$=

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The degree matrix is $D(G)$=

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

the Laplacian matrix$D(G) - A(G) = L(G)$ *Recall that the characteristic polynomial of the Laplacian matrix is* $det(L(G) - \mu_i) = 0$

$$\begin{aligned} \mu^4 - 12\mu^3 + 48\mu^2 - 64\mu &= \mu(\mu^3 - 12\mu^2 + 48\mu - 64). \\ &= \mu(\mu - 4)^3 = 0. \end{aligned}$$

Then the Laplacian spectrum is $\{0, 4, 4, 4\}$. We obtain the following table:

k	$S_k(G) = \sum_{i=1}^{k} \mu_i$	$m(G) + \binom{k+1}{2}$
1	*0*	*7*
2	*4*	*9*
3	*8*	*12*
4	*12*	*16*

We observe that $S_k(G) = \sum_{i=1}^{k} \mu_i \leq m(G) + \binom{k+1}{2}$. It holds for all $K = 1, \ldots, 4$. it is clear there is no equality .

2.5 Complete Graphs

Theorem 2.5.1. *For complete graphs, the equality holds in the Grone-Merris Conjecture for all natural numbers K.*

Proof. If G be a complete graph K_n on n vertices. The Laplacian matrix which has spectrum $(0^1, n^{n-1}) = \{0, n, n, ...\}$, and conjugate degree sequence $d^*(K_n) = \{n, n, ..., 0\}$. If $k = 1$ then the eigenvalues of L(G) to be $\mu_1 = 0$ and $d^*(G) = 0$, then the $\sum_{i=1}^{1} \mu_{1-1+1} \leq \sum_{i=1}^{1} d^*(G)$ are holds and the equality holds in $k = 1$ If $k = 2$ then the eigenvalues of L(G) is $\mu(G) = (0, 2)$. We see that the degree sequence $d(G) = (1, 1)$, and $d^*(G) = (2, 0)$ we obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{n} d_i^*$
1	0	0
2	2	2

Observe that $\mu_(G) \leq d^*(G)$ the Grone-Merris Conjecture holds. Moreover, the equality holds for all K. If $k = 3$ complete graph with three vertices. Observe that its degree sequence $d(G) = (2,2,2)$, and the conjugate of degree sequence $d^*(G) = (3,3,0)$. Computing the eigenvalues of L(G) is $\mu(G) = (3,3,0)$. We obtain the following table

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	3	3
2	6	6
3	6	6

If $K = 4$ complete graph with four vertices. Observe that its degree sequence $d(G) = (3,3,3,3)$, We see that $d^*(G) = (4,4,4,0)$. We know that the Laplacian spectrum of complete graph is $0^1, n^{n-1}$ then We have $\mu(G) = (4,4,4,0)$. We obtain the following table:

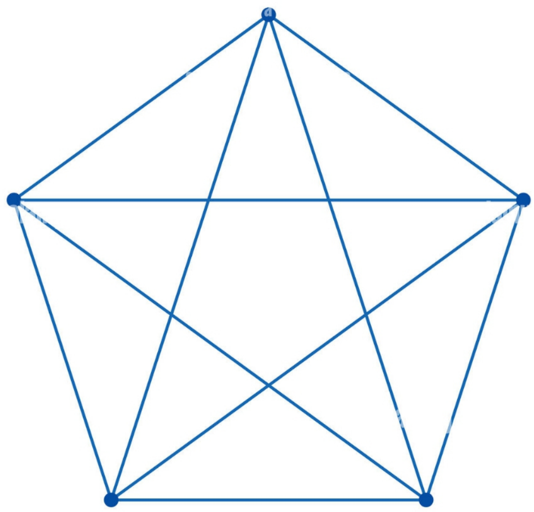
k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	4	4
2	8	8
3	12	12
4	12	12

Thus not only does the Grone-Merris Conjecture hold, but we have equality in $\sum_{=1}^{K} \mu_{n-i+1} \leq \sum_{i=1}^{k} d_i^*(G)$ for all k. So if $k = n$ complete graph with n vertices then we know that the degree sequence $d(G) = (n-1)^n$ and $d^*(G) = \mu(G) = (0^1, n^{n-1})$. We obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	$\mu_{n-1} + 1$	d_1^*
2	$\mu_{n-2} + 1$	d_2^*
$\vdots$		
n	$\mu_{n-i} + 1$	d_n^*

It is clear that the Laplacian spectrum and the conjugate degree sequence are the same. Thus, in the Grone-Merris Conjecture, equality holds for all natural numbers K. $\square$

Example 2.5.2.



Consider the above graph.We observe that the graph $G = K_5$ is the complete graph with five vertices.Take note that its degree sequence is $d(G) = (4, 4, 4, 4, 4)$, and the conjugate of degree sequence is $d^*(G) = (5, 5, 5, 5, 0)$. We know that the Laplacian spectrum of a complete graph are $(0^1, n^{n-1})$, then we have $\mu(G) = (5, 5, 5, 5, 0)$. we obtain the following table

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{n} d_i^*$
1	5	5
2	10	10
3	15	15
4	20	20
5	20	20

Thus, not only does the Grone-Merris Conjecture (theorem) hold, but we have equality in $\sum_{i=1}^{k} \mu_{n-i+1} \leq \sum_{i=1}^{k} d_i^*$ for all $k = 1, 2, \ldots, 5$

2.6 Trees and Forests

Theorem 2.6.1. *Let T be a tree with n vertices. Then $S_k(T) \leq e(T) + 2k - 1$ for $1 \leq k \leq n$.*

Proof. we prove the assertion by induction on $|V(T)|$. If T is a star them by lemma 2.1.10(ii), $S_k(T) = n + k - 1$ for $1 \leq k < n$, and we are done. Thus assume that T is not a star. Then T has an edge whose removal leaves a forest F consisting of two trees T_1 and T_2 both having at least one edge. Suppose that k_i of the k largest eigen values of F come from the Laplacian spectrum of T_i for $i = 1, 2$, where $k_1 + k_2 = k$. If one of k_i, say k_2 is zero then by $|V(T)| \geq 2$, corollary 2.1.16 and the introduction hypothesis, we conclude that $S_k(T) = S_k(F \cup k_2) \leq S_{k_1}(T_1) + S_k(K_2) \leq (e(T_1) + 2K_1 - 1) + 2 \leq n + 2k - 2 = e(T) + 2k - 1$. Otherwise, using corollary 2.1.16 and the induction hypothesis, we have $S_k(T) = S_k(T_1 \cup T_2 \cup K_2) \leq S_{k_1}(T_1) + S_{k_2}(T_2) + S_k(k_2) \leq (e(T_1) + 2k_1 - 1) + (e(T_2) + 2k_2 - 1) + 2 = e(T) + 2k - 1$. This completes the proof. □

Corollary 2.6.2. *Conjecture 2.1.2 is true for the complements of trees, uni-cyclic graphs, and bi-cyclic graphs (for all K).*

Theorem 2.6.3. *Let T be a tree then the equality holds in the Grone-Merris Theorem for all k if and only if T is a star.*

Proof. Let T be tree with n vertices the Laplacian eigen values $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots \geq \mu_{n-1} > 0$, and the degree sequence $d_1 \geq d_2 \geq d_3 \cdots \geq d_n$. We have ${d_1}^*(G) = n$

as $d_i(G) \geq 1$ for all i. If equality occurs in the Grone-Merris theorem for all i, then we must have $\mu_1 = d_1^* = n$. This implies that $mu_1 = n$. So that T is join of two graphs (n is an eigen value of G if and only if G is join of two graphs) let $T = G_1 + G_2, |V(G_i)| \geq 1$ for $i = 1, 2$. As T is a tree, so it contains no cycle . We have $G_1 = \overline{k_1}$ and $G_2 = \overline{k_{n-1}}$, which gives T $= \overline{k_1} + \overline{k_{n-1}} = k_{1,n-1}$, which clearly is a star. ☐

Theorem 2.6.4. *Let F be a Forest of order n. Then for an integer k with $1 \leq k \leq n$,*

$$s_k(F) \leq e(F) + \binom{k+1}{2} \tag{2.30}$$

Proof. Since F is a forest, the k largest Laplacian eigen values of F belong to some of its connected components, denoted by $T_1, T_2, \ldots T_r$, where $1 \leq t \leq k$. Assume that k_i of the k largest Laplacian eigen values of F come from Laplacian spectrum of T_i with n_i vertices, where $1 \leq k_i \leq n_i, i = 1, 2, \ldots t$ and $\sum\limits_{i=1}^{t} k_i = k$. Since T_i is a tree, then by inequality 2.30

$$S_{k_i}(T_i) \leq e(T_i) + \binom{k_i+1}{2},$$

Where $1 \leq k_i \leq n_i$ and $i = 1, 2 \ldots t$. Note that $\sum\limits_{i=1}^{n} e(T_i) \leq e(G)$. Hence

$$\begin{aligned} S_k(F) &= \sum_{i=1}^{n} s_{k_i}(T_i) \\ &\leq \sum_{i=1}^{t} [e(T_i) + \binom{k_i+1}{2}] \\ &\leq e(F) + \frac{\sum\limits_{i=1}^{t} k_i^2 + \sum\limits_{i=1}^{t} k_i}{2} \\ &\leq e(F) + \frac{k^2+k}{2} \end{aligned}$$

the proof of theorem is completed. ☐

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